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Mathematical Economics

Application of Fractional Calculus

Edited by

Vasily E. Tarasov

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Special Issue Editor

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About the Special Issue Editor

Vasily E. Tarasov is a leading researcher with the Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, and Professor with the Information Technologies and Applied Mathematics Faculty, Moscow Aviation Institute (National Research University). His fields of research include several topics of applied mathematics, theoretical physics, mathematical economics. He has published over 200 papers in international peer-reviewed scientific journals, including over 50 papers in mathematical economics. He has published 12 scientific books, including "Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media" (Springer, 2010) and "Quantum Mechanics of Non-Hamiltonian and Dissipative Systems" (Elsevier, 2008). He was an Editor of "Handbook of Fractional Calculus with Applications. Volumes 4 and 5" (De Gruyter, 2019). Vasily E. Tarasov is a member of the Editorial Boards of 12 journals, including *Fractional Calculus and Applied Analysis* (De Gruyter), *The European Physical Journal Plus* (Springer), *Communications in Nonlinear Science and Numerical Simulations* (Elsevier), *Entropy* (MDPI), *Computational and Applied Mathematics* (Springer), *International Journal of Applied and Computational Mathematics* (Springer), *Mathematics* (MDPI), *Journal of Applied Nonlinear Dynamics* (LH Scientific Publishing LLC), *Fractional Differential Calculus* (Ele-Math), *Fractal and Fractional* (MDPI), and others. At present, his h-index is 35 in Web of Science and Scopus with an h-index of 45 on Google Scholar Citations. Web of Science ResearcherID: D-6851-2012; Scopus Author ID: 7202004582; ORCID: 0000-0002-4718-6274; Google Scholar Citations ID: sfFN5B4AAAAJ; IstinaResearcher ID (IRID): 394056.

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Editorial

Mathematical Economics: Application of Fractional Calculus

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Mathematical economics is a theoretical and applied science in which economic objects, processes, and phenomena are described by using mathematically formalized language. In this science, models are formulated on the basis of mathematical formalizations of economic concepts and notions. An important purpose of mathematical economics is the formulation of notions and concepts in form, which will be mathematically adequate and self-consistent, and then, on their basis, to construct models of processes and phenomena of economy. The standard mathematical language, which is actively used in mathematical modeling of economy, is the calculus of derivatives and integrals of integer orders, the differential and difference equations. These operators and equations allowed economists to formulate models in mathematical form, and, on this basis, to describe a wide range of processes and phenomena in economy. It is known that the integer-order derivatives of functions are determined by the properties of these functions in an infinitely small neighborhood of the point, in which the derivatives are considered. As a result, economic models, which are based on differential equations of integer orders, cannot describe processes with memory and non-locality. As a result, this mathematical language cannot take into account important aspects of economic processes and phenomena.

Fractional calculus is a branch of mathematics that studies the properties of differential and integral operators that are characterized by real or complex orders. The methods of fractional calculus are powerful tools for describing the processes and systems with memory and nonlocality. There are various types of fractional integral and differential operators that are proposed by Riemann, Liouville, Grunwald, Letnikov, Sonine, Marchaud, Weyl, Riesz, Hadamard, Kober, Erdelyi, Caputo and other mathematicians. The fractional derivatives have a set of nonstandard properties such as a violation of the standard product and chain rules. The violation of the standard form of the product rule is a main characteristic property of derivatives of non-integer orders that allows us to describe complex properties of processes and systems.

Recently, fractional integro-differential equations are actively used to describe a wide class of economical processes with power-law memory and spatial nonlocality. Generalizations of basic economic concepts and notions of the economic processes with memory were proposed. New mathematical models with continuous time are proposed to describe the economic dynamics with a long memory.

The purpose of this Special Issue is to create a collection of articles reflecting the latest mathematical and conceptual developments in mathematical economics with memory and non-locality, based on applications of modern fractional calculus.

The proposed collection of works can be conditionally divided into three parts: historical, mathematical and applied.

This collection opens with two review articles, [1], by Vasily E. Tarasov, and [2], by Francesco Mainardi, purpose of which is a brief description of the history of the application of fractional calculus in economics and finance.

The collection continues with a review work, [3], by Vasily E. Tarasov, the purpose of which is a description of the problems and difficulties arising in the construction of fractional-dynamic analogs of standard models by using fractional calculus. In article [4], by Anatoly N. Kochubei and Yuri Kondratiev, the authors proposed correct mathematical statements for growth models with memory in general cases, for application in mathematical economics of processes with memory and distributed lag. In article [5], by Jean-Philippe Aguilar, Jan Korbel and Yuri Luchko, applications of the fractional diffusion equation to option pricing and risk calculations are described. In work [6], by Jonathan Blackledge, Derek Kearney, Marc Lamphiere, Raja Rani and Paddy Walsh, authors discuss a range of results that are connected to Einstein's evolution equation, focusing on the Lévy distribution. In article [7], by Tomas Skovranek, a mathematical model, which is based on the one-parameter Mittag-Leffler function, is proposed to describe the relation between the unemployment rate and the inflation rate, also known as the Phillips curve. In article [8], by Yingkang Xie, Zhen Wang and Bo Meng, it is considered a fractional generalization of business cycle model with memory and time delay.

Further, this collection continues with works in which fractional calculus is applied to describe economy of different countries. In article [9], by José A. Tenreiro Machado, Maria Eugénia Mata and António M. Lopes, the fractional calculus and concept of pseudo-phase space are used for modeling the dynamics of world economies and forecasting a country's gross domestic product. In work [10], by Inés Tejado, Emiliano Pérez and Duarte Valério, the fractional calculus is applied to study the economic growth of the countries in the Group of Twenty (G20). In article [11], by Hao Ming, JinRong Wang and Michal Fečkan, the application of fractional calculus to economic growth models of Chinese economy is proposed. In work [12], by Ertuğrul Karaçuha, Vasil Tabatadze, Kamil Karaçuha, Nisa Özge Önal and Esra Ergün, the fractional calculus approach and the time series modeling are applied to describe the Gross Domestic Product (GDP) per capita for nine countries (Brazil, China, India, Italy, Japan, UK, USA, Spain and Turkey) and the European Union.

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Review

On History of Mathematical Economics: Application of Fractional Calculus

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Abstract: Modern economics was born in the Marginal revolution and the Keynesian revolution. These revolutions led to the emergence of fundamental concepts and methods in economic theory, which allow the use of differential and integral calculus to describe economic phenomena, effects, and processes. At the present moment the new revolution, which can be called "Memory revolution", is actually taking place in modern economics. This revolution is intended to "cure amnesia" of modern economic theory, which is caused by the use of differential and integral operators of integer orders. In economics, the description of economic processes should take into account that the behavior of economic agents may depend on the history of previous changes in economy. The main mathematical tool designed to "cure amnesia" in economics is fractional calculus that is a theory of integrals, derivatives, sums, and differences of non-integer orders. This paper contains a brief review of the history of applications of fractional calculus in modern mathematical economics and economic theory. The first stage of the Memory Revolution in economics is associated with the works published in 1966 and 1980 by Clive W. J. Granger, who received the Nobel Memorial Prize in Economic Sciences in 2003. We divide the history of the application of fractional calculus in economics into the following five stages of development (approaches): ARFIMA; fractional Brownian motion; econophysics; deterministic chaos; mathematical economics. The modern stage (mathematical economics) of the Memory revolution is intended to include in the modern economic theory new economic concepts and notions that allow us to take into account the presence of memory in economic processes. The current stage actually absorbs the Granger approach based on ARFIMA models that used only the Granger–Joyeux–Hosking fractional differencing and integrating, which really are the well-known Grunwald–Letnikov fractional differences. The modern stage can also absorb other approaches by formulation of new economic notions, concepts, effects, phenomena, and principles. Some comments on possible future directions for development of the fractional mathematical economics are proposed.

Keywords: mathematical economics; economic theory; fractional calculus; fractional dynamics; long memory; non-locality

1. Introduction: General Remarks about Mathematical Economics

Mathematical economics is a theoretical and applied science, whose purpose is a mathematically formalized description of economic objects, processes, and phenomena. Most of the economic theories are presented in terms of economic models. In mathematical economics, the properties of these models are studied based on formalizations of economic concepts and notions. In mathematical economics, theorems on the existence of extreme values of certain parameters are proved, properties of equilibrium states and equilibrium growth trajectories are studied, etc. This creates the impression that the proof of the existence of a solution (optimal or equilibrium) and its calculation is the main aim of mathematical economics. In reality, the most important purpose is to formulate economic notions and concepts in mathematical form, which will be mathematically adequate and self-consistent, and then, on their basis to construct mathematical models of economic processes and phenomena. Moreover, it is not enough to prove the existence of a solution and find it in an analytic or numerical form, but it is necessary to give an economic interpretation of these obtained mathematical results.

We can say that modern mathematical economics began in the 19th century with the use of differential (and integral) calculus to describe and explain economic behavior. The emergence of modern economic theory occurred almost simultaneously with the appearance of new economic concepts, which were actively used in various economic models. "Marginal revolution" and "Keynesian revolution" in economics led to the introduction of the new fundamental concepts into economic theory, which allow the use of mathematical tools to describe economic phenomena and processes. The most important mathematical tools that have become actively used in mathematical modeling of economic processes are the theory of derivatives and integrals of integer orders, the theory of differential and difference equations. These mathematical tools allowed economists to build economic models in a mathematical form and on their basis to describe a wide range of economic processes and phenomena. However, these tools have a number of shortcomings that lead to the incompleteness of descriptions of economic processes. It is known that the integer-order derivatives of functions are determined by the properties of these functions in an infinitely small neighborhood of the point, in which the derivatives are considered. As a result, differential equations with derivatives of integer orders, which are used in economic models, cannot describe processes with memory and non-locality. In fact, such equations describe only economic processes, in which all economic agents have complete amnesia and interact only with the nearest neighbors. Obviously, this assumption about the lack of memory among economic agents is a strong restriction for economic models. As a result, these models have drawbacks, since they cannot take into account important aspects of economic processes and phenomena.

2. A Short History of Fractional Mathematical Economics

"Marginal revolution" and "Keynesian revolution" introduced fundamental economic concepts, including the concepts of "marginal value", "economic multiplier", "economic accelerator", "elasticity" and many others. These revolutions led to the use of mathematical tools based on the derivatives and integrals of integer orders, and the differential and difference equations. As a result, the economic models with continuous and discrete time began to be mathematically described by differential equations with derivatives of integer orders or difference equations of integer orders.

It can be said that at the present moment new revolutionary changes are actually taking place in modern economics. These changes can be called a revolution of memory and non-locality. It is becoming increasingly obvious in economics that when describing the behavior of economic agents, we must take into account that their behavior may depend on the history of previous changes in the economy. In economic theory, we need new economic concepts and notions that allow us to take into account the presence of memory in economic agents. New economic models and methods are needed, which take into account that economic agents may remember the changes of economic indicators and factors in the past, and that this affects the behavior of agents and their decision making. To describe this behavior we cannot use the standard mathematical apparatus of differential (or difference) equations of integer orders. In fact, these equations describe only such economic processes, in which agents actually have an amnesia. In other words, economic models, which use only derivatives of integer orders, can be applied when economic agents forget the history of changes of economic indicators and factors during an infinitesimally small period of time. At the moment it is becoming clear that this restriction holds back the development of economic theory and mathematical economics.

In modern mathematics, derivatives and integrals of arbitrary order are well known [1–5]. The derivative (or integral), order of which is a real or complex number and not just an integer, is called fractional derivative and integral. Fractional calculus as a theory of such operators has a long history [6–15]. There are different types of fractional integral and differential operators [1–5]. For fractional differential and integral operators, many standard properties are violated, including such properties as the standard product (Leibniz) rule, the standard chain rule, the semi-group property for orders of derivatives, the semi-group property for dynamic maps [16–21]. We can state that the violation of the standard form of the Leibniz rule is a characteristic property of derivatives of non-integer orders [16]. The most important application of fractional derivatives and integrals of non-integer order is fading memory and spatial non-locality.

The new revolution ("Memory revolution") is intended to include in the modern economic theory and mathematical economics different processes with long memory and non-locality. The main mathematical tool designed to "cure amnesia" in economics is the theory of derivatives and integrals of non-integer order (fractional calculus), fractional differential and difference equations [1–5]. This revolution has led to the emergence of a new branch of mathematical economics, which can be called "fractional mathematical economics."

Fractional mathematical economics is a theory of fractional dynamic models of economic processes, phenomena and effects. In this framework of mathematical economics, the fractional calculus methods are being developed for application to problems of economics and finance. The field of fractional mathematical economics is the application of fractional calculus to solve problems in economics (and finance) and for the development of fractional calculus for such applications. Fractional mathematical economics can be considered as a branch of applied mathematics that deals with economic problems. However, this point of view is obviously a narrowing of the field of research, goals and objectives of this area. An important part of fractional mathematical economics is the use of fractional calculus to formulate new economic concepts, notions, effects and phenomena. This is especially important due to the fact that the fractional mathematical economics is now only being formed as an independent science. Moreover, the development of the fractional calculus itself and its generalizations will largely be determined precisely by such goals and objectives in economics, physics and other sciences.

This "Memory revolution" in the economics, or rather the first stage of this revolution, can be associated with the works, which were published in 1966 and 1980 by Clive W. J. Granger [22–26], who received the Nobel Memorial Prize in Economic Sciences in 2003 [27].

The history of the application of fractional calculus in economics can be divided into the following stages of development (approaches): ARFIMA; fractional Brownian motion; econophysics; deterministic chaos; mathematical economics. The appearance of a new stage obviously does not mean the cessation of the development of the previous stage, just as the appearance of quantum theory did not stop the development of classical mechanics.

Further in Sections 2.1–2.5, we briefly describe these stages of development, and then in Section 3 we outline possible ways for the further development of fractional mathematical economics.

2.1. ARFIMA Stage (Approach)

ARFIMA Stage (Approach): This stage is characterized by models with discrete time and application of the Grunwald–Letnikov fractional differences.

More than fifty years ago, Clive W. J. Granger (see preprint [22], paper [23], the collection of the works [24,25]) was the first to point out long-term dependencies in economic data. The articles demonstrated that spectral densities derived from the economic time series have a similar shape. This fact allows us to say that the effect of long memory in the economic processes was found by Granger. Note that, he received the Nobel Memorial Prize in Economics in 2003 "for methods of analyzing economic time series with general trends (cointegration)" [27].

Then, Granger and Joyeux [26], and Hosking [28] proposed the fractional generalization of ARIMA(p,d,q) models (the ARFIMA (p,d,q) models) that improved the statistical methods for researching of processes with memory. As the main mathematical tool for describing memory, fractional differencing and integrating (for example, see books [29–34] and reviews [35–38]) were proposed for discrete time case. The suggested generalization of the ARIMA(p,d,q) model is realized by considering non-integer (positive and negative) order d instead of positive integer values of d. The Granger–Joyeux–Hosking (GJH) operators were proposed and used without relationship with the fractional calculus. As was proved in [39,40], these GJH operators are actually the Grunwald–Letnikov fractional differences (GLF-difference), which have been suggested more than a hundred and fifty years ago and are used in the modern fractional calculus [1,3]. We emphasize that in the continuous limit these GLF-differences give the GLF-derivatives that coincide with the Marchaud fractional derivatives (see Theorem 4.2 and Theorem 4.4 of [1]).

Among economists, the approach proposed by Gravers (and based on the discrete operators proposed by them) is the most common and is used without an explicit connection with the development of fractional calculus. It is obvious that the restriction of mathematical tools only to the Grunwald–Letnikov fractional differences significantly reduces the possibilities for studying processes with memory and non-locality. The use of fractional calculus in economic models will significantly expand the scope and allows us to obtain new results.

2.2. Fractional Brownian Motion (Mathematical Finance) Stage (Approach)

Fractional Brownian Motion Stage (Approach): This stage is characterized by financial models and the application of stochastic calculus methods and stochastic differential equations.

Andrey N. Kolmogorov, who is one of the founders of modern probability theory, was the first who considered in 1940 [41] the continuous Gaussian processes with stationary increments and with the self-similarity property A.N. Kolmogorov called such Gaussian processes "Wiener Spirals". Its modern name is the fractional Brownian motion that can be considered as a continuous self-similar zero-mean Gaussian process and with stationary increments.

Starting with the article by L.C.G. Rogers [42], various authors began to consider the use of fractional Brownian motion to describe different financial processes. The fractional Brownian motion is not a semi-martingale and the stochastic integral with respect to it is not well-defined in the classical Ito's sense. Therefore, this approach is connected with the development of fractional stochastic calculus [43–45]. For example, in the paper [43] a stochastic integration calculus for the fractional Brownian motion based on the Wick product was suggested.

At the present time, this stage (approach), which can be called as a fractional mathematical finance, is connected with the development of fractional stochastic calculus, the theory fractional stochastic differential equations and their application in finance. The fractional mathematical finance is a field of applied mathematics, concerned with mathematical modeling of financial markets by using the fractional stochastic differential equations.

As a special case of fractional mathematical finance, we can note the fractional generalization of the Black–Scholes pricing model. In 1973, Fischer Black and Myron Scholes [46] derived the famous theoretical valuation formula for options. In 1997, the Royal Swedish Academy of Sciences has decided to award the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel [47] to Myron S. Scholes, for the so-called Black–Scholes model published in 1973: "Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options." [47].)

For the first time a fractional generalization of the Black–Scholes equation was proposed in [48] by Walter Wyss in 2000. Wyss [48] considered the pricing of option derivatives by using the time-fractional Black–Scholes equation and derived a closed form solution for European vanilla options. The Black–Scholes equation is generalized by replacing the first derivative in time by a fractional derivative in time of the order $\alpha \in (0, 1)$. The solution of this fractional Black–Scholes equation is considered. However, in the Wyss paper, there are no financial reasons to explain why a time-fractional derivative should be used.

The works of Cartea and Meyer-Brandis [49] and Cartea [50] proposed a stock price model that uses information about the waiting time between trades. In this model the arrival of trades is driven by a counting process, in which the waiting-time between trades processes is described by the Mittag–Leffler survival function (see also [51]). In the paper [50], Cartea proposed that the value of derivatives satisfies the fractional Black–Scholes equation that contains the Caputo fractional derivative with respect to time. It should be noted that, in general, the presence of a waiting time and a delay time does not mean the presence of memory in the process.

In the framework of the fractional Brownian motion Stage, a lot of papers [50–71] and books [72,73] were written on the description of financial processes with memory and non-locality.

As a rule, in fractional mathematical finance, fractional dynamic models are created without establishing links with economic theory and without formulating new economic or financial concepts, taking only observable market prices as input data. In the fractional mathematical finance, the main requirement is the mathematical consistency and the compatibility with economic theory is not the key point.

2.3. Econophysics Stage (Approach)

Econophysics Stage (Approach): This stage is characterized by financial models and the application of physical methods and equations.

Twenty years ago, a new branch of the econophysics, which is connected with the application of fractional calculus, has appeared. In fact, this branch, which can be called fractional econophysics, was born in 2000 and it can be primarily associated with the works of Francesco Mainardi, Rudolf Gorenflo, Enrico Scalas, Marco Raberto [74–76] on the continuous-time finance.

In fractional econophysics, the fractional diffusion models [74–76] are used in finance, where price jumps replace the particle jumps in the physical diffusion model. The corresponding stochastic models are called continuous time random walks (CTRWs), which are random walks that also incorporate a random waiting time between jumps. In finance, the waiting times measure delay between transactions. These two random variables (price change and waiting time) are used to describe the long-time behavior in financial markets. The diffusion (hydrodynamic) limit, which is used in physics, is considered for continuous time random walks [74–76]. It was shown that the probability density function for the limit process obeys a fractional diffusion equation [74–76].

After the pioneering works [74–76] that laid the foundation for the new direction of econophysics (fractional econophysics), various papers were written on the application of fractional dynamics methods and physical models to describe processes in finance and economics (for example, see [77–84]). The history and achievements of the econophysics stage in the first five years are described by Enrico Scalas in the article [85] in 2016.

The fractional econophysics, as a branch of econophysics, can be defined as a new direction of research applying methods developed in physical sciences, to describe processes in economics and finance, basically those including power-law memory and spatial non-locality. The mathematical tool of this branch of econophysics is the fractional calculus. For example, application to the study of continuous time finance by using methods and results of fractional kinetics and anomalous diffusion. Another example, which is not related to finance, is the time-dependent fractional dynamics with memory in quantum and economic physics [86].

In this stage, the fractional calculus was applied mainly to financial processes. In the papers on fractional econophysics, generalization of basic economic concepts and principles for economic processes with memory (and non-locality) are not suggested.

Unfortunately, economists do not always understand the analogies with the methods and concepts of modern physics, which restricts the possibilities for economists to use this approach. As a result, it holds back and limits both the development and application of the fractional econophysics approach to describing economic processes with memory and non-locality.

It can be said that the time has come for economists and econophysicists to work together on the formulation of economic analogs of physical concepts and methods used in fractional econophysics, and linking them with existing concepts and methods of economic theory. For the development of a fractional mathematical economics, a translation should be made from the language of physics into the language of economics.

2.4. Deterministic Chaos Stage (Approach)

Deterministic Chaos Stage (Approach): This stage is characterized by financial (and economic) models and application of methods of nonlinear dynamics. Strictly speaking, this approach should be attributed to the econophysics stage/approach.

Nonlinear dynamics models are useful to explain irregular and chaotic behavior of complex economic and financial processes. The complex behaviors of nonlinear economic processes restrict the use of analytical methods to study nonlinear economic models.

In 2008, for the first time, Wei-Ching Chen proposed in [87] a fractional generalization of a financial model with deterministic chaos. Chen [87] studied the fractional-dynamic behaviors and describes fixed points, periodic motions, chaotic motions, and identified period doubling and intermittency routes to chaos in the financial process that is described by a system of three equations with the Caputo fractional derivatives. He demonstrates by numerical simulations that chaos exists when orders of derivatives are less than three and that the lowest order at which chaos exists was 2.35. The work [88] studied the chaos control method of such a kind of system by feedback control, respectively.

In the framework of the deterministic chaos stage, many papers [89–99] have been devoted to the description of financial processes with memory. In some papers [100–105], economic models were considered.

We should note that the various stages/approaches of development of fractional mathematical economics did not develop in complete isolation from each other. For example, for the fractional Chen model of dynamic chaos in the economy, Tomáš Škovránek, Igor Podlubny, Ivo Petráš [106] applied the concept of the state space (the configuration space, the phase space) that arose in physics more than a hundred years ago. As state variables authors consider the gross domestic product, inflation, and unemployment rate. The dynamics of the modeled economy in time, which is represented by the values of these three variables, is described as a trajectory in state-space. The system of three fractional order differential equations is used to describe dynamics of the economy by fitting the available economic data. Then *José* A. Tenreiro Machado, Maria E. Mata, Antonio M. Lopes suggested the development of the state space concept in the papers [107–109]. The economic growth is described by using the multidimensional scaling (MDS) method for visualizing information in data. The state space is used to represent the sequence of points (the fractional state space portrait, FSSP, and pseudo phase plane, PPP) corresponding to the states over time.

2.5. Mathematical Economics Stage (Approach)

Mathematical Economics Stage (Approach): This stage is characterized by macroeconomic and microeconomic models with continuous time and generalization of basic economic concepts and notions.

The fractional calculus approach has been used to describe the concept of memory itself for economic processes in [39,40,110–117], and to define basic concepts of economic processes with memory and non-locality in works [118–139].

From a subjective point of view, this stage began with a proposal of generalizations of the basic economic concepts and notions at the beginning of 2016, when the concept of elasticity for economic processes with memory was proposed [132–135]. Then in 2016, the concepts of the marginal values with memory [118–120], the concept of accelerator and multiplier with memory [123,130] and others were suggested [132–136,138]. These concepts are used in fractional generalizations of some standard economic models [140–167] with the continuous time, which were proposed in 2016 and subsequent years [168–189]. These dynamic models describe fractional dynamics of economic processes with memory.

The fractional calculus approach has been used to define basic concepts of economic processes with memory in works [118–139], and to describe dynamics of economic processes with memory [168–189] in the framework of the continuous time models.

It should be noted that formal replacements of derivatives of integer order by fractional derivatives in standard differential equations, which describe economic processes, and solutions of the obtained fractional differential equations were considered in papers published before 2016. However, these papers were purely mathematical works, in which generalizations of economic concepts and notions were not proposed. In these works, fractional differential equations have not been derived, since a formal replacement of integer-order derivatives by fractional derivatives cannot be recognized as a derivation of the equations. Formulations of economic conclusions and interpretations from the obtained solutions are not usually suggested in these papers. Examples of incorrectness and errors in such generalizations are given in the work [189]. In the paper [189], we formulate five principles of the fractional-dynamic generalization of standard dynamic models and then we illustrate these principles by examples from fractional mathematical economics. We can state that in the works with formal fractional generalizations of standard economic equations the Principles of Derivability and Interpretability [189] were neglected. Let us give a brief formulation of the Principles of Derivability and Interpretability.

Derivability Principle: It is not enough to generalize the differential equations describing the dynamic model. It is necessary to generalize the whole scheme of obtaining (all steps of derivation) these equations from the basic principles, concepts and assumptions. In this sequential derivation of the equations we should take into account the non-standard characteristic properties of fractional derivatives and integrals. If necessary, generalizations of the notions, concepts and methods, which are used in this derivation, should also be obtained.

Interpretability Principle: The subject (physical, economic) interpretation of the mathematical results, including solutions and their properties, should be obtained. Differences, and first of all qualitative differences, from the results based on the standard model should be described.

The most important purpose of the modern stage of development of fractional mathematical economics is the inclusion of memory and non-locality into the economic theory, into the basic economic concepts and methods. The economics should be extended and generalized such that it takes into account the memory and non-locality. Fractional generalizations of standard economic models should be constructed only on this conceptual basis. The most important purpose of studying such generalizations is the search and formulation of qualitatively new effects and phenomena caused by memory and non-locality in the behavior of economic processes. In this case, these results in mathematical economics, which are based on fractional calculus, can be further used in computer simulations of real economic processes and in econometric studies.

Let us list some generalizations of economic concepts and fractional generalizations of economic models that have already been proposed in recent years. Using the fractional calculus approach to describe the processes with memory and non-locality, the generalizations of some basic economic notions were proposed in the works [118–139]. The list of these new notions and concepts primarily include the following:

- The marginal value of non-integer order [118–122,190] with memory and non-locality;
- The economic multiplier with memory [123,124];
- The economic accelerator with memory [123,124];
- The exact discretization of economic accelarators and multiplier [125–128] based on exact fractional differences [129];
- The accelerator with memory and crisis periodic sharp bursts [130,131];
- The duality of the multiplier with memory and the accelerator with memory [123,124];
- The elasticity of fractional order [132–135] for processes with memory and non-locality;
- The measures of risk aversion with non-locality [136] and with memory [137];
- The warranted (technological) rate of growth with memory [112,170,174–176,189];
- The non-local methods of deterministic factor analysis for [138,139];

And some other.

The use of these notions and concepts makes it possible for us to generalize some classical economic models, including those proposed by the following well-known economists:

- Henry Roy F. Harrod [140–142] and Evsey D. Domar, [143,144];
- John M. Keynes [145–148];
- Wassily W. Leontief [149,150];
- Alban W.H. Phillips [151,152];
- Roy G.D. Allen [153–156];
- Robert M. Solow [157,158] and Trevor W. Swan [159];
- Nicholas Kaldor [160–162];

And other scientists.

Valentina V. Tarasova and the author built various economic models with power-law memory, which are generalizations of the classical models. For example, the following economic models have been proposed.

- The natural growth model [168,169];
- The growth model with constant pace [170,171];
- The Harrod–Domar model [172,173] and [112,174–176];
- The Keynes model [177–179];
- The dynamic Leontief (intersectoral) model [86,180–182];
- The dynamics of fixed assets (or capital stock) [170,171];
- The logistic growth model with memory [183,189];
- The model of logistic growth with memory and crises [183];
- The time-dependent dynamic intersectoral model with memory [86,182];
- The Phillips model with memory and lag [185];
- The Harrod–Domar growth model with memory and distributed lag [186];
- The dynamic Keynesian model with memory and lag [187];
- The model of productivity with fatigue and memory [188];
- The Solow–Swan model [189];
- The Kaldor-type model of business cycles (the Van der Pol model) [189];

And some other economic models.

Let us also note works, in which fractional dynamic generalizations of economic models were proposed without introducing new economic notions and concepts.

(1) Michele Caputo proposed some fractional dynamic model of economy [191–200]:

    - In the standard relaxation equation, which describes the relaxation economy to equilibrium, the memory has been introduced in the reactivity of investment to the interest rate.
    - The continuous-time IS–LM model with memory [192];
    - The tax version of the Fisher model with memory for stock prices and inflation rates [199] that can be used to predict nominal and real interest rate behavior with memory.

(2) Mathematical description of some fractional generalization of economic models was proposed by the Kabardino–Balkarian group: Adam M. Nakhushev [201,202], Khamidbi Kh. Kalazhokov [203], Zarema A. Nakhusheva [204].

(3) Mathematical description of some fractional generalization of economic models were proposed by the Kamchatka group: Viktoriya V. Samuta, Viktoriya A. Strelova, and Roman I. Parovik [205], Yana E. Shpilko, Anastasiya E. Solomko., Roman I. Parovik [206] Danil M. Makarov [207].

(4) Shiou-Yen Chu and Christopher Shane proposed the hybrid Phillips curve model with memory to describe the dynamic process of inflation with memory in the work [208].

(5) Rituparna Pakhira, Uttam Ghosh and Susmita Sarkar derived [209–213] some inventory models with memory.
(6) Computer simulation for modeling the national economies in the framework of the fractional generalizations of the Gross domestic product (GDP) model was proposed by Inés Tejado, Duarte Valério, Nuno Valério, Emiliano Pérez [214–220] in 2014–2019, and by Dahui Luo, JinRong Wang, Michal Feckan in 2018 in the paper [221].
(7) In addition, we may note the works with economic models that were proposed in [100–105] that are related to the deterministic chaos stage.

Let us note that the problems and difficulties arising in the construction of fractional-dynamic analogs of standard economic models by using the fractional calculus are described in [189] with details.

Some of proposed models can be considered as econophysics approach, which are based on fractional generalization of the standard damped harmonic oscillator equation, where the memory has been introduced in the frictional term by using fractional derivative instead of first-orderr derivative.

New principles, effects and phenomena have been suggested for fractional economic dynamics with memory and non-locality (for example, see [174–176,189,222]). Qualitatively new effects due to the presence of memory in the economic process are described in the works [174–176,189,222].

In my opinion, this stage of the development of fractional mathematical economics actually includes (absorbs) approaches based on the ARFIMA model using only the Granger–Joyeux–Hosking fractional differencing and integrating, which in really are the well-known Grunwald–Letnikov fractional differences [39]. This opinion is based on the obvious fact that the new stage allows the AFRIMA approach to go beyond the restrictions of the Grunwald–Letnikov operators, and use different types of fractional finite differences and fractional derivatives of non-integer orders.

Moreover, this stage can include (absorbs) approaches based on the fractional econophysics and deterministic chaos. For the econophysics approach, new opportunities are opening up on the way to formulating economic analogues of physical concepts and notions that will be more understandable to economists. This will significantly simplify the implementation of the concepts and methods of fractional econophysics in economic theory and application.

The most important element in the construction of the fractional mathematical economics as a new theory is the emergence and the formation of new notions, concepts, effects, phenomena, principles and methods, which are specific only to this theory. This gives rise to a new scientific direction (the fractional mathematical economics), since there is something of their own that others do not have.

We have now entered the stage of forming a new direction in mathematical economics and economic theory, when concepts and methods are not borrowed from other sciences and areas, but their own are created.

3. New Future Stages and Approaches

There is a natural question. What stages and approaches will appear and develop in the future? In this section we propose some assumptions about the future use of fractional calculus in economics and discuss the direction of development of fractional mathematical economics.

3.1. Self-Organization in Fractional Economic Dynamics

Self-organization processes play an important role in both the natural and social sciences. In the description of self-organization in economic (and social) processes, it is necessary to abandon the assumption that all economic agents suffer from amnesia. They should be considered as agents with memory that interact with each other. We can consider self-organization with memory [222] in economics and social sciences. Therefore, the fractional mathematical economics and economic theory can be developed by considering the generalization of different economic models of self-organization that are described in the books [223–225].

3.2. Distributed Lag Fractional Calculus

The continuously distributed lag has been considered in economics starting with the works of Michal A. Kalecki [163] and Alban W.H. Phillips [151,152]. The continuous uniform distribution of delay time is considered by M.A. Kalecki in 1935 [163], (see also Section 8.4 of [154], (pp. 251–254)) for dynamic models of business cycles. The continuously distributed lag with the exponential distribution of delay time is considered by A.W.H. Phillips [151,152] in 1954. In the Phillips growth models, generalization of the economic concepts of accelerator and multipliers were proposed by taking into account the distributed lag. The operators with continuously distributed lag were considered by Roy G.D. Allen [153,154], (pp. 23–29), in 1956. Currently, economic models with delay are actively used to describe the processes in the economy.

The time delay is caused by finite speeds of processes and therefore it cannot be considered as processes with long memory (for some details about concept of memory, see Section 3.12 of this paper and [29–35,112,113,116,117]). For example, in physics the propagation of the electromagnetic field with finite speed in a vacuum does not mean the presence of memory in this process. In economics and electrodynamics, processes with time delay (lag) are not referred to as processes with memory and effects of time delay are not interpreted as a memory.

Note that the operators with exponentially distributed lag, which were defined in the works of Caputo and Fabrizio [226,227], cannot be interpreted as fractional derivatives of non-integer orders and cannot describe processes with memory. Note that exponential distribution is the continuous analogue of the geometric distribution that has the key property of being memoryless. The Caputo–Fabrizio operators are integer-order derivatives with the exponentially distributed delay time [228]. The fractional generalizations of the Caputo–Fabrizio operator are proposed in [185–187,228].

The distributed lag fractional calculus was proposed by the author and Svetlana S. Tarasova in [228]. To take into account the memory and lag in economic and physical models, the fractional differential and integral operators with continuously distributed lag (time delay) were proposed in [228]. The distributed lag fractional operators are compositions of fractional differentiation or integration and continuously distributed translation (shift). The kernels of these operators are the Laplace convolution of probability density function and the kernels of fractional derivatives or integrals. The random variable is the delay time that is distributed by probability law (distribution) on positive semiaxis. Examples of economic application of the lag-distributed fractional operators have been suggested in the works [185–187], where the economic concepts of accelerator and multipliers with distributed lag and memory were proposed.

3.3. Distributed Order Fractional Calculus

In general, the order parameter α, which can be interpreted as the parameter of memory fading [112,113], can be distributed on the interval $[\alpha_1, \alpha_2]$, where the distribution is described by a weight function $\rho(\alpha)$. The functions $\varrho(\alpha)$ describes distribution of the parameter of the memory fading on a set of economic agents. This is important for the economics, since various types of economic agents may have different parameters of memory fading. In this case, we should consider the fractional operators, which depends on the weight function $\rho(\alpha)$ and the interval $[\alpha_1, \alpha_2]$.

The concept of the integrals and derivatives with distributed orders was first proposed by Michele Caputo in [229] in 1995, and then these operators are applied and developed in different works (for example see [230–234]).

Let us note the following three cases.

(1) The simplest distribution of the order of fractional derivatives and integrals is the continuous uniform distribution. The fractional operators with uniform distribution were proposed in [112, 113,129] and were called as the Nakhushev operators. Adam M. Nakhushev [235,236] proposed the continual fractional derivatives and integrals in 1998. The fractional operators, which are inversed to the continual fractional derivatives and integrals, were suggested by Arsen V.

Pskhu [237,238]. In papers [112,113,129], we proved that the fractional integrals and derivatives of the uniform distributed order could be expressed (up to a numerical factor) through the continual fractional integrals and derivatives, which have been suggested by A. M. Nakhushev [235,236]. The proposed fractional integral and derivatives of uniform distributed order have been called in our paper [129] as the Nakhushev fractional integrals and derivatives. The corresponding inverse operators, which contains the two-parameter Mittag–Leffler functions in the kernel, were called as the Pskhu fractional integrals and derivatives [129].

(2) In the papers [112,113], we proposed the concept of "weak" memory and the distributed order fractional operators with the truncated normal distribution of the order. The truncated normal distribution with integer mean and small variance can be used to describe economic processes with memory, which is distributed around the classical case.

(3) As a special case of the general fractional operators, which were proposed by Anatoly N. Kochubei, the fractional derivatives and integrals of distributed order are investigated in the works [239,240].

Fractional differential equations of distributed orders are actively used to describe physical processes. However, at the present time, equations with distributed order operators have not yet been used to describe economic processes. We hope that new interesting effects in economics can be described by using order-distributed fractional operators.

3.4. Generalized Fractional Calculus in Economics

Generalized fractional calculus was proposed by Virginia Kiryakova and described in detail in the book [2] in 1994. The brief history of the generalized fractional calculus is given in the paper [10]. Operators of generalized fractional calculus [2,10] can be use to describe complex processes with memory and non-locality in real economy. In the application of the generalized fractional operators, an important question arises about the correct economic (and physical) interpretation of these operators. It is important to emphasize that not all fractional operators can describe the processes with memory. It is important to clearly understand what type of phenomena can be described by a given operator. For example, among these types of phenomena, in addition to memory, we can specify the time delay (lag) and the scaling. Let us give a few examples to clarify this problem.

Example 1. The Abel-type fractional integral (and differential) operator with Kummer function in the kernel, which is described in the classic book [1] (see equation 37.1 in [1], (p. 731)) can be interpreted as the Riemann–Liouville fractional integral (and derivatives) with gamma distribution of delay time [187,228]. Some Prabhakar fractional operators with the three-parameter Mittag–Leffler functions in the kernel can also be interpreted as a Laplace convolution of the Riemann–Liouville (or Caputo) fractional operators with continuously distributed lag (time delay) [186,228].

Example 2. We can state that the Kober fractional integration of non-integer order [1,2,4] can be interpreted as an expected value of a random variable up to a constant factor [241] (see also Section 9 in [228]). In this interpretation, the random variable describes dilation (scaling), which has the gamma distribution. The Erdelyi–Kober fractional integration also has a probabilistic interpretation. Fractional differential operators of Kober and the Erdelyi–Kober type have analogous probabilistic interpretation, i.e., these operators cannot describe the memory. These operators describe integer-order operator with continuously distributed dilation (scaling). The fractional generalizations of the Kober and Erdelyi–Kober operators, which can be used to describe memory and distributed dilation (scaling), were proposed in [228].

Example 3. The Riesz fractional integro-differentiation (See Section 2.10 of [4]) cannot be used to describe memory since this operator violates the causality principle, if it is written in the standard form. For economic and physical processes with memory, the causality can be described by the

Kramers–Kronig relations [116]. The Riesz fractional integro-differentiation can be used to describe power-law non-locality and power-law spatial dispersion.

Therefore, an important part of the application of generalized fractional calculus is a clear understanding of what types of processes and phenomena can describe fractional operators of non-integer order.

3.5. General Fractional Calculus

The concept of general fractional calculus was suggested by Anatoly N. Kochubei [239,240] by using the differential-convolution operators. The works [239,240] describe the conditions under which the general operator has a right inverse (a kind of a fractional integral) and produce, as a kind of a fractional derivative, equations of evolution type. A solution of the relaxation equations with the Kochubei general fractional derivative with respect to the time variable is described [239]. In the works [239,240] the Cauchy problem (A) is considered for the relaxation equation $\left(D_{(k)}X\right)(t) = \lambda X(t)$, where $\lambda < 0$. This Cauchy problem has a solution $X(\lambda, t)$, which is continuous on $\mathbb{R}_+$, infinitely differentiable and completely monotone on $\mathbb{R}_+$, if the Kochubei conditions (*) are satisfied.

In the economics, various growth models are used to describe real processes in economy. Therefore, it is very important to describe conditions, for which the Cauchy problem (A) for the growth equation $\left(D_{(k)}X\right)(t) = \lambda X(t)$, where $\lambda > 0$, has a solution $X(\lambda, t)$.

The growth equation is considered in [242] for the special case of a distributed order derivative, where it was proved that a smooth solution exists and is monotone increasing. In addition, the solution of the growth equation has been proposed for the case of fractional differential operators with distributed lag in [185–187,228]. The existence of a solution in the growth case has been also proved in 2018 by Chung-Sik Sin [243] for nonlinear equation with a generalized derivative like the Kochubei fractional derivative. The growth equation for physics is discussed in [244].

Solving the problem in the general case will allow us to accurately describe the conditions on the operator kernels (the memory functions), under which equations for models of economic growth with memory have solutions. A paper dedicated to solving this mathematical problem was written by Anatoly N. Kochubei and Yuri Kondratiev [245] in 2019 for the Special Issue "Mathematical Economics: Application of Fractional Calculus" of Mathematics. The application of these mathematical results in economics and their economic interpretation is an open question at the moment.

To understand the warranted (technological) growth rate of the economy, it is important to obtain the asymptotic behavior for solutions of the general growth equation.

In the application of the general fractional operators, it is also important to have correct economic and physical interpretations that will connect the types of operator kernels with the types of phenomena. For example, it is obvious that the kernels of general fractional operators satisfying the normalization condition will describe distributed delays in time (lag), and not memory.

3.6. Partial Differential Equations in Economics

Usually the mathematical formulation of macroeconomic models is reduced to systems of difference equations or ordinary differential equations that describe the dynamics of a relatively small number of macroeconomic aggregates. However, it is known that, in macroeconomics, partial differential equations (PDEs) naturally arise, and they are used in macroeconomics [246,247]. Accounting for non-locality (for example, a power type) in the state space leads us to the necessity of using fractional partial differential equations.

3.7. Fractional Variational Calculus in Economics

In mathematical economics, theorems on the existence of extreme values of certain parameters are proved, the properties of equilibrium points and equilibrium growth trajectories are actively studied. The existence of optimal solutions for fractional differential equations should be considered for economic processes with memory and non-locality.

Methods of the fractional calculus of variations are actively developing [248,249]. However, at the present time, none of the variational problems, which are well known in economics, has been generalized to the case of processes with memory using fractional calculus.

In the variation approach, there are some problems that restrict the possibilities of its application. One of the problems associated with the property of integration in parts, which actually turns the left-second fractional derivative into a right-sided derivative. As a result, we will obtain equations in which, in addition to being dependent on the past, there is a dependence on the future, that is, the principle of causality is violated.

We assume that this problem cannot be solved within the framework of using the principle of stationarity of the holonomic functional (action). It is necessary to use non-holonomic functionals. We can also consider non-holonomic constraints with fractional derivatives of non-integer orders [250]. We can also consider variations of non-integer order [251] and fractional variational derivatives [252].

Another problem is the mathematical interpretation and the economic (and physical) interpretations of extreme values. The non-holonomic constraints and variations of fractional orders should also have a correct economic (and physical) interpretation.

However, we emphasize that for the economics, finding the optimality and stability of the solution is very important.

3.8. Fractional Differential Games in Economics

Models of differential games in which derivatives of non-integer orders are used and, thereby, the power-fading memory is taken into account were proposed in the works of Arkadiy A. Chikriy (Arkadii Chikrii), Ivan I. Matychyn and Alexander G. Chentsov [253–257] (see also [258–260]), which are clearly not related to economy. Note that the construction of models of economic behavior, using differential games with power memory, instead of games with full memory, currently remains an open question. The construction of such models requires further research on economic behavior within the framework of game theory. The basis for such constructions can be the methods and results described in [253–260].

3.9. Economic Data and Fractional Calculus in Economic Modelling

We note the importance of using fractional calculus in computer simulations and modeling of real economic data, including data related to both macroeconomics and microeconomics.

The first works that can be attributed to the mathematical economics stage/approach are works published in 2014–2016 by Inés Tejado, Duarte Valério, Nuno Valério, [214–216]. An application of fractional calculus for modeling the national economies in the framework of the fractional generalizations of the Gross domestic product (GDP) model, which are described by the fractional differential equations are used, were considered by Inés Tejado, Duarte Valério, Nuno Valério, Emiliano Pérez [214–220] in 2014–2019, and by Dahui Luo, JinRong Wang, Michal Feckan in 2018 in the paper [221]. The fractional differential equation used in the fractional GDP model was obtained by replacing first-order derivatives with fractional derivatives. Therefore, this model requires theoretical justification and consistent derivation.

To describe the real economic processes in the framework of fractional dynamic models, it is necessary to combine theoretical constructions and computer modeling.

3.10. Big Data

It is obvious that Big Data that describes behavior of peoples and other economic agents should contain information that can be considered as "Traces of People's Memory". It would be strange if these Big Data neglected memory, since people have memory if they don't suffer from amnesia. We can assume that economic modeling in the era of Big Data will describe the memory effects in microeconomics and macroeconomics. The Big Data will give us a possibility to take into account the effects of memory and non-locality in those economic and financial processes in which they were not even suspected.

3.11. Fractional Econometrics

Economic theory is a branch of economics that employs mathematical models and abstractions of economic processes and objects to rationalize, explain and predict economic phenomena.

One of the main goals of economic theory and mathematical economics is to explain the processes and phenomena in economy and make predictions. To achieve this goal, within the framework of economic theory and mathematical economics, new notions, concepts, tools and methods should be developed for describing and interpreting economic processes with memory and non-locality. Obviously, it is impossible to explain an economy with memory without having adequate concepts.

Economic theory and mathematical economics are branches of economics in which the creation of concepts and theoretical (primarily mathematical) models of phenomena and their comparison with reality is used as the main method of understanding economy. Economic theory is a separate way of studying economy, although its content is naturally formed taking into account the observations of economy. The methodology of economic theory consists in constructing main economic concepts; in formulating (in mathematical language) the principles and laws of economics connecting these concepts; in explaining observable economic phenomena and effects by using the formulated concepts and laws; in predicting new phenomena that may be discovered.

Mathematical economics, when viewed in a narrow sense as a branch of mathematics, is reduced only to the study of the properties of economic models at the mathematical level of rigor. In this approach, mathematical economics is often denied in the choice of economic concepts, interpretation and comparison of models with economic reality. For fractional mathematical economics, to describe processes with memory this view is erroneous. Obviously, it is impossible to explain economic processes with memory without having adequate concepts, since many standard concepts are not applicable.

Now the economics is undergoing a new revolution, "Memory revolution". A mathematic tool in the revolution is a new mathematics (fractional calculus), which was not previously used in mathematical economics. As a result, the present stage is a stage of the formation of new concepts and methods. In this stage, the mathematical economics cannot be isolated from the process of formation of new concepts and methods. Non-standard properties of fractional operators should be reflected in new economic concepts that take into account memory effects. Economic theory and mathematical economics can explain and predict processes with memory in real economy only if they create a solid foundation of new adequate economic concepts and principles.

To explain and predict the processes and phenomena with memory in economy, we must have a good instrument for conducting observations and their adequate description. This instrument is econometrics, or rather fractional econometrics. Econometrics is a link that connects economic theory and mathematical economics with the phenomena and processes in real economy.

Econometrics mainly based on statistics for formulating and testing models and hypotheses about economic processes or estimating parameters for them. Theoretical econometrics considers the statistical properties of assessments and tests, while applied econometrics deals with the use of econometric methods for evaluating economic models. Theoretical econometrics develops tools and methods, and also studies the properties of econometric methods. Applied econometrics uses theoretical econometrics and economic data to evaluate economic theories, develop econometric models, analyzing economic dynamics, and forecasting.

Fractional econometrics is based on statistics of long-memory processes [29–32]. Currently, fractional econometrics methods are actually related to ARFIMA models and the well-known Grunwald–Letnikov fractional differences in the form of the Granger–Joyeux–Hosking fractional differencing and integrating.

Fractional econometrics can reach new opportunities in the development of new econometric methods and their use in describing economic reality by applying methods of modern fractional calculus, various types of fractional finite differences, differential and integral operators of non-integer order.

As a result, we can state that the main goals of fractional economics such as explain the economy and to make predictions for processes with memory, and correctly describe economic events, data, processes, and to give adequate predictions, we should have fractional econometrics.

One of the main goals of fractional mathematical economics and economic theory is to explain the processes and phenomena with memory in economy and make predictions. However, to explain the economic processes with memory, it is necessary to understand what memory is and how to describe it. Note that a clear understanding of the memory does not even exist within the framework of fractional calculus approach.

3.12. Development Concept of Memory

Further development of the concept of memory for economic processes in the framework of fractional calculus is of great importance. It should be emphasized that not all types of fractional derivatives and integrals of non-integer order can describe processes with memory. The concept of memory itself for economic processes is discussed in [29–35], and [39,40,110–117]. One of the possible criteria of memory proposed in the work [116]. Let us give some comments about memory concept.

From mathematical point of view, economic models may be classified as stochastic or deterministic and as discrete or continuous. For simplification, let us consider the deterministic approach with continuous time. In this case, memory can be defined as a property of the process, when there exists at least one endogenous variable $Y(t)$, and an associated exogenous (or endogenous) variable $X(t)$, such that the variable $Y(t)$ at the time $t > t_0$ depends on the history of the change of $X(\tau)$ on the interval $\tau \in (t_0, t)$. However, not all such dependencies are due to the memory effect.

To describe processes with memory, this dependence of one variable on another should satisfy the causality principle. For economic and physical processes with memory, the causality can be described by the Kramers–Kronig relations [116].

An important property of memory is described by the principle of memory fading that was proposed by Ludwig Boltzmann in 1874 and 1876. Then, it was significantly developed by Vito Volterra in 1928 and 1930. The principle of memory fading states that the increasing of the time interval leads to a decrease in the corresponding contribution to the variable $Y(t)$.

Note that in physics the concept of fading memory assumes a set of stronger restrictions on memory. For example, it is often assumed that the memory is described by functions, which tends to zero monotonically with increasing the time variable. In this form, the principle of fading memory assumes that it is less probable to expect strengthening of the memory with respect to the more distant events. However, in some economic processes, it should be taken into account that the economic agents may remember sharp and significant changes of the exogenous variable $X(\tau)$, despite the fact that these changes were a more distant past compared to weaker changes in the near past. For this reason, in economics we can use memory functions that are not monotonic decrease.

For a simple case, we can consider the dependence in the form

$$Y(t) = \int_{t_0}^{t} M(t,\tau) X(\tau) d\tau,$$

where the kernel $M(t,\tau)$ of this integral operator is called the memory function (or the linear response function). Obviously, the derivative of the integer orders of some variable can be considered as an associated variable $X(\tau)$. We also can consider integer-order derivatives of $Y(t)$ as endogenous variables.

It is obvious that not every kernels $M(t,\tau)$ can be used to describe the memory in the economic processes. Possible restrictions on the memory function are discussed in paper [112,113].

In this paper [112,113], we describe some general restrictions that can be imposed on the structure and properties of memory. In addition, to the causality principle [116], these restrictions include the following three principles:

- The principle of fading memory;
- The principle of memory homogeneity on time (the principle of non-aging memory);
- The principle of memory reversibility (the principle of memory recovery).

Mathematically, the principle of non-aging memory means that the memory function has a property $M(t,\tau) = M(t-\tau)$. In this case, the integral operator can be described by the convolution $Y(t) = (M * X)(t)$, [112,113].

The principle of memory reversibility is connected with the principle of duality of accelerator with memory and multiplier with memory, which is proposed in [123,124]. In general, fractional calculus, which was proposed in [239,240] and based on the use of differential-convolution operators, the principle of memory reversibility means that the general operators should have a right inverse (a kind of a fractional integral).

Note that the Kober and Erdelyi–Kober fractional operators, which are interpreted in [112,113] as operators that describe memory with generalized power-law fading, really are integer-order operators with continuously distributed scaling or dilation (see [241] and Section 9 in [228]), and therefore, these operators cannot describe the memory.

Note that time delay, which is sometimes interpreted as a complete (perfect, ideal) memory [112, 113], cannot describe memory. In economics and electrodynamics, processes with time delay (lag) are not referred to as processes with memory and time delay is not interpreted as a memory. The interpretation of the time delay, which is usually called a lag in economics, as some kind of memory seems to be incorrect for the following reasons.

From economic and physical points of view, the time delay is caused by finite speeds of processes, i.e., the change of one variable does not lead to instant changes of another variable. Therefore, the time delay cannot be considered as memory in processes. This fact is well-known in physics as the retarded potential of an electromagnetic field, when a change in the electromagnetic field at the observation point is delayed with respect to the change in the sources of the field located at another point. The processes of propagation of the electromagnetic field in a vacuum are not interpreted in physics as presence of memory in these processes.

From a mathematical point of view, the kernels of integral operators for distributed time delay (lag) and fading memory are distinguished by the fact that the normalization condition holds for the time delay case. Note that the probability distribution functions as kernels, which are usually called the weighting function in economics, are actively used for macroeconomic models with distributed delay time. Equivalent differential equations of integer orders in economics are usually used instead of equations with integro-differential operators, in which the weighting function in the kernels. It is known that under certain conditions, equations with continuously distributed lag are equivalent to differential equations with standard derivatives of integer orders. Mathematically, this means that processes with time delay can be described by equations containing only a finite number of derivatives of integer orders. The integer-order derivatives of functions are determined by the properties of these functions in small neighborhood of the considered point. As a result, differential equations of integer orders cannot describe a memory. To describe processes with fading memory and distributed time delay, we should use the distributed lag fractional calculus [228], (see also [185–187]).

As a result, within the framework of fractional calculus, it is necessary to distinguish between fractional operators that describe distributed time delay and distributed scaling from operators describing memory, and the combination of memory with these phenomena. However, there are open questions about what types of memory we can describe by using fractional calculus (for example, see [116,117]), and in what directions the concept of memory for economic processes will develop.

4. Conclusions

In this brief historical description, an attempt was made to draw a sketch picture of the development of fractional calculus applications in economics, the birth of a new direction in mathematical economics, a new revolution in economic theory. Due to brevity and schematics, this picture obviously cannot reflect the fullness and complexity of the development of a fractional mathematical economics. As a result, it was possible that some directions and approaches, results and works close in the described history were missed. One can hope that the written short history will be perceived with understanding and will be supplemented in the future with new works on the history of the use of fractional calculus in the economics.

We can hope that the further development of the use of fractional calculus to describe economic phenomena and processes will take an important place with modern mathematical economics and economic theory. Generally speaking, it is strange to neglect memory in the economics, since the most important actors are people with memory.

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Review

On the Advent of Fractional Calculus in Econophysics via Continuous-Time Random Walk

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Abstract: In this survey article, at first, the author describes how he was involved in the late 1990s on Econophysics, considered in those times an emerging science. Inside a group of colleagues the methods of the Fractional Calculus were developed to deal with the continuous-time random walks adopted to model the tick-by-tick dynamics of financial markets Then, the analytical results of this approach are presented pointing out the relevance of the Mittag-Leffler function. The consistence of the theoretical analysis is validated with fitting the survival probability for certain futures (BUND and BTP) traded in 1997 at LIFFE, London. Most of the theoretical and numerical results (including figures) reported in this paper were presented by the author at the first Nikkei symposium on Econophysics, held in Tokyo on November 2000 under the title "Empirical Science of Financial Fluctuations" on behalf of his colleagues and published by Springer. The author acknowledges Springer for the license permission of re-using this material.

Keywords: econophysics; continuous-time random walk (CTRW); fractional calculus; Mittag–Leffler functions; Laplace transform; Fourier transform

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1. Introduction

As we read from Wikipedia [1], "Econophysics is a heterodox interdisciplinary research field, applying theories and methods originally developed by physicists in order to solve problems in economics, usually those including uncertainty or stochastic processes and nonlinear dynamics. Some of its application to the study of financial markets has also been termed statistical finance referring to its roots in statistical physics". Here we stress that the problems dealt with in Econophysics are essentially devoted to statistical finance. The first book on Econophysics was by R. N. Mantegna and H. E. Stanley in 2000 [2], followed by a number of books including that by Bouchaud and Potters [3].

The importance of random walks in finance has been known since the seminal work of Bachelier [4] which was completed at the end of the XIXth century, more than a hundred years ago. The ideas of Bachelier were further carried out and improved by many scholars see, for example, Mandelbrot [5], Cootner [6], Samuelson [7], Black and Scholes [8], Merton [9], Mantegna and Stanley [2], Bouchaud and Potters [3].

The term "Econophysics" was coined by H. Eugene Stanley (Boston University) to describe a number of papers written by physicists (including his Ph.D. and Post-Doc students) in the problems of stock and other markets. The inaugural meeting on Econophysics was organized in July 1997 in Budapest by János Kertész and Imre Kondor but the book of proceedings edited by them to be published with the impressive title *Econophysics, an Emerging Science* was surprisingly cancelled by the publisher Kluwer during the checking of galley proofs.

The author of this paper was alerted on this conference in Budapest from a national newspaper so he was able to get an invitation to present his current research on Lévy stable distributions with his former student Ph.D. Paolo Paradisi and his colleague, the late Professor Rudolf Gorenflo.

Indeed, the author was impressed by the derivation of Lévy distributions as fundamental solutions of a space fractional diffusion equation so generalizing the well known Gaussian distribution and the Brownian motion. However, because of the sad end of the book of Budapest proceedings, this paper was not published, but later submitted as an E-print in arXiv, see [10].

Then, after the author's seminar stable distributions held in Rome, at the University "La Sapienza", he was approached by Dr. Enrico Scalas of the University of Alessandria (Italy) on the possibility of collaborating with him and his student Marco Raberto and Prof. Rudolf Gorenflo on applications of methods of Fractional Calculus in Econophyscs. As a matter of fact, Enrico Scalas was the director of our group (in Italian called as "Il Regista", taking inspiration from directing movies) being expert of finance. So a collaboration started inside our group after an introduction paper by Scalas et al. in Physica A, see Reference [11], followed by a series of papers, see Mainardi et al. [12], Gorenflo et al. [13], Raberto et al. [14]. In these papers, the authors have argued that "the continuous-time random walk (CTRW) model, formerly introduced in Statistical Mechanics by Montroll and Weiss [15], can provide a phenomenological description of tick-by-tick dynamics in financial markets" and they have discussed some applications concerning "high frequency exchanges of bond futures". A particular mention is given to the paper by Mainardi et al. [16], being presented by the author under invitation of the organizer H. Takayasu at an international conference on Econophysics held in Tokyo, November 2000.

Later, other papers were published by our group, see References [17,18] to summarize our approach to CTRW via fractional master equations. Furthermore, Scalas published papers on the relevance of Fractional Calculus in dealing with CTRW in Econophysics, see, for example, Reference [19] and chapters in the book authored by Baleanu et al., see Reference [20,21].

The purpose of this paper is to survey our phenomenological theory of tick-by-tick dynamics in financial markets, based on the continuous-time random walk (CTRW) model, by pointing out the relevance of Fractional Calculus and of the Mittag-Leffler function, a special function almost unknown in those times. Nowadays the Mittag-Leffler function has hundreds citations and a treatise on the functions of the Mittag-Leffler type has been published in 2014 by Gorenflo et al. [22], of which a second enlarged edition will hopefully appear in 2020.

The body of our paper is essentially based on the above references and on the conferences and seminars that anybody of our group could give by invitation in several Institutions spread in the world. In particular, most of the the theoretical and numerical results reported in this paper were presented by the author at the first Nikkei symposium on Econophysics, held in Tokyo on November 2000 under the title "Empirical Science of Financial Fluctuations" on behalf of his colleagues. published in the Springer volume edited by Takayasu [16].

Our survey-article is divided as follows: Section 2 is devoted to revisit the theoretical framework of the CTRW model. We provide the most appropriate form for the general master equation, which is expected to govern the evolution of the probability density for non-local and non-Markovian processes. In Section 3 we propose a master equation containing a time derivative of fractional order to characterize non-Markovian processes with long memory. In this respect, we outline the central role played by the Mittag-Leffler function which exhibits an algebraic tail consistent with such processes. Section 4 is devoted to explain how the CTRW model can be used in describing the financial time series of the log-prices of an asset, for which the time interval between two consecutive transactions varies stochastically. In particular we test the theoretical predictions on the survival-time probability against empirical market data. The empirical analysis concerns high-frequency prices time series of German and Italian bond futures. Finally, in Section 5, we draw the main conclusions. For the sake of convenience, the Appendix A introduces in a simple way the correct notion of time derivative of fractional order.

2. The CTRW Model in Statistical Physics

We recall that the CTRW model leads to the general problem of computing the probability density function (*pdf*) $p(x,t)$ ($x \in \mathbb{R}\,,\ t \in \mathbb{R}^+$) of finding, at position x at time t, a particle (the walker) which performs instantaneous random jumps $\xi_i = x(t_i) - x(t_{i-1})$ at random instants t_i ($i \in \mathbb{N}$), where $0 = t_0 < t_1 < t_2 < \ldots$. We denote by $\tau_i = t_i - t_{1-1}$ the (so-called) *waiting times*. As usual, it is assumed that the particle is located at $x_0 = 0$ for $t_0 = 0$, which means $p(x,0) = \delta(x)$. We denote by $\varphi(\xi,\tau)$ the *joint probability density* for jumps and waiting times.

The CTRW is generally defined through the requirement that the ξ_i and τ_i are independent identically distributed (i.i.d.) random variables with pdf's independent of each other, so that we have the factorization $\varphi(\xi,\tau) = w(\xi)\,\psi(\tau)$, which implies $w(\xi) = \int_0^\infty \varphi(\xi,\tau)\,d\tau\,,\ \psi(\tau) = \int_{-\infty}^{+\infty} \varphi(\xi,\tau)\,d\xi\,.$ The marginal probability densities w and ψ are called *jump pdf* and *waiting-time pdf*, respectively.

We now provide further details on the densities $w(\xi)\,,\ \varphi(\tau)$ in order to derive their relation with the *pdf* $p(x,t)$.

The *jump pdf* $w(\xi)$ represents the pdf for transition of the walker from a point x to a point $x + \xi$, so it is also called the *transition* pdf. The *waiting-time* pdf represents the pdf that a step is taken at the instant $t_{i-1} + \tau$ after the previous one that happened at the instant t_{i-1}, so it is also called the *pausing-time* pdf. Therefore, the probability that $\tau \le t_i - t_{i-1} < \tau + d\tau$ is equal to $\psi(\tau)\,d\tau$. The probability that a given waiting interval is greater or equal to τ will be denoted by $\Psi(\tau)$, which is defined in terms of $\psi(\tau)$ by

$$\Psi(\tau) = \int_\tau^\infty \psi(t')\,dt' = 1 - \int_0^\tau \psi(t')\,dt'\,, \quad \psi(\tau) = -\frac{d}{d\tau}\Psi(\tau)\,. \tag{1}$$

We note that $\int_0^\tau \psi(t')\,dt'$ represents the probability that at least one jump is taken at some instant in the interval $[0,\tau)$, hence $\Psi(\tau)$ is the probability that the walker is sitting in x at least during the time interval of duration τ after a jump. Recalling that $t_0 = 0$, we also note that $\Psi(t)$ represents the so called *survival probability*, namely the probability of finding the walker at the initial position $x_0 = 0$ until time instant t.

Now, only based upon the previous probabilistic arguments, we can derive the evolution equation for the *pdf* $p(x,t)$, that we shall call the *master equation* of the CTRW. In fact, we are led to write

$$p(x,t) = \delta(x)\,\Psi(t) + \int_0^t \psi(t-t')\,\left[\int_{-\infty}^{+\infty} w(x-x')\,p(x',t')\,dx'\right]\,dt'\,, \tag{2}$$

where we recognize the role of the *survival probability* $\Psi(t)$ and of the pdf's $\psi(t)\,,\ w(x)$. The first term in the RHS of Equation (2) expresses the persistence (whose strength decreases with increasing time) of the initial position $x = 0$. The second term (a spatio-temporal convolution) gives the contribution to $p(x,t)$ from the walker sitting in point $x' \in \mathbb{R}$ at instant $t' < t$ jumping to point x just at instant t, after stopping (or waiting) time $t - t'$. Furthermore, as a check for the correctness of Equation (2) we can easily verify that $p(x,t) \ge 0$ for all $t \ge 0$ and $x \in \mathbb{R}$, and $\int_{-\infty}^{+\infty} p(x,t)\,dx = 1$ for all $t \ge 0$.

Originally the *master equation* was derived by Montroll and Weiss in 1965, see [15], recurring to the tools of the Fourier-Laplace transforms. These authors showed that the Fourier-Laplace transform of $p(x,t)$ satisfies a characteristic equation, now called the *Montroll-Weiss equation*, which reads

$$\widehat{\widetilde{p}}(\kappa,s) = \widetilde{\Psi}(s)\,\frac{1}{1 - \widehat{w}(\kappa)\,\widetilde{\psi}(s)}\,, \quad \text{with} \quad \widetilde{\Psi}(s) = \frac{1 - \widetilde{\psi}(s)}{s}\,. \tag{3}$$

Here, we have adopted the following standard notation for the generic Fourier and Laplace transforms:

$$\mathcal{F}\,\{f(x);\kappa\} = \widehat{f}(\kappa) = \int_{-\infty}^{+\infty} \mathrm{e}^{\,i\kappa x}\,f(x)\,dx\,, \quad \mathcal{L}\,\{g(t);s\} = \widetilde{g}(s) = \int_0^\infty \mathrm{e}^{\,-st}\,g(t)\,dt\,,$$

where $f(x)$ $(x \in \mathbb{R})$ and $g(t)$ $(t \in \mathbb{R}^+)$ are sufficiently well-behaved functions of their arguments. It is straightforward to verify the equivalence between the Equations (2) and (3) by recalling the well-known properties of the Fourier and Laplace transforms with respect to the space and time convolution.

Hereafter, we present an alternative form to Equation (2), formerly proposed by Mainardi et al. [12], which involves the first time derivative of $p(x,t)$ (along with an additional auxiliary function), so that the resulting equation can be interpreted as an *evolution* equation of *Fokker-Planck-Kolmogorov* type. To this purpose, we re-write Equation (3) as

$$\widetilde{\Phi}(s)\,\left[s\,\widehat{\widetilde{p}}(\kappa,s)-1\right] \;=\; \left[\widehat{w}(\kappa)-1\right]\,\widehat{\widetilde{p}}(\kappa,s)\,, \tag{4}$$

where

$$\widetilde{\Phi}(s) = \frac{1-\widetilde{\psi}(s)}{s\,\widetilde{\psi}(s)} = \frac{\widetilde{\Psi}(s)}{\widetilde{\psi}(s)} = \frac{\widetilde{\Psi}(s)}{1-s\widetilde{\Psi}(s)}\,. \tag{5}$$

Then our master equation reads

$$\int_0^t \Phi(t-t')\,\frac{\partial}{\partial t'}\,p(x,t')\,dt' \;=\; -p(x,t) + \int_{-\infty}^{+\infty} w(x-x')\,p(x',t)\,dx'\,, \tag{6}$$

where the "auxiliary" function $\Phi(t)$, being defined through its Laplace transform in Equation (5), is such that $\Psi(t) = \int_0^t \Phi(t-t')\,\psi(t')\,dt'$. We remind the reader that Equation (6), combined with the initial condition $p(x,0) = \delta(x)$, is equivalent to Equation (4), and then its solution represents the Green function or the fundamental solution of the Cauchy problem for Equation (6).

From Equation (6) we recognize the role of $\Phi(t)$ as a "memory function". As a consequence, the CTRW turns out to be in general a non-Markovian process. However, the process is "memoryless", namely "Markovian" if (and only if) the above memory function degenerates into a delta function (multiplied by a certain positive constant) so that $\Psi(t)$ and $\psi(t)$ may only differ by a multiplying positive constant. By appropriate choice of the unit of time we assume $\widetilde{\Phi}(s) = 1$, so $\Phi(t) = \delta(t)\,,\ t \geq 0$. In this case we derive

$$\widetilde{\psi}(s) = \widetilde{\Psi}(s) = \frac{1}{1+s}\,, \quad \text{so} \quad \psi(t) = \Psi(t) = \mathrm{e}^{-t}\,,\ t \geq 0\,. \tag{7}$$

Then Equation (6) reduces to

$$\frac{\partial}{\partial t}p(x,t) = -p(x,t) + \int_{-\infty}^{+\infty} w(x-x')\,p(x',t)\,dx'\,, \quad p(x,0) = \delta(x)\,. \tag{8}$$

This is up to a change of the unit of time (which means multiplication of the RHS by a positive constant), the most general *master equation* for a *Markovian* CTRW; it is usually called the *Kolmogorov-Feller equation*.

3. The Time-Fractional Master Equation with "Long-Memory"

Let us now consider "long-memory" processes, namely non-Markovian processes characterized by a memory function $\Phi(t)$ exhibiting a power-law time decay. To this purpose a natural choice is

$$\Phi(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}\,, \quad t \geq 0\,, \quad 0 < \beta < 1\,. \tag{9}$$

Thus, $\Phi(t)$ is a weakly singular function that, in the limiting case $\beta = 1$, reduces to $\Phi(t) = \delta(t)$, according to the formal representation of the Dirac generalized function, $\delta(t) = t^{-1}/\Gamma(0)\,,\ t \geq 0$.

As a consequence of the choice (9), we see that (in this peculiar non-Markovian situation) our *master equation* (6) contains a time fractional derivative. In fact, by inserting into Equation (4) the Laplace transform of $\Phi(t)$, $\widetilde{\Phi}(s) = 1/s^{1-\beta}$, we get

$$s^{\beta}\, \widehat{\widetilde{p}}(\kappa,s) - s^{\beta-1} = [\widehat{w}(\kappa) - 1]\; \widehat{\widetilde{p}}(\kappa,s)\,, \quad 0<\beta<1\,, \tag{10}$$

so that the resulting Equation (6) can be written as

$$\frac{\partial^{\beta}}{\partial t^{\beta}} p(x,t) = -p(x,t) + \int_{-\infty}^{+\infty} w(x-x')\, p(x',t)\, dx'\,, \quad p(x,0) = \delta(x)\,, \tag{11}$$

where $\partial^{\beta}/\partial t^{\beta}$ is the pseudo-differential operator explicitly defined in the Appendix, that we call the *Caputo* fractional derivative of order β. Thus Equation (11) can be considered as the time-fractional generalization of Equation (8) and consequently can be called the *time-fractional Kolmogorov-Feller equation*.

Our choice for $\Phi(t)$ implies peculiar forms for the functions $\Psi(t)$ and $\psi(t)$ that generalize the exponential behaviour (7) of the Markovian case. In fact, working in the Laplace domain we get from (5) and (9)

$$\widetilde{\Psi}(s) = \frac{s^{\beta-1}}{1+s^{\beta}}\,, \quad \widetilde{\psi}(s) = \frac{1}{1+s^{\beta}}\,, \quad 0<\beta<1\,, \tag{12}$$

from which by inversion we obtain for $t \geq 0$

$$\Psi(t) = E_{\beta}(-t^{\beta})\,, \quad \psi(t) = -\frac{d}{dt} E_{\beta}(-t^{\beta})\,, \quad 0<\beta<1\,, \tag{13}$$

where E_{β} denotes an entire transcendental function, known as the Mittag-Leffler function of order β, defined in the complex plane by the power series

$$E_{\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta\, n+1)}\,, \quad \beta>0\,, \quad z \in \mathbb{C}\,. \tag{14}$$

For detailed information on the Mittag-Leffler-type functions and their Laplace transforms the reader may consult e.g., the books [22,23] and the articles [24,25].

Hereafter, we find it convenient to summarize the features of the functions $\Psi(t)$ and $\psi(t)$ most relevant for our purposes. We begin to quote their series expansions and asymptotic representations:

$$\Psi(t) \begin{cases} = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{t^{\beta n}}{\Gamma(\beta\, n+1)}\,, & t \geq 0 \\ \sim \displaystyle\frac{\sin(\beta\pi)}{\pi}\, \frac{\Gamma(\beta)}{t^{\beta}}\,, & t \to \infty\,, \end{cases} \tag{15}$$

and

$$\psi(t) \begin{cases} = \displaystyle\frac{1}{t^{1-\beta}} \sum_{n=0}^{\infty} (-1)^n \frac{t^{\beta n}}{\Gamma(\beta\, n+\beta)}\,, & t \geq 0 \\ \sim \displaystyle\frac{\sin(\beta\pi)}{\pi}\, \frac{\Gamma(\beta+1)}{t^{\beta+1}}\,, & t \to \infty\,. \end{cases} \tag{16}$$

In the limit for $\beta \to 1$ we recover the exponential functions of the Markovian case. We note that for $0 < \beta < 1$ both functions $\psi(t)$, $\Psi(t)$, even if losing their exponential decay by exhibiting power-law tails for large times, keep the "completely monotonic" character. Complete monotonicity of the functions $\psi(t)$, $\Psi(t)$, $t > 0$, means:

$$(-1)^n \frac{d^n}{dt^n}\, \Psi(t) \geq 0\,, \quad (-1)^n \frac{d^n}{dt^n}\, \psi(t) \geq 0\,, \quad n = 0,1,2,\ldots \tag{17}$$

or equivalently, their representability as (real) Laplace transforms of non-negative functions. It may be instructive to note that for sufficiently small times $\Psi(t)$ exhibits a behaviour similar to that of a stretched exponential; in fact we have

$$E_\beta(-t^\beta) \simeq 1 - \frac{t^\beta}{\Gamma(\beta+1)} \simeq \exp\{-t^\beta/\Gamma(1+\beta)\}\,, \quad 0 \leq t \ll 1\,. \tag{18}$$

4. The CTRW Model in Statistical Finance

The price dynamics in financial markets can be mapped onto a random walk whose properties are studied in continuous, rather than discrete, time, see, for example [9]. As a matter of fact, there are various ways in which to embed a random walk in continuous time. Here, we shall base our approach on the CTRW discussed in Section 2, in which time intervals between successive steps are i.i.d. random variables.

Let $S(t)$ denote the price of an asset or the value of an index at time t. In finance, returns rather than prices are considered. For this reason, in the following we shall take into account the variable $x(t) = \log S(t)$, that is the logarithm of the price. Indeed, for a small price variation $\Delta S = S(t_i) - S(t_{i-1})$, the return $r = \Delta S / S(t_{i-1})$ and the logarithmic return $r_{log} = \log[S(t_i)/S(t_{i-1})]$ virtually coincide. The statistical physicist will recognize in x the position of a random walker jumping in one dimension. Thus, in the following, we shall use the language and the notations of Section 2.

In financial markets, prices are fixed when demand and offer meet and a transaction occurs. In this case, we say that a trade takes place. As a consequence, not only prices but also waiting times between two consecutive transactions can be modelled as random variables. In agreement with the assumptions of Section 2, we consider the returns $\xi_i = x(t_i) - x(t_{i-1})$ as i.i.d random variables with *pdf* $w(\xi)$ and the waiting times $\tau_i = t_i - t_{i-1}$ as i.i.d. random variables with *pdf* $\psi(\tau)$. In real processes of financial markets this independence hypothesis may not strictly hold for their duration or not be verified at all. Therefore, it may be considered with caution.

In the following, we limit ourselves to investigate the consistency of the long-memory process analyzed in Section 3 with respect to the empirical data concerning exchanges of certain financial derivatives. We have considered the waiting time distributions of certain futures traded at LIFFE in 1997 and estimated the corresponding empirical survival probabilities. LIFFE stands for *London International Financial Futures (and Options) Exchange*. It is a London-based derivative market; for further information, see http://www.liffe.com. Futures are derivative contracts in which a party agrees to sell and the other party to buy a fixed amount of an underlying asset at a given price and at a future delivery date.

As underlying assets, we have chosen German and Italian Government bonds, called BUND and BTP respectively, for both of which the delivery dates are June and September 1997. BUND and BTP (*Buoni del Tesoro Poliennali*) are respectively the German and Italian word for BOND (middle and long term Government bonds with fixed interest rate). Usually, for a future with a certain maturity, transactions begin 4 or 5 months before the delivery date. At the beginning, there are few trades a day, but closer to the delivery there may be more than 1000 transactions a day. For each maturity, the total number of transactions is greater than 160,000. Hence, these types of financial instruments are particularly interesting for the analysis of the waiting times distributions between consecutive transactions.

In Figures 1 and 2 we plot $\Psi(\tau)$ for the four cases (June and September delivery dates for BUND and BTP). The circles refer to market data and represent the probability of a waiting time greater than the abscissa τ. We have determined about 500–600 values of $\Psi(\tau)$ for τ in the interval between 1 s and 50,000 s, neglecting the intervals of market closure. The solid line is a two-parameter fit obtained by using the Mittag-Leffler type function

$$\Psi(\tau) = E_\beta\left[-(\gamma\tau)^\beta\right]\,, \tag{19}$$

where β is the index of the Mittag-Leffler function and γ is a time-scale factor, depending on the time unit. The dash-dotted line is the stretched exponential function $\exp\{-(\gamma\tau)^{\beta})/\Gamma(1+\beta)\}$, see the RHS of Equation (18), whereas the dashed line is the power law function $(\gamma\tau)^{-\beta}/\Gamma(1-\beta)$, see the RHS of the second Equation in (15), noting that $\Gamma(\beta)\,\sin(\beta\pi)/\pi = 1/\Gamma(1-\beta)$. The Mittag-Leffler function well interpolates between these two limiting behaviours—the stretched exponential for small times, and the power law for large ones.

As regards the BUND futures we can summarize as follows. For the June delivery date we get an index $\beta = 0.96$, and a scale factor $\gamma = 1/12$, whereas, for the September delivery date, we have $\beta = 0.95$, and $\gamma = 1/12$. The fits in the plots of Figure 1 have a reduced chi square $\widetilde{\chi}^2 \simeq 0.3$. As regards the BTP futures we can summarize as follows. For both the June and September delivery dates we get the same index $\beta = 0.96$, and the same scale factor $\gamma = 1/13$. The fits in the plots of Figure 2 have a reduced chi square $\widetilde{\chi}^2 \simeq 0.2$.

To the possible objection that, in all four cases here treated, β does not differ significantly from 1 and so the process still could be Markovian, we answer that then we would have $\Psi(\tau) = \exp(-\gamma\tau)$ and the graph of $\Psi(\tau)$ would look completely different for sufficiently long times.

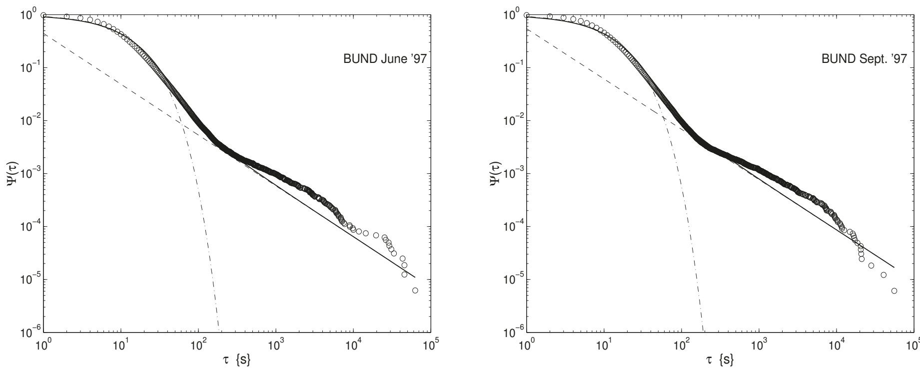

Figure 1. Survival probability for *BUND* futures with delivery date: June 1997 (**left**) September 1997 (**right**). The Mittag-Leffler function (solid line) is compared with the stretched exponential (dash-dotted line) and the power (dashed line) functions: $\{\beta = 0.96\,,\ \gamma = 1/12\}$ (**left**); $\{\beta = 0.95\,,\ \gamma = 1/12)\}$ (**right**).

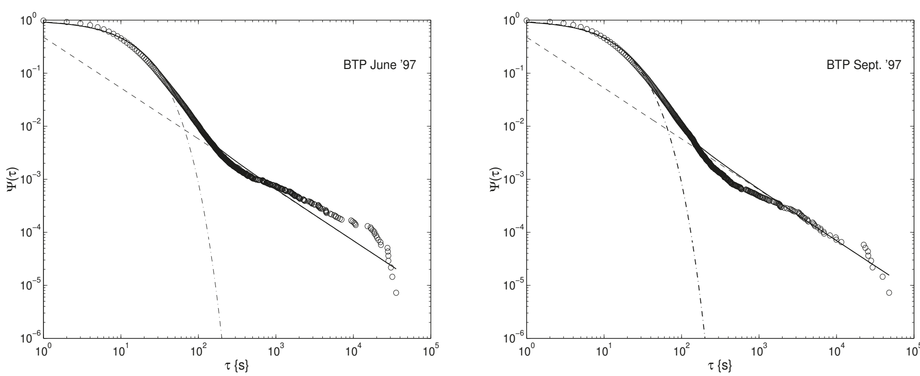

Figure 2. Survival probability for *BTP* futures with delivery date: June 1997 (**left**) September 1997 (**right**). The Mittag-Leffler function (solid line) is compared with the stretched exponential (dash-dotted line) and the power (dashed line) functions: $\{\beta = 0.96\,,\ \gamma = 1/13\}$ (**left**); $\{\beta = 0.96\,,\ \gamma = 1/13)\}$ (**right**).

5. Conclusions

In this paper, we have reviewed our phenomenological theory of tick-by-tick dynamics in financial markets, based on the continuous-time random walk (CTRW) model. The theory can take into account the possibility of the non-Markovian character of financial time series by means of a generalized master equation with a time fractional derivative. We have presented predictions of the behaviour of the waiting-time probability density by introducing a special function of Mittag-Leffler type whose decay interpolates from a stretched exponential at small times to a power-law for long times. This function has been successfully applied in the empirical analysis of high-frequency prices time series of German and Italian bond futures. We may note the common behaviour of the survival probabilities found from the trading of the above assets. This might be corroborated or not in other cases.

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Appendix A. The Caputo Fractional Derivative

For the sake of convenience of the reader here we present an introduction to the *Caputo* fractional derivative starting from its representation in the Laplace domain and pointing out its difference with respect to the standard *Riemann-Liouville* fractional derivative. So doing, we avoid the subtleties lying in the inversion of fractional integrals. If $f(t)$ is a (sufficiently well-behaved) function with Laplace transform $\mathcal{L}\{f(t);s\} = \widetilde{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt$, we have

$$L\left\{\frac{d^\beta}{dt^\beta} f(t); s\right\} = s^\beta\, \widetilde{f}(s) - s^{\beta-1} f(0^+)\,, \quad 0 < \beta < 1\,, \tag{A1}$$

if we define

$$\frac{d^\beta}{dt^\beta} f(t) := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{df(\tau)}{d\tau} \frac{d\tau}{(t-\tau)^\beta}\,. \tag{A2}$$

We can also write

$$\frac{d^\beta}{dt^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \left\{ \int_0^t [f(\tau) - f(0^+)] \frac{d\tau}{(t-\tau)^\beta} \right\}\,, \tag{A3}$$

$$\frac{d^\beta}{dt^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \left\{ \int_0^t \frac{f(\tau)}{(t-\tau)^\beta}\, d\tau \right\} - \frac{t^{-\beta}}{\Gamma(1-\beta)} f(0^+)\,. \tag{A4}$$

The reader should observe that first term in the R.H.S. of (A4) provides the most usual Riemann-Liouville fractional derivative, see e.g., [26].For more details on the *Caputo* fractional derivative we refer to Refs. [24,27].

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mathematics

Review

Rules for Fractional-Dynamic Generalizations: Difficulties of Constructing Fractional Dynamic Models

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Abstract: This article is a review of problems and difficulties arising in the construction of fractional-dynamic analogs of standard models by using fractional calculus. These fractional generalizations allow us to take into account the effects of memory and non-locality, distributed lag, and scaling. We formulate rules (principles) for constructing fractional generalizations of standard models, which were described by differential equations of integer order. Important requirements to building fractional generalization of dynamical models (the rules for "fractional-dynamic generalizers") are represented as the derivability principle, the multiplicity principle, the solvability and correspondence principles, and the interpretability principle. The characteristic properties of fractional derivatives of non-integer order are the violation of standard rules and properties that are fulfilled for derivatives of integer order. These non-standard mathematical properties allow us to describe non-standard processes and phenomena associated with non-locality and memory. However, these non-standard properties lead to restrictions in the sequential and self-consistent construction of fractional generalizations of standard models. In this article, we give examples of problems arising due to the non-standard properties of fractional derivatives in construction of fractional generalizations of standard dynamic models in economics.

Keywords: fractional calculus; fractional dynamics; fractional generalization; long memory; non-locality; mathematical economics; economic theory

MSC: 26A33 Fractional derivatives and integrals; 91B02 Fundamental topics (basic mathematics, applicable to economics in general)

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1. Introduction

In mathematics, in addition to derivatives and integrals of integer order, fractional differentiation and integration of non-integer orders (for example, see the comprehensive encyclopedic-type monograph [1], the unsurpassed monograph on generalized fractional calculus [2], the very important and remarkable books on fractional calculus and fractional differential equations [3–5]). These operators have been known for several centuries (for example, see comments to Chapters in [1], the first description of the history of fractional calculus (FC), written 150 years ago [6], brief history of FC [7–9], and the first review of history of generalized fractional calculus [10]). The recent history of fractional calculus is described in [11], the chronicles and science metrics of recent development of FC [12–14], and some pioneers in applications of FC [15]. The fractional differential equations are a powerful tool to describe processes with long memory and spatial non-locality. Recently, the fractional calculus and fractional differential equations have become actively used to describe various phenomena in natural and social sciences. The most important results in this area are collected in the eight-volume encyclopedic handbook on fractional calculus and its applications [16].

At the present time, in some works, fractional differential equations of dynamic models, which are intended to describe physical and economic processes, are proposed without carefully deducing them from some physical and economic assumptions, interpretations and generalizations of concepts. The fractional differential equations are obtained by simply replacing the integer derivatives with fractional derivatives of non-integer order in the equations of standard model. Moreover, it is usually not discussed how such fractional equations can be obtained and justified. After obtaining the solutions of fractional differential equations, which can be presented in an analytical or approximate form, the physical/economic interpretation and analysis of these solutions is not carried out. This way of obtaining fractional generalizations of standard dynamic models can be called a formal generalization, which is a mathematical exercise, and it cannot be considered as mathematical models of the natural and social processes.

In our opinion, the goals of fractional generalizations of models in natural and social sciences cannot be reduced only to a mathematical consideration of fractional differential equations and its solutions. In case of this reduction, the connection with the physics and economics is lost, and it leads to the fact that the results of such generalizations cannot be used directly in these areas of science. The mathematical analysis of fractional differential equations and its solutions should be a bridge, connecting the initial economic or physical assumptions and concepts on the one side, and economic or physical interpretations, effects and conclusions on the other side. All this leads to the need to formulate rules and principles that are important for the development of applications of fractional calculus in natural and social sciences.

Let us formulate basic rules (the principles of fractional-dynamic generalizer) for constructing fractional generalizations of standard dynamic models, i.e., models that are described by differential equations of integer orders.

(1) **Derivability Principle:** *It is not enough to generalize the differential equations describing the dynamic model. It is necessary to generalize the whole scheme of obtaining (all steps of derivation) these equations from the basic principles, concepts and assumptions. In this sequential derivation of the equations we should take into account the non-standard characteristic properties of fractional derivatives and integrals. If necessary, generalizations of the notions, concepts and methods, which are used in this derivation, should also be obtained.* The derivability principle states that we should realize a correct fractional generalization of the derivation of the model equations. It is necessary to generalize not only and not so much the differential equation of the model itself, but a generalization of all steps of deriving the standard (non-fractional) equations of the model. In the general case, this will not be an equation that is obtained by simply replacing the integer derivatives with fractional derivatives of non-integer order. Often, the consistent construction of a fractional-dynamic model is associated with the need to introduce new concepts and notions that generalize the concepts and notions of standard models. Note that fractional generalizations of basic concepts are not so much a part of this particular model, but in fact are the common basis of different models, and basis of all fractional dynamics (fractional mathematical economics), and not just the model. An important part of this derivation is the need to take into account the non-standard characteristic properties of fractional derivatives and integrals [17–22]. These properties include (a) violation of the standard chain rule (for example, see [3], pp. 97–98, [5], pp. 35–36, [19] and Section 2.1); (b) violation of the standard semi-group property for orders of derivatives (see [1], pp. 46–47, [5], p. 30, and Section 2.2); (c) violation of the standard product (Leibniz) rule (for example, see [1], pp. 280–284, [3], pp. 91–97, [5], pp. 33, 59, [17,20,22] and Section 2.3); (d) violation of the standard semi-group property for dynamic maps (see the explanations and references in Section 2.4). These properties narrow the field for maneuver and make it difficult to obtain fractional generalizations. These non-standard properties are obstacles that must be overcome to build correct fractional dynamic models. At the same time, these non-standard properties

allow us to get correct fractional dynamic models to describe non-standard effects, processes and phenomena. Schematically, this principle is represented by Figure 5.

(2) **Multiplicity Principle:** *For one standard model, there is a set of fractional dynamical generalizations, due to the existence of various types of fractional operators and violation of s-equivalence for fractional differential equations.* In addition to existence a large number of different types of fractional derivatives and integrals, the violation of the standard rules generate an additional uncertainty of fractional generalizations. Fractional generalizations of solution-equivalent (s-equivalent) representations of integer-order differential equations of standard models, as a rule, lead to different fractional differential equations that have non-equivalent solutions. This situation is partially analogous to the fact that quantization of equivalent classical models leads to nonequivalent quantum theories. As a result, fractional generalizations of one standard model (which is represented by s-equivalent differential equations of integer order) can lead to different fractional-dynamic models that will predict different behaviors of a process and only some of them may be useful in a given context. We can state that for one standard model, there is a family of fractional dynamical generalizations, due to the existence of various types of fractional operators and violation of s-equivalence for fractional differential equations. In this regard, it is important to investigate and describe the properties of solutions of fractional dynamic equations, which are (qualitatively and/or quantitatively) the same, and the properties of solutions that are (first of all, qualitatively) different. Schematically, this principle is given by Figure 1.

(3) **Solvability Principle:** *The properties of process types (such as long memory, spatial nonlocality, distributed delay, distributed scaling) and the properties of the corresponding types of fractional operators must be taken into account in the existence of solutions, and in obtaining correct analytical and numerical solutions.* The solvability principle states that the existence of solution, and the possibility of obtaining an exact analytical solution or correct numerical solutions for some conditions. Obviously, the existence conditions should allow us to obtain solutions for those cases and properties that the described process has. In addition, we should take into account that different types of fractional derivatives and integrals are known in fractional calculus [1,2,4]. Therefore, in fractional dynamic generalization, it is important that type of fractional operators correspond to the type of natural or social process. It should be noted that not all well-known fractional operators can describe the long memory and spatial non-locality (see Section 2.5 of this paper). For example, some fractional operators can be used to describe the distributed lag (time delay) and the distributed scaling (dilation) and they are not suitable for memory and non-locality. Additionally, we need to verify the existence condition for properties of solutions obtained. For example, if we describe processes with long memory then derivation of numerical solution must take into account not only local information, but the numerical scheme must contain memory terms. Schematically, this principle is represented by Figure 4.

(4) **Correspondence Principle:** *The limiting procedure, when orders of fractional derivatives tend to integer values, applied to the equations of the fractional dynamic model and their solutions, should give the standard model equations and their solutions.* The correspondence principle means a possibility of obtaining equations and solutions of standard model by using a limit procedure, when the orders of the fractional derivatives tend to an integer values. The principle of correspondence must be fulfilled both for the equation itself and for its solution. It should be noted if the order of the derivative tends to the integer value, then the limit on the left and the limit on the right can give different results in the general case. Schematically, this principle is depicted in Figure 2. The Correspondence Principle can also be represented by the formal expression:

$$\lim_{\alpha\to n} \mathrm{Frac-Eq}[\alpha] = \mathrm{Int-Eql}[\mathrm{n}], \tag{1}$$

$$\lim_{\alpha\to n} \mathrm{Frac-Sol}[\alpha] = \mathrm{Int-Sol}[\mathrm{n}], \tag{2}$$

where $n \in \mathbb{N}$. It should be noted that the limit on the left and the limit on the right do not coincide in the general case:

$$\lim_{\alpha \to n-} \mathrm{Frac} - \mathrm{Sol}[\alpha] \neq \lim_{\alpha \to n+} \mathrm{Frac} - \mathrm{Sol}[\alpha]. \tag{3}$$

(5) **Interpretability Principle:** *The subject (physical, economic) interpretation of the mathematical results, including solutions and its properties, should be obtained. Differences, and first of all qualitative differences, from the results based on the standard model should be described.* The subject interpretation of the solutions should be obtained. The properties of solutions should be described in details with their economic or physical meaning (interpretation). It is important to have an interpretability of mathematical results. The differences between results, which were obtained for the proposed generalization and the standard model, should be clearly indicated. An important purpose is to find qualitative differences between the properties of solutions for the fractional dynamic model and the properties of the solutions of the standard model. Schematically, this principle is given by Figure 3.

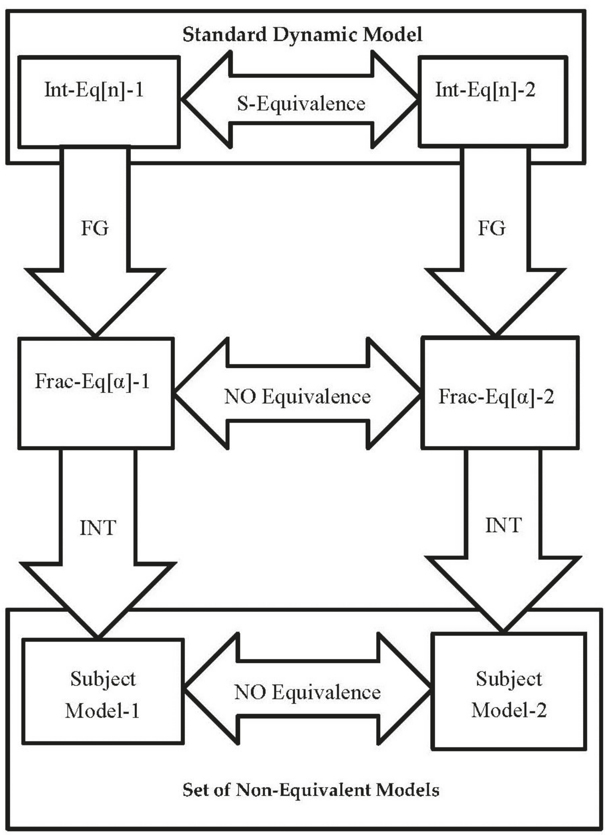

Figure 1. The non-equivalence (multiplicity) diagram. This diagram is non-commutative. The following notation is used in the diagram: Int-Eq[n] is a set of differential and/or integral equations of integer orders that describe the standard dynamic model; Frac-Eq[α] is a set of fractional differential and/or integral equations of non-integer orders that describe the fractional dynamic model. The S-equivalence of some equations of standard models and non-equivalence of fractional generalizations of these equations are considered in Section 5.

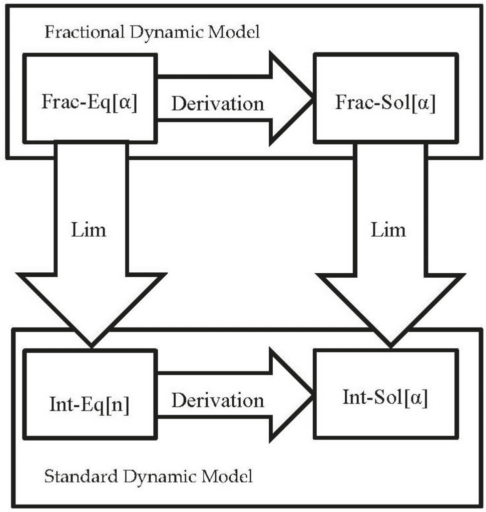

Figure 2. The correspondence diagram. This diagram should be commutative. The following notation is used in the diagram: Int-Eq[n] is a set of differential and/or integral equations of integer orders that describe the standard dynamic model; Int-Sol[n] denotes solutions of differential and/or integral equations of the standard dynamic model; Frac-Eq[α] is a set of fractional differential and/or integral equations of non-integer orders that describe the fractional dynamic model; Frac-Sol[n] denotes solutions of fractional differential and/or integral equations of the fractional dynamic model; Lim is a limit transition when the non-integer orders α tend to integer values n from the left ($\alpha \rightarrow n-$) or form the right ($\alpha \rightarrow n+$).

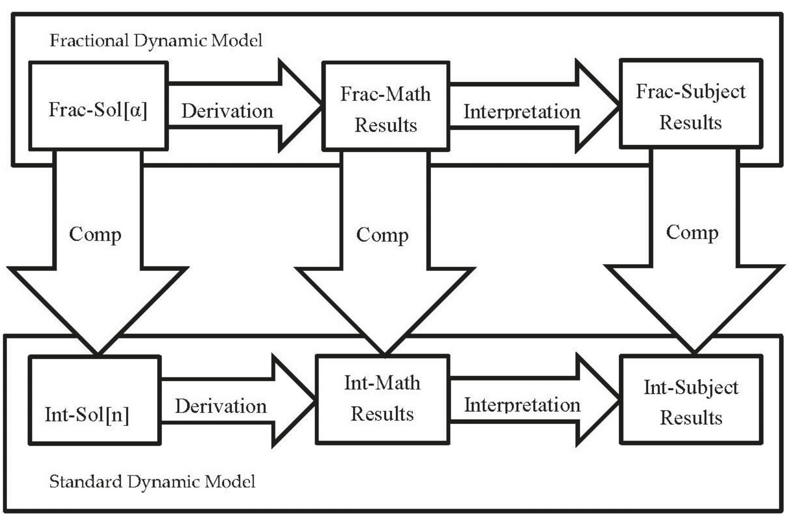

Figure 3. The interpretability diagram. The following notation is used in the diagram: Int-Sol[n] denotes solutions of differential and/or integral equations of the standard dynamic model; Frac-Sol[n] denotes solutions of fractional differential and/or integral equations of the fractional dynamic model; "Int-Math Results" and "Frac-Math Results" denote mathematical results (for example, asymptotic behaviors) obtained from solutions of integer-order and fractional-order differential and/or integral equations; "Comp" denotes a comparison of solutions, mathematical results and subject (economic, physical) results based on the interpretation.

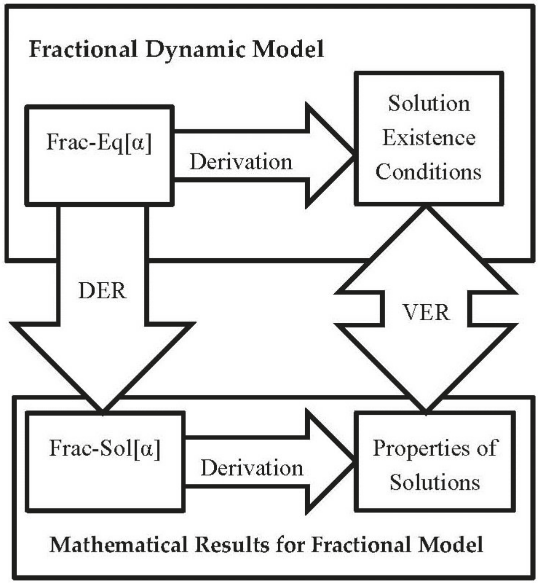

Figure 4. The solvability diagram. This diagram should be commutative. The following notation is used in the diagram: Frac-Eq[α] is a set of fractional differential and/or integral equations of non-integer orders that describe the fractional dynamic model; Frac-Sol[n] denotes solutions of fractional differential and/or integral equations of the fractional dynamic model; DER denoted a derivation of exact analytical solutions or correct numerical solutions for the fractional differential and/or integral equations; VER is a verification of the fulfillment of the condition of existence of solutions for properties of solutions obtained.

The proposed five principles are designed primarily to eliminate errors that are usually made when building fractional dynamic generalizations of standard models. The most important element is the requirement that in fractional generalization of economic (or physical) model the "output" of the research should be an economic (physical) conclusions (phenomena, effects) and new economic (physical) effects that are a consequence of subject assumptions on the "input". Here, mathematics (fractional calculus) is the tool that mathematically strictly connects "economic/physical input" and "economic/physical output". If mathematical equations and solutions are not rigidly connected with subject "input" and "output", they will fly away into "airless space". In this case, the results will turn from economics and physics into formal manipulations, which may not even have mathematical value from the point of view of pure mathematics (fractional calculus).

An important goal of fractional generalizations is to obtain qualitatively new effects and phenomena in natural and social sciences. The results obtained in a science by using the new mathematical apparatus (fractional calculus) should give qualitatively new results and predict new effects and phenomena for this science. First of all, it is precisely such qualitatively new results are interesting in the first place.

In this paper, we illustrate these rules (principles) by using examples of fractional generalizations of standard economic models.

In Section 2 of this paper, we describe the non-standard rules for fractional operators of non-integer orders. The violation of the standard chain rule is described in Section 2.1. The violation of the standard semi-group property for orders of derivatives is discussed in Section 2.2. We consider the violation of the standard product (Leibniz) rule in Section 2.3. The violation of the standard semi-group property for dynamic maps is described in Section 2.4. A correspondence between the types of fractional operators of non-integre orders and the types of phenomena is discussed in Section 2.5.

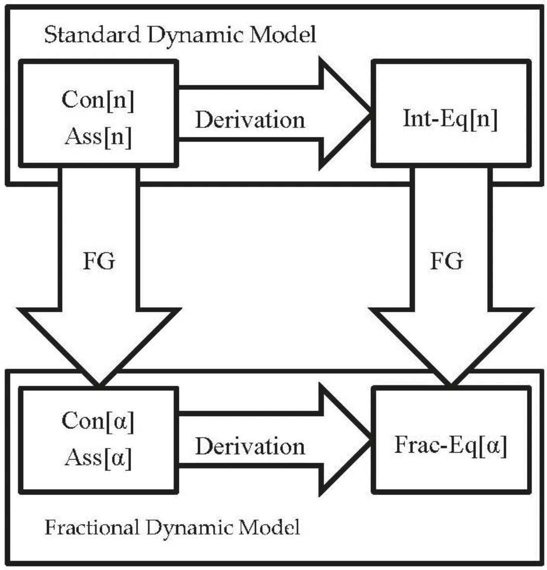

Figure 5. The derivability diagram. This diagram is non-commutative in general. The reason for this noncommutativity is a violation of standard rules and properties for fractional derivatives of non-integer order. The following notation is used in the diagram: Con[n] and Con[α] are concepts of the standard dynamic model and their fractional generalization; Ass[n] and Ass[α] are assumptions that are used in the standard dynamic model and in its fractional generalization; FG is fractional generalization based on the fractional calculus; Int-Eq[n] is a set of differential and/or integral equations of integer orders that describe the standard dynamic model; Frac-Eq[α] is a set of fractional differential and/or integral equations of non-integer orders that describe the fractional dynamic model.

In Section 3, we consider an application of the Derivability Principle and we give examples of the problem with the violation of the standard rules for fractional operators of non-integer orders. In Section 3.1, to illustrate problems that are connected with the non-standard form of the chain rule, we consider a fractional generalization of the Kaldor-type model of business cycles. In Section 3.2, problem with violation of the standard semi-group rule for orders of derivatives is shown for the fractional generalization of the Phillips model of the multiplier-accelerator. In Section 3.3, to illustrate the problems arising from the non-standard form of the product (Leibniz) rule, we consider the fractional generalization of the standard Solow–Swan model. In Section 3.4, the problem with the violation of the standard semi-group property of dynamic map is described using the examples of fractional generalization of the dynamic Leontief (intersectoral) model and logistic growth model. In Section 3.4, the definitions of new economic concepts and notions are described.

In Section 4, the Solvability Principle and the Correspondence Principle are discussed and some examples are suggested. In Section 4.1, we discuss the Solvability Principle by using the general fractional calculus as an example. In Section 4.2, for illustration we consider the distributed lag fractional calculus and growth-relaxation equations with gamma distributed delay time. In Section 4.3, a simple example of the Correspondence Principle for the case, when the order of the derivative tends toward integer values from the left and from the right, is considered. In Section 4.4, the Solvability Principle is discussed by using example from numerical simulation of fractional differential equations.

In Section 5, we describe some problems ("Non-Equivalence" and "Unpredictability") of fractional generalizations that are associated with non-equivalent fractional equations, which are formal generalization of equivalent differential equations of integer orders. In Section 5.1, we give definitions of equivalence of equations by solutions (s-equivalence). In Section 5.2, we illustrate non-equivalence of fractional generalization for relaxation and growth differential equations. In Section 5.3, we describe non-equivalence of fractional generalization of the fractional logistic equation that in economics describes growth in a competitive environment with memory. In Section 5.4, we formulate that fractional generalization of standard model can generate non-equivalent models.

In Section 6, we consider example of application of the Interpretability Principle by describing some examples of new effects and phenomena in economics. In Section 6.1, we describe a simple economic model with memory. Fractional differential equation, its solution and asymptotic behavior are proposed. In Section 6.2, we give an interpretation of the mathematical results by using suggested new concept of the warranted rate of growth with memory. In Section 6.3, we describe the interpretation of mathematical results in the form of economic phenomena for economic growth and decline with memory. In Section 6.4, we describe an interpretation of relaxation of economic processes with memory.

In Section 7, we give a short conclusion.

2. Non-Standard Properties of Fractional Derivatives

In this section we describe some properties (rules) of fractional derivatives causing problems when constructing fractional generalizations of standard dynamic models.

The fractional derivatives of non-integer orders have a set of non-standard properties and rules such as the violation of the standard product (Leibniz) and the standard chain rules, the violation of semigroup rules for orders of the derivatives and the violation of semigroup rules for dynamical map. The non-standard properties of fractional derivatives should be taken into account, when constructing fractional generalization of dynamic models. These properties create problems in realization of the derivability principle.

2.1. Violation of Standard Chain Rule

The standard chain rule for the first order derivative has the form:

$$D^1_t f(g(t)) = (D^1_g f(g))_{g=g(\mathrm{t})}\, D^1_t g(\mathrm{t}), \tag{4}$$

where $D^1_t = d/dt$ is the derivative of first order. The standard chain rule for the derivative of integer order $n \in \mathbb{N}$ can be written by the equation:

$$D^n_t f(g(\mathrm{t})) = n! \sum_{m=1}^{n} (D^m_g f(g))_{g=g(t)} \sum \prod_{r=1}^{n} \frac{1}{a_r!} \left(\frac{1}{r!} D^r_t g(\mathrm{t})\right)^{a_r}, \tag{5}$$

which is called the Faá di Bruno's formula [23].

The standard chain rules shown in Equations (4) and (5) are not satisfied for fractional derivatives of non-integer orders. Foe example, the chain rule for the Riemann–Liouville fractional derivative of the order $\alpha > 0$ (see equation (2.209) in section 2.7.3 of [3], pp. 97–98, [5], pp. 35–36, and [19]) has the form:

$$\begin{gathered} D^{\alpha}_{RL,0+} f(g(t)) = \tfrac{t^{\alpha}\, f(g(t))}{\Gamma(1-\alpha)} + \\ \sum_{k=1}^{\infty} C^{\alpha}_k \tfrac{k!\, t^{k-\alpha}}{\Gamma(k-\alpha+1)} \sum_{m=1}^{k} \left(D^m_g f(g)\right)_{g=g(t)} \sum \prod_{r=1}^{k} \tfrac{1}{a_r!} \left(\tfrac{1}{r!} D^r_t g(\mathrm{t})\right)^{a_r}, \end{gathered} \tag{6}$$

where $t > 0$, D^m_g and D^r_t are derivatives of integer orders, $\sum$ extends over all combinations of non-negative integer values of $a_1, a_2, \ldots, a_k$ such that $\sum_{r=1}^{k} r a_r = k$ and $\sum_r^k a_r = m$.

The chain rules for other type of fractional derivatives have a similar form. We see that standard chain rules (4) and (5) do not satisfied for fractional derivatives of non-integer order.

2.2. Violation of Semi-Group Rule for Orders of Derivatives

The standard semi-group rule for orders of integer-order derivatives has the form of the equality:

$$D_t^n\, D_t^m\, X(t) = D_t^{n+m}\, X(t), \tag{7}$$

which holds for $n, m \in \mathbb{N}$, if the function $X(t)$ is smooth or $X(t)$ is a continuous function that has continuous first $n+m$ derivatives (for example, $X(t) \in C^{n+m}\,(\mathbb{R})$). It is well known that this property may be broken for discontinuous functions $X(t)$ and if the derivatives are not continuous.

For fractional derivatives, the standard semi-group rule (7) is not satisfied in the general case (for example, see [1], pp. 46–47 and [5], p. 30). For example, the Caputo fractional derivatives of the orders $0 < \alpha, \beta < 1$ satisfy the equality:

$$\left(D^{\alpha}_{C,0+}\, D^{\beta}_{C,0+}\, X\right)(t) = \left(D^{\alpha+\beta}_{C,0+}\, X\right)(t) + \frac{1}{\Gamma(2-\alpha-\beta)}\, X^{(1)}\,(0) t^{1-\alpha-\beta}, \tag{8}$$

where $D^{\alpha}_{C,0+}$ and $D^{\beta}_{C,0+}$ are the Caputo fractional derivative of the orders $0 < \alpha, \beta < 1$ is defined by the equation:

$$\left(D^{\alpha}_{0+}\, X\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha}\, X^{(1)}\,(\tau) d\tau, \tag{9}$$

where $\Gamma(\alpha)$ is the gamma function. Equality (8) means the violation of the semi-group property for orders of derivatives, i.e., in general, we have the inequality:

$$\left(D^{\alpha}_{C,0+}\, D^{\beta}_{C,0+}\, X\right)(t) \neq \left(D^{\alpha+\beta}_{C,0+}\, X\right)(t), \tag{10}$$

if the orders of these fractional derivatives are non-integer. In the order α of the Caputo fractional derivative in (10) is non-integer and the order $\beta = n \in \mathbb{N}$, then we have the equality $\left(D^{\alpha}_{C,0+} D_t^n\, X\right)(t) = \left(D^{\alpha+n}_{C,0+}\, X\right)(t)$. If the order $\alpha = n \in \mathbb{N}$ and β is non-integer, then the standard semi-group property is violated, i.e., the inequality $\left(D_t^n\, D^{\beta}_{C,0+}\, X\right)(t) \neq \left(D^{n+\beta}_{C,0+}\, X\right)(t)$ holds in general.

2.3. Violation of the Standard Product Rule

The standard product (Leibniz) rule for first-order derivative (for $n = 1$) has the form:

$$D_t^1 (f(\mathrm{t})\, g(\mathrm{t})) = \left(D_x^1 \mathrm{f}(x)\right) g(x) + f(x) \left(D_x^1 g(x)\right). \tag{11}$$

The standard product rule for the derivative of integer order $n \in \mathbb{N}$ has the form:

$$D_t^n (f(t) g(t)) = \sum_{k=0}^{n} \frac{n!}{(n-k)!\, k!} \left(D_t^{n-k} f(t)\right)\left(D_t^k g(t)\right). \tag{12}$$

The Leibniz rule for derivative of non-integer order $\alpha \neq 1$ cannot have the simple form:

$$D_t^{\alpha} (f(t)\, g(t)) = \left(D_t^{\alpha} f(t)\right) g(t) + f(t) \left(D_t^{\alpha} g(t)\right). \tag{13}$$

A violation of relation in Equation (13) is a characteristic property of all derivatives of integer-orders $n \in \mathbb{N}$ greater than one and for all types derivatives of the non-integer order $\alpha > 0$ (for example, see [1], pp. 280–284, [3], pp. 91–97, [5], p. 33, 59, and [17,20,22]). In [17], the following theorem has been proved:

Theorem 1 ("No violation of the Leibniz rule. No fractional derivative"). *If a linear operator D^{α}_t satisfies the product rule in the form of Equation (13), then the operator D^{α}_t is the differential operator of first order, that can be represented in the form $D^{\alpha}_t = a(t)D^1_t$, where $a(t)$ is function on $\mathbb{R}$.*

As a result, we can states that derivatives of non-integer orders $\alpha \neq 1$ cannot satisfy the standard product rule of Equation (13). For example, the fractional generalization of the Leibniz rule for the Riemann–Liouville derivatives has the form (see section 15 in [1], pp. 277–284), of the infinite series:

$$D^{\alpha}_{RL}(f(t)g(t)) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} \Big(D^{\alpha-k}_{RL} f(t)\Big) \Big(D^k_t g(t)\Big), \tag{14}$$

where $f(\mathrm{t})$ and $g(\mathrm{t})$ are analytic functions on $[a, b]$ (see theorem 15.1 in [1]), D^{α}_{RL} is the Riemann–Liouville derivative; D^k is derivative of integer order $k \in \mathbb{N}$. It should be noted that the sum of Equation (14) is infinite and it contains the fractional integrals $I^{k-\alpha}_{RL} = D^{\alpha-k}_{RL}$ of non-integer orders $(k-\alpha)$ for the values $k > [\alpha] + 1$.

2.4. Violation of the Standard Semi-Group Rule for Dynamic Maps

Let us consider linear ordinary differential equation equations of first order in the form:

$$\frac{dX(t)}{dt} = A\, X(t), \tag{15}$$

where $X(t)$ is an unknown function (with values in a Banach space) and A is a constant linear bounded operator acting in the space (or A is the linear operator having an everywhere dense domain of definition $D(A)$ in the Banach space). We can consider the Cauchy problem of finding a solution of Equation (15) for $0 < t < \infty$, satisfying the given initial condition $X(0) = X_0 \in D(A)$. A unique solution of the Cauchy problem exists for the differential equation of first order (Equation (15)) with a constant bounded operator A and it can be written (for example, see [24], pp. 119–157) in the form:

$$X(t) = U(t)\, X(0), \tag{16}$$

where the operator $U(t)$ is defined by the series:

$$U(t) = exp(t\,A) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k, \tag{17}$$

which converges in the operator norm. The operator $U(t)$ is called the dynamic map or the phase flow [25].

A family of bounded linear operators $U(t)$, depending on the parameter $0 < t < \infty$, forms a semi-group if the condition $U(0) = I$ and the equality:

$$U(t_1)\, U(t_2) = U(t_1 + t_2) \tag{18}$$

hold for all t_1, t_2 where $0 < t_1, t_2 < \infty$. Equation (18) is the standard semi-group rule for dynamical map. The set $\{U(t), t > 0\}$ is called one-parameter dynamical semi-group. In quantum theory the operator A is called the infinitesimal generator of the quantum dynamical semi-group (see classical papers [26–29]). The class of differential equations for which A is a generator for a semigroup of class (C_0) coincides with the class of differential equations for which the Cauchy problem is uniformly correct [24].

Daftardar-Gejji and Babakhani [30] (see also [31] and [4], p. 142) have studied the existence, uniqueness, and stability of solutions for the fractional differential equations:

$$\Big(D^{\alpha}_{C,0+} X\Big)(t) = A\, X(t), \tag{19}$$

where $D^{\alpha}_{C,0+}$ is the Caputo fractional derivative of the order $0 < \alpha < 1$, $X(t)$. is the column vector and A. is real square $N\ x\ N$ matrix. They obtained the unique solution of Equation (19) in the form:

$$X(t) = U_{\alpha}(t)\ X(0), \tag{20}$$

where the operator $U_{\alpha}(t)$ is defined by the series:

$$U_{\alpha}(t) = E_{\alpha}[t^{\alpha} A] = \sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(\alpha k + 1)} A^k. \tag{21}$$

Here, $E_{\alpha}[t^{\alpha} A]$ is the Mittag–Leffler function with matrix arguments [32].

For $\alpha = 1$, we have $E_1[z] = exp(z)$. Therefore, we have $U_1(t) = U(t) = exp(t\,A)$.

The standard semi-group rule (Equation (18)) for dynamical maps $U_{\alpha}(t)$ does not hold for non-integer values of $\alpha \in (0, 1)$, i.e., we have the inequality:

$$U_{\alpha}(t_1)\ U_{\alpha}(t_2) \neq U_{\alpha}(t_1 + t_2) \tag{22}$$

that follows from the property of the Mittag–Leffler function (for example, see [33,34], and some additional information in [35–37]) in the form:

$$E_{\alpha}\Big[t_1^{\alpha} A\Big]\ E_{\alpha}\Big[t_2^{\alpha} A\Big] \neq E_{\alpha}\Big[(t_1 + t_2)^{\alpha} A\Big]. \tag{23}$$

As a result, the dynamical maps $U_{\alpha}(t)$ with $\alpha \notin \mathbb{N}$ cannot form a semigroup.

The operator $U_{\alpha}(t)$ describes the dynamical map with power-law fading memory for non-integer values of α. The violation of the standard semigroup rule for dynamical maps is a characteristic property of dynamics with memory. We can only state that the set $\{U_{\alpha}(t), t > 0\}$ of the dynamical map with memory forms a dynamical groupoid [34,37] for on-integer values of $\alpha \in (0, 1)$.

It should be noted that the fractional differential Equation (19) describes the fractional generalization of N-level open quantum system and the Leontief dynamic model of N-sectors in economy, in which the power-law memory is taken into account (see Section 3.4.1).

2.5. What Effects Are Fractional Derivatives Described?

In fractional calculus, many different types of fractional derivatives and integrals are known [1–4]. In construction of a fractional generalization of a standard dynamic model, an important part of the work is an adequate choice of the type of the fractional derivative or/and integral. First of all, fractional operators must correspond to the type of process to be described. It is well known that fractional derivatives and integrals are a powerful tool for describing processes with memory and nonlocality. However, not all fractional operators can describe the effects of memory (or non-locality). In application of the generalized and general fractional operators, an important question arises about the correct subject interpretation of these operators (for example, see informational [38], physical [39], and economic [40–42] interpretations). It is important to emphasize that not all fractional operators can describe the processes with memory (for example, see [43–46]). It is important to clearly understand what type of phenomena a given operator can describe. Let us give some examples for illustration.

2.5.1. First Example: Kober and Erdelyi–Kober Operators

The Kober fractional integration of non-integer order [1,2,4] can be interpreted as an expected value of a random variable up to a constant factor (for example, see [43,45] and section 10 in [46]), where the random variable describes scaling (dilation) with the gamma distribution. The Erdelyi–Kober integral operator, the differential operators of Kober and Erdelyi–Kober type have analogous interpretation [43,45,46]. As a result, these operators are integer-order operator with continuously distributed scaling (dilation), and these operators cannot describe the memory. Note that the fractional generalizations of the Kober and Erdelyi–Kober operators, which can be used to describe memory and distributed scaling (dilation) simultaneously, were proposed in [46].

The Kober fractional integral of the order $\alpha > 0$ [4], p. 106, is defined as:

$$\left(I^{\alpha}_{K;0+;\eta}\varphi\right)(t) = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^t \tau^{\eta}\, (t-\tau)^{\alpha-1}\varphi(\tau)d\tau, \tag{24}$$

where $\alpha > 0$ is the order of integration and $\eta \in \mathbb{R}$. Using the variable $x = \tau/t$, this operator can be represent by the equation:

$$\left(I^{\alpha}_{K;0+;\eta}\varphi\right)(t) = \frac{\Gamma(\eta+\alpha+1)}{\Gamma(\eta+1)} \int_0^1 f_{\eta+1;\alpha}(x)(S_x\varphi)(t)dx, \tag{25}$$

where S_x is the operator [1], pp. 95–96 and [4], p. 11 such that $(S_x\varphi)(t) = \varphi(x\,t)$ and $f_{\alpha;\beta}(x)$ is the probability density function (pdf) of the beta-distribution such that:

$$f_{\alpha;\beta}(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \tag{26}$$

for $x \in [0,1]$ and $f_{\alpha;\beta}(x) = 0$ if $x \notin [0,1]$, where $B(\alpha,\beta)$ is the beta function. We see that the Kober integral operator describes beta distributed scaling up to numerical factor. For details see [43,45] and section 10 in [46].

2.5.2. Second Example: Causality Principle and Kramers–Kronig Relations

To describe processes with memory [47–49], the operators should satisfy the causality principle. For natural and social processes with memory, the causality can be described by the Kramers–Kronig relations [50]. The Riesz fractional operators (see section 2.10 of [4]) cannot be used to describe memory since this operator violates the causality principle. The Riesz fractional operators can be used to describe power-law non-locality and power-law spatial dispersion (for example, see [51,52]).

The principle of causality is represented in the form of the Kramers–Kronig relations (the Hilbert transform pair) by using the Fourier transforms. Let us consider the Fourier transform $\widetilde{M}(\omega)$ of the memory function $M(t)$. In general, $\widetilde{M}(\omega)$ is the complex function $\widetilde{M}(\omega) = \widetilde{M}_1(\omega) + i\,\widetilde{M}_2(\omega)$, where the real part $\widetilde{M}_1(\omega) = Re\big[\widetilde{M}(\omega)\big]$ and the imaginary part $\widetilde{M}_2(\omega) = Im\big[\widetilde{M}(\omega)\big]$ are real-valued functions. The Kramers–Kronig relations state that the real part and the imaginary parts of the memory function are not independent, and the full function can be reconstructed given just one of its parts. Let us assume that the function $\widetilde{M}(\omega)$ is analytic in the closed upper half-plane of frequency ω and vanishes like $1/|\omega|$ or faster as $|\omega| \to \infty$. The Kramers–Kronig relations are given by:

$$\widetilde{M}_1(\omega) = \frac{1}{\pi} P.V. \int\limits_{-\infty}^{+\infty} \frac{1}{\Omega-\omega} \widetilde{M}_2(\Omega)d\Omega, \tag{27}$$

$$\widetilde{M}_2(\omega) = -\frac{1}{\pi} P.V. \int\limits_{-\infty}^{+\infty} \frac{1}{\Omega-\omega} \widetilde{M}_1(\Omega)d\Omega, \tag{28}$$

where P.V. denotes the Cauchy principal value. For details see [50].

2.5.3. Third Example: Abel-type operator with Kummer Function in Kernel

The Abel-type fractional integral (and differential) operator with Kummer function (or the three parameter Mittag–Leffler functions) in the kernel (see the classic book [1] and equation (37.1) in [1], p. 731) can be interpreted as the Riemann–Liouville fractional integral (and derivatives) with gamma distribution of delay time [43,53,54].

It is known that the Abel-type (AT) fractional integral operator with Kummer function in the kernel (see equation (37.1) in [1], p. 731) is defined by the equation:

$$\left(\boldsymbol{I}^{\alpha,\beta,\gamma,}_{AT;0+}Y\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_{1,1}(\beta;\alpha;\gamma(t-\tau))Y(\tau)\, d\tau, \tag{29}$$

and $F_{1,1}(a;b;z)$ is the confluent hypergeometric Kummer function. Using equality $F_{1,1}(a;c;z) = \Gamma(c)E^{a}_{1,c}(z)$ the memory kernel in Equation (29) can be expressed through the three parameter Mittag–Leffler functions $E^{\gamma}_{\alpha,\beta}(z)$.

The fractional integration with the gamma distributed lag in the form:

$$\left(\boldsymbol{I}^{\lambda,a;\alpha}_{T;RL;0+}Y\right)(t) = \left(M^{\lambda,a}_{T}(\tau) * \left(I^{\alpha}_{RL,0+}Y\right)\right)(t) = \int_0^t M^{\lambda,a}_{T}(\tau)\left(I^{\alpha}_{RL,0+}Y\right)(t-\tau)\, d\tau, \tag{30}$$

where * denotes the Laplace convolution, $\left(I^{\alpha}_{RL,0+}Y\right)(t)$ is the Riemann–Liouville fractional integral [1,4], $M^{\lambda,a}_{T}(\tau)$ is the probability density function (weighting functions) of the gamma distribution:

$$M^{\lambda,a}_{T}(\tau) = \frac{\lambda^a\, \tau^{a-1}}{\Gamma(a)}\, exp(-\lambda\, \tau) \tag{31}$$

for $\tau > 0$ and $M^{\lambda,a}_{T}(\tau) = 0$ for $\tau \le 0$, where $a > 0$ is the shape parameter and $\lambda > 0$ is the rate parameter. Equation (30) can be written thought the Laplace convolution of memory and weighting functions:

$$\left(\boldsymbol{I}^{\lambda,a;\alpha}_{T;C;0+}Y\right)(t) = \left(M^{\lambda,a}_{T} * \left(M^{\alpha}_{RL} * Y\right)\right)(t), \tag{32}$$

where $M^{\alpha}_{RL}(t) = (t-\tau)^{\alpha-1}/\Gamma(\alpha)$ is the kernel of the Riemann–Liouville fractional integral. The associativity of the Laplace convolution allows us to represent operator in the form:

$$\left(\boldsymbol{I}^{\lambda,a;\alpha}_{T;C;0+}Y\right)(t) = \int_0^t M^{\lambda,a;\alpha}_{TRL}(t-\tau)Y(\tau)\, d\tau, \tag{33}$$

where $M^{\lambda,a;\alpha}_{TRL}(t) = \left(M^{\lambda,a}_{T} * M^{\alpha}_{RL}\right)(t)$ is the memory-and-lag function of the form:

$$M^{\lambda,a;\alpha}_{TRL}(t) = \frac{\lambda^a\, \Gamma(a)}{\Gamma(a+n-\alpha)}\, t^{a+\alpha-1} F_{1,1}(a;a+\alpha;-\lambda t), \tag{34}$$

where $F_{1,1}(a;b;z)$ is the confluent hypergeometric Kummer function.

As a result, we obtain the relation:

$$\left(\boldsymbol{I}^{a+\alpha,a,-\lambda,}_{AT;0+}Y\right)(t) = \frac{1}{\lambda^a\, \Gamma(a)} \cdot \left(I^{\lambda,a;\alpha}_{T;RL;0+}Y\right)(t). \tag{35}$$

This equation shows that the AT fractional integral can be expressed through the Riemann–Liouville fractional integral with gamma distributed lag for wide range of parameters.

2.5.4. Fourth Example: Abel-type Operator with Kummer Function in Kernel

In application it is important to have conditions for the operator kernel, which make it possible to assign this operator to one or another type of phenomena or processes. For example, it is obvious that the kernels of general fractional convolution operators satisfying the normalization condition will describe distributed delays in time (lag), and not memory (for example, see [44,46], and some additional comments in [53–55]). It is well known in physics that the time delay is related to the finite speed of the process and not to the memory. For example, the Caputo–Fabrizio operators, which were misinterpreted as fractional derivatives of non-integer orders, are integer-order derivatives with the exponentially distributed delay time [43,44]. Therefore, these operators cannot be used to describe processes with memory. Note that the fractional derivatives with exponentially distributed is suggested in [43] and then applied in economics [53–55].

2.5.5. Fifth Example: Fractional operators with Uniform Distributed Order

The continual fractional derivatives and integrals were proposed by A.M. Nakhushev [56,57]. The fractional operators, which are inversed to the continual fractional integrals and derivatives, have been proposed by A.V. Pskhu [58,59]. In papers [47,60], we proved that the fractional integrals and derivatives of the uniform distributed order can be expressed (up to a numerical factor) thought the continual fractional integrals and derivatives that were suggested by A.M. Nakhushev. Therefore, the proposed fractional integral and derivatives of uniform distributed order we called in our paper [60] as the Nakhushev fractional integrals and derivatives. The corresponding inverse operators, which contain the two parameter Mittag–Leffler functions in the kernel, were called as the Pskhu fractional integrals and derivatives [60].

For example, the Riemann–Liouville fractional integral of distributed order is defined as:

$$\left(I^{[\alpha_1,\alpha_2]}_{RL,0+}X\right)(t) = \int_{\alpha_1}^{\alpha_2} \rho(\alpha)\ (I^{\alpha}_{RL,0+}X)(t)d\alpha, \tag{36}$$

where $\alpha_2 > \alpha_1 \geq 0$, and the weight function $\rho(\alpha)$ satisfies the normalization condition:

$$\int_{\alpha_1}^{\alpha_2} \rho(\alpha)d\alpha = 1. \tag{37}$$

In Equation (36) the integration with respect to time and the integration with respect to order can be permuted for a wide class of functions $X(\tau)$. As a result, Equation (36) is written in the form:

$$\left(I^{[\alpha_1,\alpha_2]}_{RL,0+}X\right)(t) = \int_0^t M^{[\alpha_1,\alpha_2]}_{\rho(\alpha)}(t-\tau)X(\tau)d\tau, \tag{38}$$

where the kernel $M^{[\alpha_1,\alpha_2]}_{\rho(\alpha)}(t-\tau)$ is defined by the equation:

$$M^{[\alpha_1,\alpha_2]}_{\rho(\alpha)}(t-\tau) = \int_{\alpha_1}^{\alpha_2} \frac{\rho(\alpha)}{\Gamma(\alpha)} \frac{1}{(t-\tau)^{1-\alpha}} d\alpha, \tag{39}$$

where $\alpha_2 > \alpha_1 \geq 0$. In the simplest case, we can use the continuous uniform distribution (CUD) that is defined by the expression:

$$\rho(\alpha) = \begin{cases} \frac{1}{\alpha_2-\alpha_1} & for\ \alpha \in [\alpha_1,\alpha_2] \\ 0 & for\ \alpha \in (-\infty,\alpha_1) \cup (\alpha_2,\infty) \end{cases}. \tag{40}$$

For the probability density function (Equation (40)), the memory function (Equation (39)) has the form:

$$M^{[\alpha_1,\alpha_2]}_{CUD}(t) = W(\alpha_1,\alpha_2,t) = \frac{1}{(\alpha_2-\alpha_1)\ t} \int_{\alpha_1}^{\alpha_2} \frac{t^{\xi}d\xi}{\Gamma(\xi)}. \tag{41}$$

As a result, the fractional integral of uniform distributed order is defined by the equation:

$$I^{[\alpha,\beta]}_{N,RL}X(t) = \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \left(I^{\xi}_{RL,0+}X\right)(t)d\xi = \int_0^t W(\alpha,\beta,t-\tau)\ X(\tau)d\tau, \tag{42}$$

where $\beta > \alpha > 0$. The fractional integrals and derivatives of the uniform distributed order can be expressed thought the continual fractional integrals and derivatives, which have been suggested by A.M. Nakhushev [56,57].

2.5.6. Sixth Example: Left-Sided and Right-Sided Fractional Operators

The right-sided Riemann–Liouville, Liouville, and Caputo fractional derivatives [4] cannot describe the memory processes. Using only the left-sided derivatives of bon-integer orders, we take into account the history of changes of variable in the past, that is for $\tau < t$. The right-sided operators are defined by integration over $\tau > t$, where t is the present moment of time. Using right-sided operators actually means that the present state depends on the future states, and not on the past states of the process.

2.5.7. Seventh Example: Fading Memory, Spetial Non-Locality, Time Delay (Lag), Scaling

Fractional calculus approach allows us to describe the spatial non-locality and fading memory of power-law type, the openness of processes and systems, intrinsic dissipation, long-range interactions, and some other type of phenomena. The most well-known phenomena in physics that can be described by fractional differential equations, are the fractional relaxation-oscillation, fractional diffusion-wave, fractional viscoelasticity, spatial and frequency dispersion of power type, nonexponential relaxation, anomalous diffusion, and some others [61,62].

As a result, we can state that the following type of phenomena can be independent of each other:

- fading memory (forgetting) (for example, see [47–50] and references therein) and power-law frequency dispersion;
- spatial non-locality (for example, see [63]) and power-law spatial dispersion (for example, see [64]);
- lag (time delay) (for example, see [43,53–55,65] and references therein); and
- scaling (dilation) (for example, see section 9 in [43] and references therein).

As a result, these phenomena are described by certain types of operator kernels. For other types of processes and phenomena, we do not have mathematical conditions on the kernel of operators, which allow us to uniquely identify one or another type of process. In this part of applied mathematics, the fractional calculus requires its development. Mathematically strict conditions on the operator kernels are necessary to initially distinguish between various types of processes and phenomena. It should be emphasized that we must first clearly distinguish between the types of processes and phenomena, but simply list various examples of their specific manifestations in the reality surrounding us, described by the natural and social sciences. It is necessary to establish a clear correspondence between the types of operator kernels and the types of phenomena.

3. Examples of Problems from Non-Standard Properties of Fractional Derivatives

In this section, we present examples illustrating the problems and difficulties of fractional generalization of standard dynamic models, which arise from non-standard properties of fractional derivatives. As an example of the problem with the non-standard form of the chain rule, we consider a fractional generalization of the Kaldor-type model of business cycles. Problem with the violation of the standard semi-group rule for orders of derivatives is shown for the fractional generalization of the Phillips model of the multiplier-accelerator. To illustrate the problems arising from the non-standard form of the product (Leibniz) rule, we consider the fractional generalization of the standard Solow–Swan model. Problem with the violation of the standard semi-group property of dynamic map is described on the examples of fractional generalization of the dynamic Leontief (intersectoral) model and logistic growth model.

3.1. Example of Problems with Chain Rule: Kaldor-Type Model of Business Cycles and Slutsky Equation

In this subsection, we demonstrate that the violation of the standard chain rule gives a restriction in fractional generalization of dynamic models. For this purpose, we consider a fractional generalization of the Kaldor-type model of business cycles and the economic model [66–68] based on the van der Pol equation [69,70].

Economic models, which are based on the van der Pol equation, are considered as prototypes of model for complex economic dynamics [69,70]. Nonlinear dynamic models are used to explain irregular and chaotic behavior of complex economic and financial processes (for example, see the business cycle theory [71,72], nonlinear economic dynamics and chaos [73,74], and stabilization [75]). Some models of business cycles, which are based on the Kaldor nonlinear investment-savings functions [69–72] and the Goodwin nonlinear accelerator-multiplier (for example, see the Goodwin's paper [76], and [77–79]), can be reduced to the van der Pol equation, which describes damped oscillations [69–72].

3.1.1. Standard Kaldor-Type Model of Business Cycles

In the framework of Keynesian approach to theory of national income, Nicholas Kaldor formulated [66–68] the first nonlinear model of endogenous business cycles in 1940. Kaldor consider the interactions between the investment $I(Y)$ and the savings $S(Y)$, where $Y = Y(t)$ denotes national income. Using the fact that the linear functions $I(Y)$ and $S(Y)$ cannot describe processes of business cycle, Kaldor proposed nonlinear form for $I(Y)$ and $S(Y)$, which leads to oscillatory processes of business cycles [69,70].

Let us derive the equation of the Kaldor model of business cycles by using approach proposed by Chang and Smyth [68] (see also [70–72]). In the Kaldor model, instead of the standard accelerator equation $I(t) = vY^{(1)}(t)$ the dependence of investments on the rate of change of national income is considered in the form:

$$I(Y,K) - S(Y,K) = v\, Y^{(1)}(t), \tag{43}$$

which takes into account the savings, where $K = K(t)$ denotes the capital stock, $Y = Y(t)$ is the national income, v is the accelerator coefficient and $Y^{(1)}(t)$ denotes its time derivatives of first order. The parameter $a = 1/v$ is an adjustment coefficient. In this model assumes that $I_K(Y,K) = \partial I(Y,K)/\partial K < 0$ and $S_K(Y,K) = \partial S(Y,K)/\partial K > 0$.

Differentiation of Equation (43) with respect to time and using the standard chain rule, we obtain:

$$v\, Y^{(2)}(t) = (I_Y(Y,K) - S_Y(Y,K))Y^{(1)}(t) + (I_K(Y,K) - S_K(Y,K))K^{(1)}(t). \tag{44}$$

In the paper [68] it is assumed that the actual change in the capital stock is determined by savings decisions, such that:

$$S(Y,K) = K^{(1)}(t), \tag{45}$$

where $K^{(1)}(t)$ denotes the time derivatives of first order of the capital stock $K(t)$. Substitution of Equation (45) into Equation (44) gives:

$$v\, Y^{(2)}(t) = (I_Y(Y,K) - S_Y(Y,K))\, Y^{(1)}(t) + (I_K(Y,K) - S_K(Y,K))\, S(Y,K). \tag{46}$$

In the paper [68], it is also assumed that the function $I(Y,K)$ is linear in $K(t)$ and savings is independent of the capital stock, i.e., the function $S(Y,K) = S(Y)$. In this case, the expression $(I_K(Y,K) - S_K(Y,K))$ is independent of the capital stock $K(t)$ and Equation (46) takes the form:

$$v\, Y^{(2)}(t) = (I_Y - S_Y)(Y)\, Y^{(1)}(t) + I_K(Y)S(Y). \tag{47}$$

Using the variable $y(t) = Y(t) - \overline{Y}$, where $\overline{Y}$ is the equilibrium value, Equation (47) can be rewritten [70] in the form of the Lienard equation:

$$y^{(2)}(t) + g(y(t))y^{(1)}(t) + f(y(t)) = 0, \tag{48}$$

which is used in mechanics to describe the dynamics of a spring-mass system.

Assuming symmetric shapes of the investment and savings functions, the parabolic form of the function of their difference, $g(y) = \mu\left(y^2 - 1\right)$, and the linear form of $f(y) = y$, we obtain the Van der Pol equation:

$$y^{(2)}(t) + \mu\left(y^2(t) - 1\right) y^{(1)}(t) + y(t) = 0. \tag{49}$$

This equation is used in economic modeling of the business cycles in the framework of nonlinear economic models with continuous-time. The Van der Pol Equation (49) can be written in the two-dimensional form:

$$\begin{cases} y^{(1)} = x, \\ x^{(1)} = \mu\left(1 - y^2\right)x - y(t). \end{cases} \tag{50}$$

This form of the Van der Pol equation is used in computer simulation on the phase space.

3.1.2. Fractional Generalization of Kaldor-Type Model of Business Cycles

To generalize Equation (49) for the case of processes with memory, we cannot simply replace the derivatives of integer order by fractional derivatives to get the fractional Van der Pol equation:

$$\left(D^{\alpha}_{C,0+}X\right)(t) + \mu\left(y^2(t) - 1\right)\left(D^{\beta}_{C,0+}X\right)(t) + y(t) = 0, \tag{51}$$

where $\alpha > \beta > 0$. The fractional generalization of the Van der Pol equation are considered in physics (for example, see [80–82]) and in economics [83,84].

To correctly generalize the standard model, it is necessary to take into account the process of obtaining Equations (49) and (50) from Equation (43). Note that the replacement of the derivatives of the integer order in Equations (43) and (44) by fractional derivatives also does not allow obtaining the fractional differential Equation (51). This is because, when deriving Equation (49) from Equations (43) and (44), we must use the standard chain rules in the form:

$$D^1_t F(Y(t), K(t)) = F_Y(Y, K)Y^{(1)}(t) + F_K(Y, K)\, K^{(1)}(t), \tag{52}$$

where $D^1_t = d/dt$.

The chain rule for fractional derivative has more complicated form (see equation (2.209) in section 2.7.3 of [3,19]). As a result, we should restrict ourselves to the assumption of the presence of a memory only for Equation (43). Let us assume that the excess of investment over saving, i.e., the difference $I(Y, K) - S(Y, K)$ is determined by changes in the growth rate of the national income in the past:

$$I(Y(t), K(t)) - S(Y(t), K(t)) = \int_0^t v(t - \tau)Y^{(1)}(\tau)d\tau, \tag{53}$$

where the time variable is considered as dimensionless variable. For the case $v(t - \tau) = v\, \delta(t - \tau)$ Equation (53) gives Equation (43) of the standard model.

The memory with one-parameter power-law fading is described [47,48,60] by the function:

$$v(t - \tau) = \frac{v_\alpha}{\Gamma(1 - \alpha)}(t - \tau)^{-\alpha}, \tag{54}$$

where $\Gamma(\alpha)$ is the gamma function and $0 < \alpha \leq 1$, and $\left(D^{\alpha}_{C,0+} Y\right)(t)$ is the Caputo fractional derivative:

$$\left(D^{\alpha}_{C,0+} Y\right)(t) = \left(I^{n-\alpha}_{RL,0+} Y^{(n)}\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} Y^{(n)}(\tau) d\tau, \tag{55}$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$, and the function $Y(\tau)$ has integer-order derivatives $Y^{(j)}(\tau)$, $j = 1, \ldots, (n-1)$, that are absolutely continuous.

Equation (53) with kernel (54) can be rewritten through the Caputo fractional derivative:

$$I(Y(t), K(t)) - S(Y(t), K(t)) = v_\alpha \left(D^{\alpha}_{C,0+} Y\right)(t). \tag{56}$$

Action of the first-order derivative $D^1_t = d/dt$. with respect to time on Equation (56) and using the standard chain rule, we obtain:

$$\begin{aligned} v_\alpha D^1_t \left(D^{\alpha}_{C,0+} Y\right)(t) = (I_Y(Y,K) - S_Y(Y,K)) Y^{(1)}(t) + \\ (I_K(Y,K) - S_K(Y,K)) K^{(1)}(t) \end{aligned} \tag{57}$$

Substituting Equation (45) into Equation (57) gives:

$$\begin{aligned} v_\alpha D^1_t \left(\mathrm{D}^{\alpha}_{C,0+} Y\right)(t) = (I_Y(Y,K) - S_Y(Y,K)) Y^{(1)}(t) + \\ (I_K(Y,K) - S_K(Y,K)) S(Y,K). \end{aligned} \tag{58}$$

Using the assumptions that are proposed in the paper [68], Equation (58) takes the form:

$$v_\alpha D^1_t \left(D^{\alpha}_{C,0+} Y\right)(t) = (I_Y - S_Y)(Y)\, Y^{(1)}(t) + I_K(Y) S(Y), \tag{59}$$

and:

$$D^1_t \left(D^{\alpha}_{C,0+} y\right)(t) + g(y) y^{(1)}(t) + f(y) = 0. \tag{60}$$

Note that $D^1_t \left(D^{\alpha}_{C,0+} y\right)(t) \neq \left(D^{\alpha+1}_{C,0+} y\right)(t)$ since the standard semi-group rule for order of derivatives is violated in general.

To obtain two-dimensional form of fractional differential Equation (60), we can use the Riemann–Liouville fractional derivative that is defined by the equation:

$$\left(D^{\alpha}_{RL,0+} Y\right)(t) = D^n_t \left(I^{n-\alpha}_{RL,0+} Y\right)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} Y(\tau) d\tau. \tag{61}$$

Using Equation (61), we can get the equalities:

$$D^1_t \left(\mathrm{D}^{\alpha}_{C,0+} Y\right)(t) = D^1_t \left(I^{1-\alpha}_{RL,0+} Y^{(1)}\right)(t) = \left(\left(D^1_t I^{1-\alpha}_{RL,0+}\right) Y^{(1)}\right)(t) = \left(D^{\alpha}_{RL,0+} Y^{(1)}\right)(t). \tag{62}$$

This allows us to rewrite Equation (60) as:

$$\left(D^{\alpha}_{RL,0+} y^{(1)}\right)(t) + g(y) y^{(1)}(t) + f(y) = 0. \tag{63}$$

As a result, the Kaldor-type model of business cycles with power-law memory can be described by the fractional Van der Pol Equation (63). Equation (63) can be written in the two-dimensional form:

$$\begin{cases} D^1_t y = x, \\ D^{\alpha}_{RL,0+} x = \mu\left(1 - y^2\right) x - y. \end{cases} \tag{64}$$

This form of the fractional Van der Pol equation can be used in computer simulation of the Kaldor-type model of business cycles with power-law memory by analogy with the papers in physics [80–82], and in economics [83,84].

3.1.3. Fractional Generalization of Slutsky Equation

The Slutsky Equation (see classical paper [85], and its available copies [86–89]), which is used in microeconomics [90–92], allows us to calculate the unobservable functions (compensated (Hicksian) demand function) from observable functions such as the derivatives of the ordinary (Marshallian) demand function with respect to price and income. The difficulties of the fractional generalization of the standard Slutsky equation is connected with the using the chain rule in the derivation of this equation in microeconomics.

Let us describe the derivation of the standard Slutsky equation. For simplification we will assume that there are only two goods (x and y). In microeconomics, two type demand function are used: the compensated demand function, $x_c\left(p_x,\ p_y,\ U\right)$, and the ordinary (uncompensated) demand function, $x\left(p_x,\ p_y, I\right)$. The compensated (Hicksian) demand function describes the demand of a consumer over a bundle of goods (x and y) that minimizes their expenditure while delivering a fixed level of utility. The compensated demand functions are convenient from a mathematical point of view since these functions do not require income or wealth to be represented. In addition, the function $x_c\left(p_x,\ p_y,\ U\right)$ is linear in (x, y), which gives a simpler optimization problem. Unfortunately these functions are not directly observable. The uncompensated (Marshallian) demand functions $x\left(p_x,\ p_y, I\right)$ are convenient from an economic point of view. However, this convenience is due to the fact that the uncompensated demand function $x\left(p_x,\ p_y, I\right)$ describes demand given prices $p_x,\ p_y$ and income I that are easier to observe directly in economics.

The compensated (Hicksian) demand function is defined by the equation

$$x_c\left(p_x,\ p_y,\ U\right) = x\left(p_x, p_y, E\left(p_x, p_y, U\right)\right), \tag{65}$$

where $E\left(p_x, p_y, U\right)$ is the expenditure function that gives the minimum wealth required to get to a given utility level. Equation (65) is obtained by inserting that expenditure level into the demand function, $x\left(p_x,\ p_y, I\right)$. Note that the variables $p_x,\ p_y$ enter into the ordinary demand function in (65) in two places.

In 1915, Evgeny E. Slutsky proposed [85–89] an equation that allows us to calculate the compensated (Hicksian) demand function from observable functions, namely, the derivative of the Marshallian demand with respect to price and income.

To derive the Slutsky equation, we apply the partial differentiation of Equation (65) with respect to p_x. This allows us to obtain the equation:

$$\frac{\partial x_c\left(p_x,\ p_y,\ U\right)}{\partial p_x} = \frac{\partial\, x\left(p_x, p_y, E\left(p_x, p_y, U\right)\right)}{\partial p_x} + \frac{\partial\, x\left(p_x, p_y, E\left(p_x, p_y, U\right)\right)}{\partial E} \frac{\partial E\left(p_x, p_y, U\right)}{\partial p_x}, \tag{66}$$

where we use the standard chain rule. Then we should change the notation and taking into account two following economic effects. The first, we take into account the substitution effect that mathematically is represented by the equality:

$$\frac{\partial x_c\left(p_x,\ p_y,\ U\right)}{\partial p_x} = \left(\frac{\partial\, x\left(p_x, p_y, E\left(p_x, p_y, U\right)\right)}{\partial p_x}\right)_{U=const}, \tag{67}$$

that indicates movement along a single indifference curve ($U = const$). The second, we take into account the income effect in the form:

$$\frac{\partial\, x\left(p_x, p_y, E\left(p_x, p_y, U\right)\right)}{\partial E} = \frac{\partial\, x\left(p_x, p_y, I\right)}{\partial I}, \tag{68}$$

because changes in income or expenditures is the same thing in the function $x\big(p_x,\ p_y, I\big)$. Then we can use the Shephard's lemma in the form:

$$\frac{\partial E\big(p_x, p_y, U\big)}{\partial p_x} = x_c\big(p_x,\ p_y,\ U\big). \tag{69}$$

Substitution of Equations (67)–(69) into Equation (66) gives the Slutsky equation:

$$\frac{\partial x_c\big(p_x,\ p_y,\ U\big)}{\partial p_x} = \left(\frac{\partial\, x\big(p_x, p_y, E\big(p_x, p_y, U\big)\big)}{\partial p_x}\right)_{U=const} + \frac{\partial\, x\big(p_x, p_y, I\big)}{\partial I} x_c\big(p_x,\ p_y,\ U\big), \tag{70}$$

where we should see that $x_c\big(p_x,\ p_y,\ U(x^*, y^*)\big) = x_c\big(p_x,\ p_y,\ I\big)$ at the utility-maximizing point (x^*, y^*).

In fractional generalization of the Slutsky equation, the violation of the standard chain rule leads to the equation:

$$\begin{gathered} D^{\alpha}_{p_x} x_c\big(p_x,\ p_y,\ U\big) = \frac{p_x^{\alpha}}{\Gamma(1-\alpha)} x_c\big(p_x,\ p_y,\ U\big) + \Big(D^{\alpha}_{p_x} x\big(p_x, p_y, E\big(p_x, p_y, U\big)\big)\Big)_{U=const} + \\ \sum_{k=1}^{\infty} C^{\alpha}_{k} \frac{k!\ p_x^{k-\alpha}}{\Gamma(k-\alpha+1)} \sum_{m=1}^{k} \frac{\partial^m\ x\big(p_x, p_y, I\big)}{\partial I^m} \sum \prod_{r=0}^{k-1} \frac{1}{a_r!} \left(\frac{1}{r!} \frac{\partial^r x_c\big(p_x, p_y, U\big)}{\partial p_x^r}\right)^{a_r}, \end{gathered} \tag{71}$$

which has a significant complication of the form in compared to the standard equation. In the fractional Slutsky equation $\sum$ extends over all combinations of non-negative integer values of $a_1, a_2, \dots, a_k$ such that $\sum_{r=1}^{k} r a_r = k$ and $\sum_{r}^{k} a_r = m$.

In addition, the fractional Slutsky equation does not make much sense from an economic point of view, if we consider it as a description of the relationship of compensated (Hicksian) demand function and ordinary (Marshallian) demand function. The standard equation describes the connection these functions in full and this connection is local.

However, the Slutsky fractional equation is important from the other point of view. It is known that the standard Slutsky equation can be represented in terms of elasticity. In this form the Slutsky equation describes a connection of the compensated (Hicksian) price elasticity, the (uncompensated) price elasticity, and the income elasticity of goods. The proposed fractional Slutsky equation describes a connection of the fractional Hicksian elasticity of non-integer order [93–96] and the Marshallian (uncompensated) price and income elasticities, which are special cases of the fractional elasticity [93–96] for $\alpha \in \mathbb{N}$.

In this regard, we note that the fractional elasticity of a non-integral order can be represented as an infinite sum of elasticities of a higher order, using an equation expressing a fractional derivative in view of the infinite sum of the derivatives of integer orders (see lemma 15.3 in [1], p. 278).

3.2. Example of Problems with Semi-Group Rule for Orders of Derivatives: Phillips Model of Multiplier-Accelerator

Let us consider a fractional generalization of the standard Phillips model of the multiplier-accelerator to demonstrate the fact that the semi-group rule for orders of fractional derivatives gives a restriction in the construction of such generalizations.

The Phillips model of the multiplier-accelerator has been proposed by Alban W.H. Phillips [97,98] (see also [55,78,79,99]) in 1954 as a generalization of the Harrod–Domar macroeconomic growth model with continuous time. The standard Phillips model is described by the ordinary differential equation of second order in the form:

$$Y^{(2)}(t) + a\, Y^{(1)}(t) + bY(t) = \lambda_1 \lambda_2 A, \tag{72}$$

where $a = \lambda_2 s + \lambda_1 - \lambda_1 \lambda_2 v$ and $b = \lambda_1 \lambda_2 s$; $Y(t)$ is the national income; $0 < s < 1$ is the marginal propensity to save; v is the investment coefficient; λ_1 is the speed of response of output to changes in demand; λ_2 is the speed of response of induced investment to changes in output. The autonomous expenditure $A(t)$ is assumed [78,79] to be constant ($A(t) = A$).

The formal generalization of Equation (72) by replacing the derivatives of integer orders by fractional derivatives has the form:

$$\left(D^{\beta}_{C,0+}Y\right)(t) + a\left(D^{\alpha}_{C,0+}Y\right)(t) + bY(t) = \lambda_1\lambda_2 A, \tag{73}$$

where $\beta > \alpha > 0$ and $D^{\alpha}_{C,0+}$ is the Caputo fractional derivative, for example. Such a generalization does not take into account how the standard Phillips equation was obtained. It does not take into account what assumptions are used in the basis and what economic concepts were applied for the derivation of equation of standard model.

Let us briefly describe the process of obtaining the standard equation. The first assumption is form of equation of the investment accelerator [78], p. 72. The value of the actual induced investment $I(t)$ at time t in response to changes in output $Y(t)$ is given by:

$$I^{(1)}(t) = -\lambda_1\left(I(t) - v\, Y^{(1)}(t)\right). \tag{74}$$

The second assumption is the equation for the total demand $Z(t)$ in the form:

$$Z(t) = C(t) + I(t) + A(t), \tag{75}$$

where $C(t) = cY(t)$ is the planned consumption, and we can use $s = 1 - c$, the marginal propensity to save instead of the marginal propensity to consume $c \in (0, 1)$. Then we have the equation:

$$Z(t) = cY(t) + I(t) + A(t). \tag{76}$$

The third assumption is the multiplier equation [78], p. 73, in the form

$$Y^{(1)}(t) = -\lambda_2(Y(t) - Z(t)). \tag{77}$$

The equations of the standard model are Equations (74), (76)–(77). A differential equation for income $Y(t)$ is obtained by eliminating $Z(t)$ and $I(t)$ from the system of Equations (75)–(77). Substitution of Equation (76) into Equation (77) allows us to obtain the expression for the induced investment in the form:

$$I(t) = \lambda_2^{-1}Y^{(1)}(t) + sY(t) - A(t). \tag{78}$$

Substituting Equation (78) into Equation (76) under the assumption that the autonomous expenditure $A(t) = A$ is constant, we obtain Equation (72) of the standard Phillips model by the first-order differentiation.

The type of Equations (74), (76), and (77), which are used in the derivation of the standard model Equation (72), gives an impression that it is possible to propose a fractional generalization of the standard model using a formal replacement of the derivatives of first order by fractional derivatives in Equations (74) and (77). This gives the following system of equations:

$$\begin{cases} \left(D^{\alpha_1}_{C,0+}I\right)(t) = -\lambda_1\left(I(t) - v\left(D^{\alpha_2}_{C,0+}Y\right)(t)\right), \\ Z(t) = cY(t) + I(t) + A(t), \\ \left(D^{\alpha_3}_{C,0+}Y\right)(t) = -\lambda_2(Y(t) - Z(t)), \end{cases} \tag{79}$$

where the orders of fractional derivatives do not necessarily coincide, and $0 < \alpha_1, \alpha_2, \alpha_3 \leq 1$.

The last two equations of system (79) give an expression for the function $I(t)$ in the form:

$$I(t) = \lambda_2^{-1}\left(D^{\alpha_3}_{C,0+}Y\right)(t) + sY(t) - A(t). \tag{80}$$

Substituting Equation (80) in the first equation of system (79) under the assumption that the autonomous expenditure $A(t) = A$ is constant, we obtain the equation:

$$\begin{gathered}\left(D^{\alpha_1}_{C,0+}\left(D^{\alpha_3}_{C,0+}Y\right)\right)(t) + \lambda_2 s\left(D^{\alpha_1}_{C,0+}Y\right)(t) - \lambda_1\lambda_2 v\left(D^{\alpha_2}_{C,0+}Y\right)(t) + \lambda_1\left(D^{\alpha_3}_{C,0+}Y\right)(t) + \\ \lambda_1\lambda_2 sY(t) = \lambda_1\lambda_2 A.\end{gathered} \tag{81}$$

For the case $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, we have the equation:

$$\left(D^{\alpha}_{C,0+}\left(D^{\alpha}_{C,0+}Y\right)\right)(t) + a\left(D^{\alpha}_{C,0+}Y\right)(t) + bY(t) = \lambda_1\lambda_2 A, \tag{82}$$

where

$$a = \lambda_2 s + \lambda_1 - \lambda_1\lambda_2 v\,;\; b = \lambda_1\lambda_2 s. \tag{83}$$

As a result, we see that in general case the fractional generalization of the Phillips model can be described by Equation (82) instead of Equation (73). We can also see that Equation (82) cannot contain $\beta = 2\alpha$ as it used in Equation (73). In addition, the violation of the standard semi-group rule for the orders of derivatives led us to the fact that we have $D^{\alpha}_{C,0+}\left(D^{\alpha}_{C,0+}Y\right)$ instead of $D^{2\alpha}_{C,0+}Y$.

It should be emphasized that the generalization given by equation system (79) is formal and does not reflect the economic sense of the original Equations (74) and (77) of the standard model. In Equations (74) and (77), the derivatives of the functions to the left of the equal sign in reality are part of the operator of the exponential distributed lag [78], pp. 72–74.

The standard Phillips model of the multiplier-accelerator takes into account two continuously distributed lags. The first lag characterize the output responding to demand with speed λ_1. The second lag describes the induced investment responding to changes in output with speed λ_2. These economic accelerator and multiplier can be described by the following operators.

The integer-order derivative with exponentially distributed lag can be defined [46] by the first-order equation:

$$\left(\boldsymbol{D}^{\lambda,n}_{T,C}Y\right)(t) = \lambda\int_{-\infty}^{t} exp\{-\lambda\,(t-\tau)\}\,Y^{(n)}(\tau)d\tau,. \tag{84}$$

where $n \in \mathbb{N}_0$. For $n = 0$, we have:

$$\left(\boldsymbol{D}^{\lambda,0}_{T,C}Y\right)(t) = \lambda\int_{-\infty}^{t} exp\{-\lambda\,(t-\tau)\}\,Y(\tau)d\tau. \tag{85}$$

In reality, the first and third assumptions of the standard model, which are described by Equations (74) and (77), should be written [78], pp. 25–27, in the form of the equations:

$$\mathrm{I(t)} = v\left(\mathbf{D}^{\lambda_1,1}_{\mathbf{T,C}}\,\mathrm{Y}\right)\mathrm{(t)}, \tag{86}$$

and:

$$\mathrm{Y(t)} = \left(\mathbf{D}^{\lambda_2,0}_{\mathbf{T,C}}\,\mathrm{Z}\right)\mathrm{(t)}. \tag{87}$$

In standard macroeconomic models, the differential equations of exponentially distributed lag are used in the form of Equations (74) and (77) instead of equations with integro-differential operators in the form of Equations (86) and (87). Equations (74) and (77) are called the differential equations of the exponential lag [78], p. 27. In economics, the use of differential equations of integer orders instead of the integro-differential operators (86) and (87) is caused by the fact that there are considerable difficulties in handling the integrals in Equations (86) and (87). It is seen that equations with continuously distributed lag are equivalent to differential equations of integer orders under certain conditions. These differential equations are easier to handle in comparison with equations that contain integro-differential operators of the distributed lag.

As a result, to obtain a correct generalization of the standard Philips model, we should use the fractional derivative with exponentially distributed lags [46,55] instead of the integer-order operators with exponentially distributed lags. For example, we can use the Caputo fractional derivative with exponentially distributed lag:

$$\left(\boldsymbol{D}^{\lambda,\alpha}_{T,C} Y\right)(t) = \lambda \int_0^t exp\{-\lambda\,(t-\tau)\}\left(D^{\alpha}_{C,0} Y\right)(\tau) d\tau, \tag{88}$$

where $\lambda > 0$ is the rate parameter of exponential distribution and $\left(D^{\alpha}_{C,0} Y\right)$ is the Caputo fractional derivative of the order $\alpha > 0$.

Another generalization method is to account for memory effects instead of the distributed lag effect [55]. This generalization assumes to use fractional derivatives (without distributed lag) instead of integer-order operators (86) and (87).

Self-consistent constructions of different fractional generalizations of the standard Phillips model of the multiplier-accelerator were proposed in the work [55].

At the same time, Equation (73), which is a formal fractional generalization of the equation of the standard Phillips model, does not have economic significance due to the violation of the principle of derivability.

3.3. Example of Problems with Product Rule: Solow–Swan Model

In this subsection, we consider a fractional generalization of the standard Solow–Swan model (see, classical papers [100–102], and books [103,104]) to demonstrate the fact that the violation of the standard product (Leibniz) rule for fractional derivatives [17,20,22], which is main characteristic property of these operators, gives a restriction in the construction of such generalizations.

The standard Solow–Swan model with continuous time is represented in the form of the single nonlinear ordinary differential equation:

$$k^{(1)}(t) \;= -(a+b)k(t) + \, pf(k(t)), \tag{89}$$

which describes how an increase of capital stock leads to an increase of per capita production, when the supply of labor changes as $L(t) = \; L_0\, exp\;(at)$ at a constant rate $a \in (-1,+1)$. Here $k(t) = K(t)/L(t)$ is the per capita capital; $K(t)$ is capital expenditure; $b \in (0,1)$ is the capital retirement ratio; $p \in (0,1)$ is the rate of accumulation. The function $f(k(t))$ describes the labor productivity, which is usually considered in the form $f(k(t)) = Ak^{\gamma}(t)$ with $\gamma \in (0,1)$.

The formal generalization, which is realized by replacing the first-order derivative by the fractional derivative in Equation (89), has the form:

$$\left(D^{\alpha}_{C,0+} k\right)(t) = -(a+b)k(t) + \, pf(k(t)), \tag{90}$$

where $D^{\alpha}_{C,0+}$ is the Caputo fractional derivative, for example.

Unfortunately, the consistent construction of the fractional generalization of the standard model Equation (89) cannot give a fractional differential equation in the form of Equation (90). In order to prove this statement, we first briefly describe the consistent construction of the equation for the standard Solow model.

3.3.1. Standard Solow Model with Continuous Time

The Solow model, which is also called the Solow–Swan model, is a dynamic single-sector model of economic growth (see, Solow and Swan articles [100–102], and books [103,104]). In this model, the economy is considered without structural subdivisions. The economy produces only universal products, which can be consumed both in the non-production and production sectors. As a universal product, one can consider a monetary value of the entire economy. Exports and imports are not taken into account. This model describes the capital accumulation, labor or population growth, and increases in productivity, which is commonly called the technological progress. The Solow model can be used to estimate the separate effects on economic growth of capital, labor and technological change.

The Solow model is a generalization of the Harrod–Domar model, which includes a productivity growth as new effect. This relatively simple growth models was independently proposed by Robert M. Solow and Trevor W. Swan in 1956 [100,101]. In 1987 Solow was awarded the Nobel Memorial Prize in Economic Sciences for his contributions to the theory of economic growth [105]. Mathematically, the Solow–Swan model is actually represented by one nonlinear ordinary differential equation (Equation (89), which describes the evolution of the per capita stock of capital. Now it is a classical nonlinear economic model that is actively used in economics [106–109].

In the Solow model, the state of the economy is given by the following five endogenous state variables (defined within the model): $Y(t)$ is the final product (production capacity), $L(t)$ is the labor input (available labor resources), $K(t)$ describes the capital reserves (capital expenditure, production assets), $I(t)$ is the investment (investment rates), and $C(t)$ is the amount of non-productive consumption (instant consumption). All variables are functions of time t, which is assumed to be continuous. In addition, the Solow model uses exogenous indicators (defined outside the model): $a \in (-1, +1)$ is the rate of increase in labor resources; $b \in (0, 1)$ is the capital retirement ratio; $p \in (0, 1)$ is the rate of accumulation (the share of the final product used for investment). These exogenous indicators are considered constant in time. The rate of accumulation is considered as a controlling parameter. It is assumed that the production and labor resources are fully used in the production of the final product. The final product at each moment in time is a function of the capital and labor: $Y = F(K(t), L(t))$. This production function $F\ (K,\ L)$ of the national economy is often specified to be a function of the Cobb–Douglas type. It is assumed that $Y = F(K, L)$ is a linearly homogeneous function satisfying the constant scale, i.e.:

$$F(zK,\ zL) = zF\ (K,\ L). \tag{91}$$

The final product is used for non-productive consumption and investment: $Y(t) = C(t) + I(t)$. The accumulation rate $p \in (0, 1)$ is the fraction of the final product used for investment, i.e., $I(t) = pY(t)$. Therefore, we have the multiplier equation $C(t) = (1 - p)\ Y(t)$.

If we assume that the increase in labor resources is proportional to the available labor resources, then taking into account the growth rate of employed $a \in (-1, +1)$, we can write the differential equation:

$$L^{(1)}(t) = aL(t), \tag{92}$$

where $L^{(1)}(t) = dL(t)/dt$ is the derivative of first order. Equation (92) with the initial condition $L(0) = L_0$, has the solution $L(t) = L_0\ exp\ (at)$, where L_0 is the labor resources at the beginning of observation at $t = 0$. The equation of labor resources can also be considered in the form of the logistic equation (for example, see [106]).

Capital stock may change for two reasons: investment causes an increase in capital stock; depreciation or disposal of capital causes a decrease in its reserves. If we assume that the retirement of capital occurs with a constant retirement rate of $b \in (0, 1)$, then the capital dynamics is described by the equation $K^{(1)}(t) = I(t) - bK(t)$. Finally, taking into account $I(t) = pY(t)$ and $Y = F(K(t), L(t))$, we obtain:

$$K^{(1)}(t) = pF(K(t), L(t)) - bK(t). \tag{93}$$

To obtain the equation of the standard Solow model, the following relative variables are introduced. The per capita capital (capital endowment) is defined as $k(t) = K(t)/L(t)$. The labor productivity is:

$$y(t) = Y(t)/L(t) = F(K(t), L(t))/L(t) = F(K(t)/L(t), 1) = f(k), \tag{94}$$

where we use the property (Equation (91)) of the linear homogeneity of the production function.

The dynamics of the output of the final product depends on the amount of the capital per employed person, the per capita capital $k(t) = K(t)/L(t)$.

Substitution of $K(t) = k(t)\, L(t)$ into Equation (93) gives:

$$(k(t)\, L(t))^{(1)} = pF(k(t)\, L(t), L(t)) - bk(t)\, L(t). \tag{95}$$

Using the standard product (Leibniz) rule:

$$(k(t)\, L(t))^{(1)} = k^{(1)}(t)\, L(t) + k(t)\, L^{(1)}(t), \tag{96}$$

and the property of the linearly homogeneity (Equation (91)), Equation (95) is rewritten in the form:

$$k^{(1)}(t)\, L(t) + k(t)\, L^{(1)}(t) = pf(k(t))\, L(t) - bk(t)\, L(t). \tag{97}$$

Using Equation (92) for the labor resources, we obtain:

$$k^{(1)}(t) = -(a+b)k(t) + pf(k(t)). \tag{98}$$

Equation (98) is the standard Solow–Swan model.

The behavior of the indicators of the standard Solow–Swan model is determined by the ordinary differential equation (Equation (98)) of the first order and the dynamics of labor resources (Equation (92)). The Cauchy problem, which consists of Equation (97) and an initial condition, has a unique solution.

3.3.2. Fractional Generalization of Solow Model

A fractional generalization of the labor resource Equation (92) and obtaining a solution to this fractional differential equation is not difficult. If we take into account this consistent derivation of Equation (98) of the standard model, we see that we cannot use the standard product (Leibniz) rule for fractional derivative. Therefore, we cannot obtain a fractional generalization of the differential Equation (98) for the per capita capital $k(t) = K(t)/L(t)$ because of a violation of the standard Leibniz rule for fractional derivatives of non-integer orders.

We emphasize that the violation of the standard product rule is a characteristic property of all derivatives of non-integer order. Note that the implementation of the standard product rule for an operator means that this operator is a differential operator of integer order [17], and such operators cannot describe the effects of memory and nonlocality.

As a result, the fractional generalization of the standard Solow–Swan model, which will take into account the power-law memory effects, should be represented as the system of the fractional differential equation:

$$\begin{cases} \left(D^{\alpha}_{C,0+}L\right)(t) = aL(t), \\ \left(D^{\beta}_{C,0+}K\right)(t) = pF(K(t), L(t)) - bK(t). \end{cases} \tag{99}$$

The fractional dynamics of the per capita capital $k(t)$ will be described as the ratio $K(t)/L(t)$ of solutions of these two fractional differential equations.

For production function of the national economy in the form the Cobb–Douglas function $F(K,\ L) = A\, K^{\gamma}(t)L^{1-\gamma}(t)$, we have the system (99) in the form:

$$\begin{cases} \left(D^{\alpha}_{C,0+}L\right)(t) = aL(t), \\ \left(D^{\beta}_{C,0+}K\right)(t) = pA\, K^{\gamma}(t)L^{1-\gamma}(t) - bK(t). \end{cases} \tag{100}$$

The fractional differential equation with $n-1 < \alpha \leq n$, which describes the labor resources, has the solution (theorem 5.15 of [4], p. 323) in the form:

$$L(t) = \sum_{k=0}^{n-1} L^{(k)}(0)t^k E_{\alpha,k+1}[a\, t^{\alpha}] \tag{101}$$

where $L^{(k)}(0)$ is integer-order derivatives of orders $k \geq 0$ at $t = 0$, and $E_{\alpha,k+1}[a\, t^{\alpha}]$ is the two-parameter Mittag–Leffler function [32]. In the case $0 < \alpha \leq 1$ ($n = 1$) Equation (101) takes the form:

$$L(t) = L(0)E_{\alpha,1}[\lambda\, t^{\alpha}]. \tag{102}$$

For $\alpha = 1$, we obtain the standard solution $L(t) = L_0\, exp\,(at)$, where $L_0 = L(0)$.

Using Equation (102) that describes fractional dynamics of the labor resources, we can obtain the nonlinear fractional differential equation for the capital expenditure $K(t)$ in the form:

$$\left(D^{\beta}_{C,0+}K\right)(t) = pA\, K^{\gamma}(t)L_0^{1-\gamma}\, (\mathrm{E}_{\alpha,1}[\lambda\, \mathrm{t}^{\alpha}])^{1-\gamma} - bK(t). \tag{103}$$

In the case $\alpha = 1$, this equation takes the form:

$$\left(D^{\beta}_{C,0+}K\right)(t) = pA\, K^{\gamma}(t)L_0^{1-\gamma}\, e^{a(1-\gamma)t} - bK(t). \tag{104}$$

The question of the existence of solutions of nonlinear fractional differential Equations (103) and (104) and computer modeling of capital expenditure dynamics remains open at the present time.

Note that the nonlinear fractional differential Equations (103) and (104) can be represented as Volterra integral equations by using the results of the papers of Kilbas and Marzan [110,111]. In the space $C^r[0,T]$ of continuously differentiable function the Cauchy problem for fractional differential equation:

$$\left(D^{\beta}_{C,0+}K\right)(t) = G(t, K(t)), \tag{105}$$

where $n - 1 < \beta \leq n$, is equivalent (see Theorem 3.24 of [4], pp.199–202, to the Volterra integral equation:

$$K(t) = \sum_{m=0}^{n-1} \frac{K^{(m)}(0)}{m!}\, t^m + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\beta-1} G(\tau, K(\tau))d\tau, \tag{106}$$

if the function $G(t, K(t)) \in C_{\gamma}[0,T]$ with $0 \leq \gamma < 1$ and $\gamma \leq \beta$, the variable $K(t) \in C^r[0,T]$, where $r = n$ for integer values of β ($\beta \in \mathbb{N}$) and $r = n - 1$ for non-integer values of β ($\beta \notin \mathbb{N}$).

At the same time, Equation (90), which is a formal fractional generalization of the equation (Equation (89)) of the standard model, does not have economic significance due to the violation of the principle of derivability.

3.4. Example of Problem with Semi-Group Rule of Dynamic Map: Dynamic Leontief Model and Logistic Growth Model

In this subsection, we consider fractional generalizations of the standard dynamic Leontief model and logistic growth model to demonstrate that the violation of the standard semi-group rule of dynamic map for fractional derivatives creates a restriction in the construction of such generalizations.

3.4.1. Dynamic Leontief (Intersectoral) Model

One of the famous multidimensional economic models is the dynamic intersectoral model that was proposed Wassily W. Leontief [112,113] in 1951. The Royal Swedish Academy of Sciences has awarded the 1973 year's Prize in Economic Science in Memory of Alfred Nobel to W.W. Leontief for "the development of the input-output method and for its application to important economic problems" [114]. The Leontief dynamic model is an economic model of growth of gross national product and national income [115,116].

The fractional generalization of the dynamic Leontief (intersectoral) model was proposed in [117,118] in 2017 and in the works [119,120] for the case of time-dependent direct material costs and the incremental capital intensity of production.

Let us give the first example from the econophysics approach based on [117,118], and [119,120]. The fractional generalization of the equation for the dynamic Leontief (intersectoral) model [92,93] has the form:

$$\left(D^{\alpha}_{t_0+}X\right)(t) = H(t)\, X(t), \tag{107}$$

where the vector $X(t) = (X_k(t))$ describes the gross product (gross output) in monetary terms, where $k = 1, \ldots, n$ are production sectors; the matrix $A = \left(a_{ij}\right)$ describes the direct material costs; the matrix $B = \left(b_{ij}\right)$ describes the incremental capital intensity of production; the matrix E is the unit diagonal matrix of n-th order; the matrix H is defined by the equation $H = B^{-1}(E - A)$.

Equation (107) describes dynamics of the sectoral structure of the gross products in the closed dynamic intersectoral model with power-law memory (for details, see [117,118], and [119,120]). The solution of Equation (107) with constant operator $H(t) = H = const$ has the form:

$$X(t) = U_{\alpha}(t)\, X(0), \tag{108}$$

where the operator $U_{\alpha}(t)$ is defined through the Mittag–Leffler function with matrix arguments by the equation $U_{\alpha}(t) = E_{\alpha}[t^{\alpha}\mathrm{H}]$. Therefore, for the operator $U_{\alpha}(t)$, which describes the dynamic map with power-law memory, we have the inequality:

$$U_{\alpha}(t_1)U_{\alpha}(t_2) \neq U_{\alpha}(t_1 + t_2), \tag{109}$$

which means the violation of the standard semi-group rule for non-integer values of α (for example, $0 < \alpha < 1$).

For the general case of the time-dependent matrix $H(t)$, the solutions of Equation (110) are given in [119,120]. To obtain these solutions, we proposed new concepts of the memory-ordered exponential and memory-ordered product, which are a generalization to processes with memory of such well-known concepts in quantum physics as time-ordered exponential (T-exponential) and time-ordered product (T-product) [121,122].

3.4.2. Logistic Growth with Memory

The second example is taken from the economic model of logistic growth [104,123]. In economic growth models, the competition effects are taken into account by assuming that price is a function of the value of output. Model of natural growth in a competitive environment is often called a model of logistic growth. The variables of this model are the function $Y(t)$ that describes the value of output at time t; the price $P(t)$ is considered as a function of released product $Y(t)$, i.e., $P = P(Y(t))$. It is often assumed that this function is linear, i.e., $P(Y(t)) = b–a\ Y(t)$, where b is the price, which is independent of the output and the parameter a is the margin price. In addition, it is assumed that all manufactured products are sold (the assumption of market unsaturation). The equation of this model is the differential equation of the first order in the form:

$$\frac{dY(t)}{dt} = \frac{m}{v}\,(b - a\,Y(t))\,Y(t), \tag{110}$$

where $v > 0$ is the accelerator coefficient, $1/v$ is the marginal productivity of capital (rate of acceleration), m is the norm of net investment $(0 < m < 1)$ that describes the share of income, which is spent on the net investment.

If $a \neq 0$ and $b \neq 0$, we can use the variable $Z(t)$ and the parameter r, which are defined by the equations $Z(t) = \frac{a}{b}Y(t)$, and $r = \frac{m}{v}$. Then Equation (110) of the logistic growth model is represented in the form:

$$\frac{dZ(t)}{dt} = r\,(1 - Z(t))\,Z(t), \tag{111}$$

The fractional generalization of the logistic growth model with power-law memory [123] gives the fractional differential equation:

$$\left(D^{\alpha}_{C,0+}Z\right)(t) = r\,(1 - Z(t))\,Z(t), \tag{112}$$

where $D^{\alpha}_{C;0+}$ is the Caputo fractional derivative. Equation (112) is the logistics fractional differential equation.

The solution of nonlinear fractional differential equations is a difficult problem. Recently, Bruce J. West has published the paper [124], where he proposed an analytical expression of the solution for the fractional logistic equation with $\alpha \in (0,1)$ in the form:

$$Z(t) = \sum_{k=0}^{\infty}\left(\frac{Z(0)-1}{Z(0)}\right) E_{\alpha}[k\,r(\alpha)\,t^{\alpha}], \tag{113}$$

where $E_{\alpha}(z)$ is the Mittag–Leffler function [4], p. 42. As it has been proved by I. Area, J. Losada, J. Nieto in [125], the function (113) is not the solution to (11). The main reason is the violation the semi-group property by the Mittag–Leffler function, i.e., we have (for example, see [33,34], and [35–37]) the inequality:

$$E_{\alpha}\Big[\lambda\,(t+s)^{\alpha}\Big] \neq E_{\alpha}[\lambda\,t^{\alpha}]\,E_{\alpha}[\lambda\,s^{\alpha}] \tag{114}$$

for $\alpha \in (0,1)$, and real constant λ. In [125] it has been proved that Equation (113), which is proposed in [124], is not an exact solution of the fractional logistic Equation (112).

As a result, we see that the violation of the standard semi-group property for the dynamic map us an important property of processes with memory that should be taken into account in dynamic models. Neglect of this non-standard property of the dynamical map can lead to errors.

3.4.3. Principle of Optimality for Processes with Memory

The principle of optimality, which was originally proposed for dynamic programming by Belmann, is very important for describing economic processes. The Bellman principle of optimality states that any tail of an optimal trajectory is optimal too.

In considering optimal growth trajectories of economy, a concept known as the optimality principle is very useful. Let us give the standard principle of optimality that describes processes without memory (for example, see section 11.2 of [92]):

Principle of Optimality. *Any optimal behavior has the property that whatever the initial state and corresponding (initial) solution are, the subsequent solutions must constitute the optimal behavior with regards to the state resulting from the initial solution.*

Applied to economic growth theories, the optimality principle leads to the following conclusion. If the trajectory is optimal, starts from point $X(0)$ and passes through $X(t)$ on the way to the end point $X(T)$, then part of the trajectory from $X(t)$ to $X(T)$ will be optimal with respect to the initial point $X(t)$.

The implementation of the principle of optimality is based on the semi-group rule of dynamic map. The violation of the standard semi-group rule of dynamic map for dynamics with memory leads to violation of the standard principle of optimality.

Mathematically, the violation of the standard optimality principle is represented by the violation of the semi-group rule of dynamic map.

Economically the reason for the violation of the standard principle of optimality is the cutting off of part of the history of this process (that is, starting from the beginning of this process, but at a different time point). In other words, if you put in place of the general director, whose age is 40 years old, his same age 15 years, the company will develop differently.

As a result, we can formulate the following statement:

Principle of Optimality in Processes with Memory. *For economic processes with memory, any optimal behavior has the property that whatever the initial state and corresponding (initial) solution are, the subsequent solutions cannot constitute the optimal behavior with regards to the state resulting from the initial solution. We can state that if there is no violation of the standard principle of optimality, then there is no memory in the process "No Violation of Optimality Principle. No Memory".*

For economic growth models, the suggested optimality principle in processes with memory leads to the following statement. If the trajectory is optimal, starts from point $X(0)$ and passes through $X(t)$ on the way to the end point $X(T)$, then part of the trajectory from $X(t)$ to $X(T)$ cannot be optimal with respect to the initial point $X(t)$ in general.

This principle actually means that the implementation of the standard optimality principle for economic processes with memory in the general case is equivalent to the lack or absence of memory in this process.

3.5. Generalizations of Economic Notions and Concepts

Derivability Principle states that it is not enough to get a fractional generalization of the differential equations of economic model. It is necessary to generalize the whole scheme (all steps) of obtaining these equations from the basic principles, concepts and assumptions that is used in economic theory for standard model. In this sequential derivation of the equations, we should take into account the non-standard characteristic properties of fractional derivatives and integrals. Another important requirement of the derivability principle is the need to generalize economic the notions, concepts and methods, which were used in the derivation of standard model.

It should be noted that formal replacements of derivatives of integer order by fractional derivatives in standard differential equations, and then solutions of these fractional differential equations cannot be considered as a correct and self-consistent fractional generalization of the standard dynamic models in different sciences.

A very important part of the fractional generalization of dynamic models is the inclusion of memory and non-locality into the economic theory and into the basic economic concepts and methods. A fractional generalizations of basic economic concepts and notions are not so much a part of this particular economic model, but in fact are the common basis of different models, and basis of fractional mathematical economics, and not just an economic model.

The concept of memory for economics is considered in [47–50] and [126–131]. The fractional dynamic models should be constructed on this conceptual basis. The most important task of studies of such fractional generalizations is also the search for qualitatively new effects and phenomena caused by memory and non-locality in the behavior of processes.

Let us give a list of some standard notions of economic theory, the generalization of which were proposed to describe the processes with memory and non-locality in the last years.

The list of these new notions and concepts primarily include the following:

- the marginal value of non-integer order [132–134], (see also [40,41]) with memory and nonlocality;
- the multiplier with memory [60,135–139];
- the accelerator with memory [60,135–139] (see also [140,141]);
- the duality of the multiplier with memory and the accelerator with memory [60,135];
- the elasticity of fractional order [93–96];
- the measures of risk aversion with memory [142] and non-locality [143];
- the warranted (technological) rate of growth with memory [144–146]; and
- the non-local fractional deterministic factor analysis [147,148], and other.

The use of these notions and concepts makes it possible us to construct fractional generalizations of some economic models. A brief description of the history of the use of fractional calculus in economics is proposed as a separate article [149].

4. Example of Application of the Solvability and Correspondence Principles

The Solvability Principle assumes the existence of solution, and the possibility of obtaining an exact analytical solution or a correct numerical solution for some conditions. Obviously, these conditions for the existence of solutions should allow us to describe the processes considered in natural and social sciences.

The Correspondence Principle assumes that in the limit cases of integer orders the solution (and equation) should exist and the expression of this solution (and equation) should give expression of the standard solution. The principle of correspondence must be performed both for the fractional differential equation itself and for its solution.

4.1. Solvability Principle: Example from General Fractional Calculus

A general concept of fractional calculus was proposed by Anatoly N. Kochubei [150] on the basis of the differential-convolution operator. The general fractional calculus is described in the works [150,151], where author describes the conditions under which the general operator has a right inverse (a kind of a fractional integral) and produce, as a kind of fractional derivative, equations. A solution of the relaxation equations with the Kochubei fractional derivative with respect to the time variable is described. As a special case of the general fractional operators, the fractional derivatives and integrals of distributed order are considered in [150,151].

In the works about the general fractional calculus [150,151] the Cauchy problem (A) is considered for Equation $(D_{(k)}X)(t) = \lambda X(t)$, where $\lambda < 0$ (see [151], p. 112). In Section 6 "Relaxation equations", Theorem 4 states that this Cauchy problem has a solution $X(\lambda, t)$, which is continuous on $\mathbb{R}_+$, infinitely differentiable and completely monotone on $\mathbb{R}_+$, if the Kochubei conditions (*) hold. The works [150,151] consider only the case of relaxation, i.e., $\lambda < 0$. The case of growth ($\lambda > 0$) is not discussed.

In the economics, different growth models are actively studied. In the simplified form, these growth models can be described by the ordinary differential equation $D^1_t X(t) = \lambda X(t)$, where $\lambda > 0$. The fractional generalization of these models, in which the memory function $k(t)$ is taken into account, can be described by the Equation $(D_{(k)}X)(t) = \lambda X(t)$ with $\lambda > 0$, i.e., "relaxation equations" is replaced by "growth equations".

It is known that for the Caputo fractional derivative, which is a special case of the Kochubei fractional derivatives, the Cauchy problem (A) has a solution $X(\lambda, t)$, for all real $\lambda \in \mathbb{R}$, i.e., for $\lambda < 0$ and $\lambda > 0$ (see theorem 4.3 in [4], p. 231).

Therefore, the following questions, which are important for describing processes with memory in economics, arise within the framework of general fractional calculus.

1. Is there a mathematical reason for using only the condition $\lambda < 0$ in general calculus, when the Caputo fractional derivative there is no such restriction?
2. Could we tell something under what conditions on the memory function, which is described by the kernel $k(t)$ of the general fractional derivative, the solution exists for $\lambda > 0$?
3. Is it possible to specify a wider class of operators than the fractional Caputo derivative for the existence of solutions of growth equations?
4. Do the conditions of existence of solutions for the general relaxation equation and the general growth equation coincide?
5. What types of asymptotic behavior of solutions of general growth equations and type of growth rates exist?

The growth equation is considered in [150] for the special case of a distributed order derivative (see also [152]). In this paper it was proved that a smooth solution exists and is non-decreasing belongs to $C^{\infty}(0, \infty)$. To understand the warranted (technological) growth rate of the economy, it is important to know the asymptotic behavior of this solution. The description of the asymptotic behavior of such solutions is an open question at the moment. This complicates the economic interpretation of solutions and, thus, prevents the implementation of the interpretability principle, when writing works on the economics of processes with distributed memory fading parameter.

In addition, the solution of the growth equation has been proposed for the case of fractional differential operators with distributed lag in [46,53,54]. This case will be briefly described in the next subsection.

The existence of a solution in the growth case has been also considered by Chung-Sik Sin [153] in 2018 for a much more general case of a nonlinear equation with a generalized derivative like the Kochubei fractional derivative.

The solution of the Cauchy problem for general growth equation is an open question at the present moment. The growth case of general fractional calculus was discussed by Kochubei and Kondratiev [154] as a part of the intermittency property in fractional models of statistical mechanics. Unfortunately, the results were not formulated separately for the general fractional growth equation in [154].

Solving the existence problem in the general case will allow us to accurately describe the conditions on the operator kernels (the memory functions), under which equations for models of economic growth with memory have solutions. The asymptotic behavior of these solutions allows us to describe the warranted (technological) growth rate in the economy, in which we take into account this type of memory. An article dedicated to solving this mathematical problem was written by Anatoly N. Kochubei and Yuri Kondratiev [155] in 2019 for Special Issue "Mathematical Economics: Application of Fractional Calculus" of Mathematics. The application of these mathematical results in economics and their economic interpretation is an open question at the moment.

4.2. Distributed Lag Fractional Calculus: Growth-Relaxation Equations

The fractional calculus with continuously distributed lag is proposed in [46]. In the papers [46,53–55], we consider an application of this fractional calculus to describe economic growth with power-law memory and distributed lag.

Let us consider the fractional integration with the gamma distributed lag that is defined by the equation [46,53–55] in the form:

$$\left(\boldsymbol{I}^{\lambda,a;\alpha}_{T;RL;0+}Y\right)(t) = \left(M^{\lambda,a}_{T}(\tau) * \left(I^{\alpha}_{RL,0+}Y\right)\right)(t) = \int_0^t M^{\lambda,a}_{T}(\tau)\left(I^{\alpha}_{RL,0+}Y\right)(t-\tau)\, d\tau, \tag{115}$$

where $M^{\lambda,a}_{T}(\tau)$ is the probability density function of the gamma distribution:

$$M^{\lambda,a}_{T}(\tau) = \begin{cases} \frac{\lambda^a\ \tau^{a-1}}{\Gamma(a)}\, exp(-\lambda\ \tau) & if\ \tau > 0 \\ 0 & if\ \tau \leq 0 \end{cases} \tag{116}$$

with the shape parameter $a > 0$ and the rate parameter $\lambda > 0$. If $a = 0$, Equation (116) describes the exponential distribution. The function $M^{\lambda,a}_{T}(\tau)$ describes the distribution of the delay time τ, which is considered as a random variable.

In the papers [46,53,54], we prove that the Riemann–Liouville fractional integral with gamma distribution of delay time can represented [40] by the equation:

$$\left(\boldsymbol{I}^{\lambda,a;\alpha}_{T;RL;0+}Y\right)(t) = \frac{\lambda^a\,\Gamma(a)}{\Gamma(a+\alpha)} \int_0^t (t-\tau)^{\alpha+a-1} F_{1,1}(a; a+\alpha; -\lambda(t-\tau))\, Y(\tau)\, d\tau, \tag{117}$$

where $F_{1,1}(a;c;z)$ is the confluent hypergeometric Kummer function, $\alpha > 0$ is the order of integration and the parameters $a > 0$, $\lambda > 0$ describe the shape and rate of the gamma distribution, respectively. Note that the kernel of Equation (117) can be represented through the three parameter Mittag–Leffler function instead of the confluent hypergeometric Kummer function [53,54].

The fractional integral (Equation (117)) is the Abel-type fractional integral operator with Kummer function in the kernel (see equation (37.1) in [1], p. 731, and [53]). This kernel can be considered as a new memory function.

Note that the fractional integral (Equation (117)) with gamma distributed lag (Equation (116)) can be represented as the series of the Riemann–Liouville fractional integrals:

$$\left(\boldsymbol{I}^{\lambda,a;\alpha}_{T;RL;0+}Y\right)(t) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(k+1)}(-1)^k \lambda^{k+a}\left(I^{\alpha+a+k}_{RL,0+}Y\right)(t). \tag{118}$$

The Caputo fractional differential operator with gamma distributed lag can be expressed through the Riemann–Liouville fractional integral operator in the form:

$$\left(\boldsymbol{D}^{\lambda,a;\alpha}_{T;C;0+}Y\right)(t) = \left(M^{\lambda,a}_{T}(\tau) * \left(D^{\alpha}_{C,0+}Y\right)\right)(t) = \left(\boldsymbol{I}^{\lambda,a;n-\alpha}_{T;RL;0+}Y^{(n)}\right)(t), \tag{119}$$

where $n-1 < \alpha \leq n$, the parameters $a > 0$ and $\lambda > 0$ describe the shape and rate of the gamma distribution of delay time, respectively.

Let us consider the growth-relaxation equation with the fractional operator (Equation (119)). In the works [46,53,54], for the fractional differential equation:

$$\left(\boldsymbol{D}^{\lambda,a;\alpha}_{T;C;0+}Y\right)(t) = \omega Y(t), \tag{120}$$

we proposed has the solution:

$$Y(t) = \sum_{j=0}^{n-1} S^{\alpha-j-1}_{\alpha,a}[\omega\, \lambda^{-a}, \lambda|t]\, Y^{(j)}(0), \tag{121}$$

where $n = [\alpha]+1$, and $S^{\gamma}_{\alpha,\delta}\,[\mu,\lambda\big|t]$ is the special function that is defined by the expression:

$$S^{\gamma}_{\alpha,\delta}\,[\mu,\lambda\big|t] = -\sum_{k=0}^{\infty} \frac{t^{\delta(k+1)-\alpha k-\gamma-1}}{\mu^{k+1}\Gamma(\delta(k+1)-\alpha k-\gamma)}\, F_{1,1}(\delta(k+1);\delta(k+1)-\alpha k-\gamma,-\lambda t), \tag{122}$$

where $F_{1,1}(a;b;z)$ is the confluent hypergeometric Kummer function.

To prove this statement we can use the Laplace transform method [46]. Using the Laplace transform of Equation (120), we obtain:

$$\frac{\lambda^a}{(s+\lambda)^a}\left(s^{\alpha}(\mathcal{L}Y)(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1}Y^{(j)}(0)\right) = \omega(\mathcal{L}Y)(s). \tag{123}$$

Then we can write:

$$(\mathcal{L}\,Y)(s) = \sum_{j=0}^{n-1} \frac{s^{\alpha-j-1}}{s^{\alpha}-\mu(s+\lambda)^a}Y^{(j)}(0), \tag{124}$$

where $\mu = \omega\, \lambda^{-a}$. Using equation (5.4.9) of [156] and [157] in the form:

$$\left(\mathcal{L}^{-1}\left(\frac{s^a}{(s+b)^c}\right)\right)(s) = \frac{1}{\Gamma(c-a)}t^{c-a-1}F_{1,1}(c;c-a,-bt), \tag{125}$$

where $Re(\mathrm{c}-\mathrm{a}) > 0$, we obtain [46,53,54] the inverse Laplace transform:

$$\left(\mathcal{L}^{-1}\left(\frac{s^{\gamma}}{s^{\alpha}-\mu(s+\lambda)^{\delta}}\right)\right)(s) = S^{\gamma}_{\alpha,\delta}\,[\mu,\lambda\big|t]. \tag{126}$$

Then using equality (126), the solution of Equation (120) takes the form of Equation (121).

Note that we can use Equation (1.9.13) [4], p. 47 (see also [32,158,159]) in the form:

$$\left(\mathcal{L}\left(t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha})\right)\right)(\mathrm{s}) = \frac{\mathrm{s}^{\alpha\gamma-\beta}}{(\mathrm{s}^{\alpha}-\lambda)^{\gamma}}, \tag{127}$$

where $\lambda \in \mathbb{C}$ $Re(s) > 0$, $Re(\beta) > 0$ and $|s|^{\alpha} > |\lambda|$, instead of Equation (126) to get the representation of Equation (121) though three parameter Mittag–Leffler functions instead of the confluent hypergeometric functions.

Equation (1.9.3) of [4], p. 45, in the form:

$$E^{\gamma}_{1,\beta}(z) = \frac{1}{\Gamma(\beta)} F_{11}(\gamma, \beta; z), \tag{128}$$

where $E^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha})$ is the three parameter Mittag–Leffler function [31] allows us to represent of the fractional differential Equation (120) with $0 < \alpha < 1$ in the form:

$$Y_0(t) = -\sum_{k=1}^{\infty} \omega^{-k}\, \lambda^{ak} t^{(\alpha+a)k} E^{ak}_{1,(\alpha+a)k+1}(-\lambda t)\; Y(0). \tag{129}$$

Note that the solution of homogeneous fractional differential equation that describes economic growth with memory in absence of time delay (lag) is expressed thought the two parameter Mittag–Leffler function, where the argument depends on $\omega\, t^{\alpha}$ [144–146], instead of the rate parameter $\lambda > 0$ of gamma distribution.

4.3. Correspondence Principle: Order of Derivative Tends to Integer Value

Let us give an example to illustrate that if the order of the fractional derivative tends to the integer value, then the limit on the left and the limit on the right can give different results in the general case.

The fractional differential equation:

$$\left(D^{\alpha}_{C,0+}Y\right)(t) = \lambda\, Y(t) \tag{130}$$

has the following solution. If $\alpha \in (0,1)$ the solution takes the form:

$$Y_1(t) = Y(0) E_{\alpha,1}[\lambda\, t^{\alpha}]. \tag{131}$$

For $\alpha \in (1,2)$ Equation (130) has the solution:

$$Y_2(t) = Y(0)\, E_{\alpha,1}[\lambda\, t^{\alpha}] + Y^{(1)}(0)\, t\, E_{\alpha,2}[\lambda\, t^{\alpha}], \tag{132}$$

where $Y^{(1)}(0)$ is first-order derivative of $Y(t)$ at $t = 0$.

Using the equalities 1.8.2, 1.8.18, 1.8.19 of the book [4] in the form:

$$E_{1,1}[z] = E_1[z] = \exp(z), \tag{133}$$

$$E_{1,2}[z] = E_1[z] = \frac{1}{z}(\exp(z) - 1), \tag{134}$$

we obtain:

$$\lim_{t\to 1-} Y_1(t) = Y(0) E_{1,1}[\lambda\, t] = Y(0) \exp(\lambda\, t), \tag{135}$$

$$\lim_{t\to 1+} Y_2(t) = Y(0) \exp(\lambda\, t) + \frac{1}{\lambda} Y^{(1)}(0)(\exp(\lambda\, t) - 1). \tag{136}$$

The correct solution of growth Equation (130) for the case $\alpha = 1$ is Equation (135). We can see that in the limit the solutions (135) and (136) of the growth equation (Equation (130)) coincide only if the derivative $Y^{(1)}(0)$ is equal to zero.

4.4. Solvability Principle: Examples from Numerical Simulation

Computer simulation of processes with memory and non-locality should use such methods for the numerical solution of fractional differential equations with derivatives of non-integer order, which take into account non-locality and memory. Numerical approximation should not use only local information. Numerical scheme should contain a term of the memory (or non-locality). Numerical methods that neglect non-locality and memory are not reliable and often lead to incorrect results, since the non-local nature of fractional differential operators with non-integer orders cannot be neglected. Examples of such errors are given in the work of Roberto Garrappa [160].

5. "Non-Equivalence" and "Unpredictability" of Fractional Generalization

5.1. Equivalence of Equations by Solutions (s-Equivalence)

Differential equations will be called equivalent by solution (s-equivalent) if these equations have the same solutions for a sufficiently wide class of functions and initial conditions [29,161,162]. Let us give some detalizations of this notion.

Let us consider two differential equations:

$$E_1\Big[x, f(x), f^{(1)}(x), \ldots, f^{(n)}(x)\Big] = 0, \tag{137}$$

$$E_2\Big[x, u(x), u^{(1)}(x), \ldots, u^{(n)}(x)\Big] = 0. \tag{138}$$

Differential Equations (137) and (138) will be called equivalent by solution (s-equivalent) if there exists a certain function $g : u(x) = g(f(x))$ such that the solution of Equation (138), which is expressed through the function $f(x)$, coincides with the solution of Equation (137).

For simplicity, we consider the first-order ordinary differential equation:

$$\frac{df(x)}{dx} = E_1[f(x), x, \lambda], \ and \ f(0) = C_1 \ (x \in \mathbb{R}_+), \tag{139}$$

where λ denotes a set of parameters, and $E_1[f(x), x, \lambda]$ is such that Equation (139) has a unique solution for $x \geq 0$ or $x \in \mathbb{R}$. The solution of Equation (139) will be denoted as:

$$f(x) = S_1(x, \lambda, C_1) \tag{140}$$

with the initial condition:

$$S_1(0, \lambda, C_1) = C_1. \tag{141}$$

Let us consider the second differential equation:

$$\frac{du(x)}{dx} = E_2[u(x), x, \lambda], \ and \ u(0) = C_2 \ (x \in \mathbb{R}_+), \tag{142}$$

where $E_2[u(x), x, \lambda]$ is such that Equation (142) has a unique solution for $x \geq 0$ or $x \in \mathbb{R}$. The solution of Equation (142) will be denoted as:

$$u(x) = S_2(x, \lambda, C_2) \tag{143}$$

with the condition:

$$S_2(0, \lambda, C_2) = C_2. \tag{144}$$

Let us give the concept an equivalence of the solutions of these two differential equations.

Definition 1. *Equations (139) and (142) are called equivalent by solution (s-equivalent) if there exists a function* $g : u(x) = g(f(x))$ *such that:*

$$S_2(x, \lambda, C_2) = g(S_1(x, \lambda, C_1)), \tag{145}$$

and $C_2 = g(c_1)$ *for a sufficiently wide class of functions and initial conditions.*

An s-equivalence can be considered as a map of differential equation into another differential equation such that the solutions of these equations are also transformed by the same map. In the next subsections we give simple examples to illustrate this concept.

5.2. Relaxation and Growth Differential Equations

Let us consider the "relaxation" differential equation:

$$\frac{df(x)}{dx} = -\lambda\, f(x)\ (x \in \mathbb{R}_+) \tag{146}$$

with $\lambda > 0$. The solution of Equation (146) has the form:

$$f(x) = f(0)\, \exp\,(-\lambda\, x). \tag{147}$$

Let us consider also the "growth" differential equation:

$$\frac{du(x)}{dx} = \lambda\, u(x)\ (x \in \mathbb{R}_+) \tag{148}$$

with $\lambda > 0$. The solution has the form:

$$u(x) = u(0)\, \exp\,(\lambda\, x). \tag{149}$$

We can consider the function $u(x) = g(ufx))$ in the form:

$$u(x) = \frac{1}{f(x)}. \tag{150}$$

In this case, using Equations (146) and (150), we can get Equation (148):

$$\frac{du(x)}{dx} = \frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f^{(1)}(x)}{f^2(x)} = \frac{\lambda\, f(x)}{f^2(x)} = \lambda \frac{1}{f(x)} = \lambda\, u(x), \tag{151}$$

and the solution (149) can be obtained from (147) by the obvious transformations:

$$u(x) = \frac{1}{f(x)} = \frac{1}{f(0)\ \exp\,(-\lambda\, x)} = u(0)\frac{1}{\exp\,(-\lambda\, x)} = u(0)\exp\,(\lambda\, x). \tag{152}$$

Let us consider the "relaxation" fractional differential equation:

$$\left(D^{\alpha}_{C,0+} f\right)(t) = -\lambda\, f(x)\ (x \in \mathbb{R}_+) \tag{153}$$

with $\lambda > 0$, where $D^{\alpha}_{C,0+}$ is the Caputo fractional derivative of order $0 < \alpha < 1$. The solution has the form:

$$f(x) = f(0)\, E_{\alpha}[-\lambda\, x^{\alpha}], \tag{154}$$

where $E_{\alpha}[t^{\alpha} \mathrm{A}]$ is the Mittag–Leffler function [32]. Let us consider the "growth" differential equation:

$$\left(D^{\alpha}_{C,0+} u\right)(x) = \lambda\, u(x)\ (x \in \mathbb{R}_+) \tag{155}$$

with $\lambda > 0$. The solution has the form:

$$u(x) = u(0)E_{\alpha}[\lambda\, x^{\alpha}]. \tag{156}$$

These fractional differential equations are not equivalent:

$$u(x) = \frac{1}{f(x)} = \frac{1}{f(0)\, E_{\alpha}[-\lambda\, x^{\alpha}]} = u(0)\frac{1}{E_{\alpha}[-\lambda\, x^{\alpha}]} \neq \mathrm{u}(0)E_{\alpha}[\lambda x \alpha]. \tag{157}$$

As a result, the solutions of fractional differential Equation (155) with $\lambda > 0$ and $\lambda < 0$ cannot be considered as s-equivalent equations.

The fact of violation of s-equivalence is caused by the violation of the standard chain rule, which is used in (151), for fractional derivatives of non-integer orders. As a result, the derivative $(D^{\alpha}_{C,0+} f^{-1})(x)$ cannot be represented through $(D^{\alpha}_{C,0+} f)(x)$ in the simple form. From the point of view of solutions, this nonequivalence is caused by the properties of the Mittag–Leffler functions and the violation of group (and semi-group) property of dynamic maps. This allows us to formulate the following statement:

Principle "Violation of s-Equivalence by Fractional Generalization". *The s-equivalence property of differential equations of integer order is violated by formal fractional generalization of these equations. As a result, the equivalence of dynamic models is violated by the fractional dynamic generalization.*

Another example is given in the next subsection.

5.3. Fractional Logistic Equation: Growth in Competitive Environment with Memory

The logistic differential equation can be derived from economic model of natural growth in a competitive environment. This model is described in Section 3.4.2 of this paper. Differential equation that describes logistic growth in competitive environment without memory (110) has the form:

$$\frac{dY(t)}{dt} = \frac{m}{v}\,(b - a\, Y(t))Y(t). \tag{158}$$

Equation (158) is the logistic differential equation, i.e., the ordinary differential equation of first order that describes the logistic growth. For $a = 0$, Equation (158) describes the natural growth in the absence of competition. If $a \neq 0$ and $b \neq 0$, we can use the variable $f(t)$ and the parameter μ that are defined by the expressions:

$$f(t) = \frac{a}{b}\, Y(t) \text{ and } \mu = \frac{m}{v}. \tag{159}$$

Then Equation (158) of the logistic growth is represented in the form:

$$\frac{df(t)}{dt} = \mu\, f(t)(1 - f(t)). \tag{160}$$

This is the standard logistics differential equation. The solution of this logistic equation has the form:

$$f(t) = \frac{f(0)}{f(0) + (1 - f(0))\exp(-\mu\, t)} = \frac{f(0)\exp(\mu\, t)}{1 + f(0)\,(\exp(\mu\, t) - 1)}. \tag{161}$$

Using the variable:

$$u(t) = \frac{1}{f(t)}, \tag{162}$$

Equation (160) can be represented as:

$$\frac{du(t)}{dt} = \mu \, (1 - u(t)). \tag{163}$$

Equations (160) and (163) are integer-order differential equations that are s-equivalent.

Let us consider the fractional generalization of these equations, which are represented in the form:

$$\left(D^{\alpha}_{C,0+} f\right)(t) = \mu \, f(t)(1 - f(t)), \tag{164}$$

$$\left(D^{\alpha}_{C,0+} u\right)(x) = \mu \, (1 - u(t)).) \tag{165}$$

Equations (164) and (165) cannot be considered as s-equivalent equations. It is well known that the analytical expression for solution of fractional differential Equation (164) for the function $f(t)$ is still unknown at the moment. The solution of equation of linear Equation (165) for the function $u(t)$ is given in the form:

$$u(t) = 1 + (u(0) - 1) E_{\alpha}[-\mu \, t^{\alpha}], \tag{166}$$

where $\mu > 0$. Therefore, we obtain:

$$f(t) = \frac{1}{u(t)} = \frac{f(0)}{f(0) + (1 - f(0)) \, E_{\alpha}[-\mu \, t^{\alpha}]} \, . \tag{167}$$

For $\alpha = 1$, using that $E_1[-\mu \, t] = exp(-\mu \, t)$, Equation (167) is the standard solution (161).

As a result, we can see that Equations (164) and (165) have different solutions and the fractional generalization violates the s-equivalence of differential equations. The fractional generalizations of equivalent models can give non-equivalent fractional dynamic models.

5.4. Fractional Generalization Generates Nonequivalent Models

In Sections 6.2 and 6.3, we prove that fractional generalizations of equivalent representations of standard dynamic models, which are described by s-equivalent differential equations, as a rule, lead to different fractional dynamic models that have non-equivalent solutions. This, in a sense, is analogous to the situation in quantum theory when quantization of equivalent classical models leads to nonequivalent quantum theories.

As a result, we can formulate the following statement:

Principle "Non-Equivalence of Equivalent". *Fractional generalizations of s-equivalent differential equations of integer order are not equivalent in general.*

This property of fractional generalization is caused by the violation of the standard rule (the chain rule and other rules) for fractional derivatives of non-integer order. This non-equivalence of equations in natural and social sciences generates uncertainty in the description of the processes. Note that an additional unpredictability of fractional generalizations creates the presence of a large number of different types of fractional derivatives and integrals. This fact of mathematical non-equivalence allows us to formulate the following principle:

Principle "Unpredictability of Fractional Generalization". *A fractional generalization of one standard model (which is represented by s-equivalent differential equations of integer order) can lead to different fractional-dynamic models that will predict different behaviors of a process.*

Due to this, the correct and self-consistent derivation of fractional differential equations and the economic justification of existence of memory (or nonlocality) for one or another endogenous variable, are of fundamental importance.

As a result, the importance of the Derivability Principle and Interpretability Principle, which are proposed at the beginning of this work, increases substantially.

6. Example of Application of the Interpretability Principle: Effects, Phenomena, and Principles

In this section, some effects and phenomena are considered only for illustration and explanation of the interpretability principle. As an example, we give an economic interpretation of the solutions of fractional differential equations, which describe the fractional generalization of the standard Harrod–Domar growth model.

6.1. Economic Model with Memory: Equation, Solution, and Asymptotic Behavior

One of the simple models of economic growth was proposed by Roy Harrod [163] and Evsey Domar [164,165] in 1946–1947. The fractional generalization of the standard Harrod–Domar growth models was proposed in papers [166,167] in 2016 (see also [47,54,144,145]).

Let us consider the fractional generalization of the standard Harrod–Domar growth model, which is described in Allen's book [78], (pp. 64–65). The fractional generalization of this model was proposed in section 3 of [54]. The fractional differential equation, which describes the Harrod–Domar model with power-law memory, has the form:

$$\left(D^{\alpha}_{C,0+}Y\right)(t) = \lambda\, Y(t) + C(t), \tag{168}$$

where the function $Y(t)$ describes the national income; $C(t) = -v^{-1}A(t)$ is the exogenous variable that is independent of the national income $Y(t)$; the function $A(t)$ is the autonomous investment; the parameter $s \in (0,1)$ is the marginal propensity to save; $v > 0$ is the investment coefficient indicating the power of accelerator; $B = v/s$ describes the capital intensity of the national income; $\lambda = B^{-1} = s/v$ [115]. In Equation (168), we use the Caputo fractional derivative $\left(D^{\alpha}_{C,0+}Y\right)(t)$ of the order $0 < \alpha < 2$. This order of the Caputo fractional derivatives is interpreted as the memory fading parameter [47]. The absence of memory corresponds to the positive integer values of α. For $\alpha = 1$, Equation (168) gives the differential equation of the first order that describes the standard Harrod–Domar model.

Equation (168) has the solution (Theorem 5.15 of [4], p. 323) in the form:

$$Y(t) = \sum_{k=0}^{n-1} Y^{(k)}(0)t^k E_{\alpha,k+1}[\lambda\, t^{\alpha}] + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda(t-\tau)^{\alpha}\right] C(\tau)d\tau, \tag{169}$$

where $Y^{(k)}(0)$ is integer-order derivatives of the orders $k \geq 0$ at $t = 0$, and $E_{\alpha,\beta}[z]$ is the two-parameter Mittag–Leffler function [32].

In the case $0 < \alpha \leq 1$ (n = 1) Equation (169) takes the form:

$$Y(t) = Y(0)E_{\alpha,1}[\lambda\, t^{\alpha}] + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda\, (t-\tau)^{\alpha}\right] C(\tau)d\tau. \tag{170}$$

For $1 < \alpha \leq 2$ (n = 2) Equation (169) gives:

$$Y(t) = Y(0)\, E_{\alpha,1}[\lambda\, t^{\alpha}] + Y^{(1)}(0)\, t\, E_{\alpha,2}[\lambda\, t^{\alpha}] + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda\, (t-\tau)^{\alpha}\right] C(\tau)d\tau, \tag{171}$$

where $Y^{(1)}(0)$ is first-order derivative of $Y(t)$ at $t = 0$.

The asymptotic behavior of solution (169) of the fractional differential Equation (168) with $\lambda > 0$ and $C(t) = 0$ at $t \to \infty$ is described by the expression:

$$Y(t) = exp\left(\lambda^{1/\alpha} t\right) \sum_{k=0}^{n-1} Y^{(k)}(0) \frac{\lambda^{-k/\alpha}}{\alpha} + \sum_{k=0}^{n-1} \left(\sum_{j=1}^{m} \frac{Y^{(k)}(0)\lambda^{-j}}{\Gamma(k+1-\alpha\, j)}\, t^{k-\alpha j} + O\left(\frac{1}{t^{\alpha\,(m+1)-k}}\right)\right), \tag{172}$$

where $n-1 < \alpha < n$. For the non-integer values of the fading parameter $0 < \alpha < 2$ the behavior of the national income $Y(t)$ is determined by the term with $\exp\left(\lambda^{1/\alpha} t\right)$. The power-law terms $t^{k-\alpha j}$ of (172) do not determine the dominant behavior at $t \to \infty$.

6.2. Interpretation: Warranted Rate of Growth with Memory

An important economic concept of growth models is the technological growth rate [115], p. 49, which is also called the Harrod's warranted rate of growth [78], p. 67, of endogenous variables (for example, national income). The technological (warranted) growth rate describes the growth rate in the case of the constant structure of the economy and the absence of external influences. The constant structure means that the parameters of the model do not change over time (for example, s, v are constants). The absence of external influences means the absence of exogenous variables ($C(t) = 0$). Mathematically, the technological growth rate is described by the asymptotic behavior of the solution of homogeneous differential equations for the economic model.

In the standard Harrod–Domar model, the solution of Equation (168) with $\alpha = 1$ and $C(t) = 0$ has the form $Y(t) = Y(0)\, exp(\lambda\, t)$. Therefore, the technological growth rate of this model is described by the value $\lambda = s/v$. The capital intensity of the national income $B = \lambda^{-1} = v/s$ is the characteristic time $\tau = B = \lambda^{-1}$ of growth without memory.

Using Equation (172) of the asymptotic behavior of the solution, we can formulate new economic concept, which can be called the warranted (technological) rate of growth with memory [144–168]. This concept allows us to characterize the processes of economic growth with memory not only in the fractional generalizations of the standard Harrod–Domar model, but also for a wide range of other models described by fractional differential equations [144–146,168].

The warranted (technological) rate of growth with memory is defined by the equation:

$$\lambda(\alpha) = \lambda^{1/\alpha} = B^{-1/\alpha}. \tag{173}$$

Note that for parameter $\alpha = 1$ this growth rate is equal to the standard warranted rate of growth without memory, $\lambda_{eff}(1) = \lambda$. It can be seen that the warranted (technological) rates of growth (Equation (173)) with one-parametric memory do not coincide with the growth rates $\lambda = B^{-1}$ of standard models without memory ($\alpha = 1$).

As a result, we can formulate [144–146,168] the following principles, which gives an economic interpretation of obtained mathematical results:

Principle of Changing of the Warranted Growth Rate by Memory. *The power-law memory with the non-integer fading parameter* $0 < \alpha < 2$ *change of the warranted rate of growth with memory according to the equation:*

$$\lambda(\alpha) = \lambda^{1/\alpha}, \tag{174}$$

where $\lambda = s/v$ *is the warranted growth rate without memory (*$\alpha = 1$*) for the same values of other parameters.*

Using this concept and principle, we can give examples of the economic interpretation of solution (169) of fractional differential Equation (168) that describes the fractional generalization of the standard Harrod–Domar model.

Let us consider two phenomena that follow from the suggested principle of changing of the warranted growth rate by memory for the case $0 < \alpha < 1$.

Acceleration Phenomenon. *The memory effect with* $\alpha \in (0,1)$ *increases the warranted growth rate of economic processes if growth rate of the processes without memory is* $\lambda > 1$. *In the case* $\lambda > 1$, *the memory effect can increase the growth rate by many orders of magnitude.*

Let us give a numerical example of acceleration phenomenon. For $\alpha = 0.1$ *and* $\lambda_1 = 10 > 1$, the warranted rate of growth with memory ($\alpha = 0.1$) is equal to $\lambda_1(0.1) = 10000000000$ instead of $\lambda_1(1) = \lambda_1 = 10$ for process without memory ($\alpha = 1$), that is, the memory effect can increase the growth rate by nine orders of magnitude.

Slowdown Phenomenon. *The memory effect with* $\alpha \in (0,1)$ *decreases the warranted growth rate of economic processes if growth rate of the processes without memory is small* $0 < \lambda < 1$. *In the case* $0 < \lambda < 1$, *the memory effect can decrease the growth rate by many orders of magnitude.*

Let us give a numerical example of slowdown phenomenon. For $\alpha = 0.1$ and $\lambda_2 = 0.1 < 1$, the warranted rate of growth with memory ($\alpha = 0.1$) is equal to $\lambda_2(0.1) = 0.00000000001$ instead of $\lambda_2(1) = \lambda_2 = 0.1$ for the process without memory ($\alpha = 1$), that is, the memory effect can decrease the growth rate by nine orders of magnitude.

As a result, these examples demonstrate that the memory effect can significantly change the warranted growth rate by many orders of magnitude for the case $0 < \alpha < 1$.

Note that the concept of the "*warranted characteristic times*" of processes with memory is proposed in [168] for processes of growth ($\lambda > 0$), which can be called the amplification, and for processes of relaxation ($\lambda < 0$).

6.3. Interpretation: Growth and Decline with Memory

Let us consider an economic interpretation of solution (169) of the fractional differential Equation (168) with $\lambda > 0$ for the case of constant autonomous investment $A(t) = const$, i.e., $C(t) = C = const$. This solution for $n - 1 < \alpha < n$ has the form:

$$Y(t) = \frac{1}{\lambda}\, C\,(E_{\alpha,1}[\lambda\, t^{\alpha}] - 1) + \sum_{k=0}^{n-1} Y^{(k)}(0) t^{k} E_{\alpha,k+1}[\lambda\, t^{\alpha}]. \tag{175}$$

For $0 < \alpha < 1$ the behavior of solution (175) with $\lambda > 0$ at $t \to \infty$ is described by the equation:

$$\begin{gathered} Y(t) = -\tfrac{1}{\lambda}\, C + \tfrac{1}{\alpha}\Big(Y(0) + \tfrac{1}{\lambda}\, C\Big) exp\Big(\lambda^{1/\alpha} t\Big) - \\ \Big(Y(0) + \tfrac{1}{\lambda}\, C\Big)\left(\sum_{j=1}^{m} \tfrac{\lambda^{-j}}{\Gamma(1-\alpha j)} + O\Big(\tfrac{1}{t^{\alpha(m+1)}}\Big)\right). \end{gathered} \tag{176}$$

For $1 < \alpha < 2$ the behavior of solution (175) with $\lambda > 0$ at $t \to \infty$ has the form:

$$\begin{gathered} Y(t) = -\tfrac{1}{\lambda}\, C + \tfrac{1}{\alpha}\Big(Y(0) + \tfrac{1}{\lambda}\, C + \lambda^{-1/\alpha} Y^{(1)}(0)\Big) \exp\Big(\lambda^{1/\alpha} t\Big) - \\ \sum_{j=1}^{m} \Big(Y(0) + \tfrac{1}{\lambda}\, C + \tfrac{Y^{(1)}(0)\, t}{1-\alpha\, j}\Big) \tfrac{\lambda^{-j}}{\Gamma(1-\alpha j)} t^{-\alpha j} + O\Big(\tfrac{1}{t^{\alpha(m+1)-1}}\Big). \end{gathered} \tag{177}$$

For processes without memory ($\alpha = 1$), we have:

$$Y(t) = -\frac{1}{\lambda}\, C + \Big(Y(0) + \frac{1}{\lambda}\, C\Big) exp(\lambda t). \tag{178}$$

Using Equation (177), we can formulate conditions of growth and decline for economic processes with memory that is described by solution (175) with $\lambda > 0$ and the memory fading parameter $\alpha \in (0,2)$.

For the case $\alpha \in (0,1)$ and $\alpha = 1$, the condition of growth with memory is represented by the inequality:

$$Y(0) + \frac{1}{\lambda} C > 0. \tag{179}$$

For the case $\alpha \in (0,1)$ and $\alpha = 1$, the condition of decline with memory is represented by the inequality:

$$Y(0) + \frac{1}{\lambda} C < 0. \tag{180}$$

We see that conditions (179) and (180) do not depend on the value of the memory fading parameter $\alpha \in (0,1)$. As a result, the conditions of growth and decline with memory have the same form as for process without memory ($\alpha = 1$). However, the growth rates of the processes with memory and without memory may differ greatly.

Let us describe the conditions of growth and decline for the case $1 < \alpha < 2$.

For the case $\alpha \in (1,2)$, the condition of growth with memory is given by the inequality:

$$Y(0) + \frac{1}{\lambda} C + \lambda^{-1/\alpha} Y^{(1)}(0) > 0. \tag{181}$$

For the case $\alpha \in (1,2)$, the condition of decline with memory is represented by the inequality:

$$Y(0) + \frac{1}{\lambda} C + \lambda^{-1/\alpha} Y^{(1)}(0) < 0. \tag{182}$$

From inequalities (181) and (182), we see that these conditions of the growth and decline for economy are determined not only by the initial conditions, but also by the memory fading parameter $\alpha \in (1,2)$, if $Y^{(1)}(0) \neq 0$.

Let us consider some special cases. If the condition (181) is satisfied, the effects of memory with $\alpha \in (1,2)$, can lead to faster growth, i.e., to increase of the warranted growth rate, if $0 < \lambda < 1$, since $\lambda(\alpha) > \lambda$. The effects of memory with $\alpha \in (1,2)$, can lead to a slowing of the decline, i.e., to decrease of the warranted growth rate, if $0 < \lambda < 1$, since $\lambda(\alpha) < \lambda$.

Let us also note an important special case, when we have a decline for process without memory and a growth for process with memory for the same other parameters [145,146,168].

For example, we can consider the case $C(t) = -v^{-1}A < 0$, $\lambda > 0$ and $\alpha \in (1,2)$.

Phenomenon of Replacing Economic Decline with Growth. *The process without memory shows decline, while the process with memory (at the same other parameters) demonstrates a growth, if the inequalities:*

$$Y(0) < \frac{1}{\lambda}|C| < Y(0) + \lambda^{-1/\alpha} Y^{(1)}(0) \tag{183}$$

holds for $C < 0$, $Y^{(1)}(0) > 0$, $\lambda > 0$ *and* $1 < \alpha < 2$.

Condition (183) means that decline is replaced by the growth, when the memory effect is taken into account [145,146,168]. As a result, memory effects can change the decline by growth. In processes with memory with $\alpha \in (1,2)$, we can have a growth instead of decline, when the other process parameter is unchanged. The decline of economic processes can be replaced by the growth, when the memory effect is taken into account [145,146,168].

As an example, the mathematical results, which is represented by the solution of the fractional differential equation allows us to give the following economic interpretation of these results for the case $\lambda \in (0,1)$ and $Y^{(1)}(0) > 0$.

Phenomenon of Amplification of Economy by Memory. *For small values of warranted growth rates* $\lambda \in (0,1)$ *and* $Y^{(1)}(0) > 0$, *the effects of memory with* $\alpha \in (1,2)$ *positively affect the economy, and lead to an improvement in economic dynamics. In other words, for the case* $\lambda \in (0,1)$ *and* $Y^{(1)}(0) > 0$ *effects of memory* *with $\alpha \in (1, 2)$ lead to positive results, such as a slowdown in the rate of decline, a replacement of the economic decline by its growth, an increase in the rate of economic growth.*

Finding qualitative differences in the behavior of the processes described by generalizations of standard models is an important part of building fractional dynamic generalizations. The phenomenon of replacing economic decline with growth demonstrates the qualitative difference (from an economic point of view) of the behavior of the standard model from the fractional dynamic model.

6.4. Interpretation: Relaxation with Memory

The fractional differential equation with $\lambda < 0$ describes a relaxation process. As an example of relaxation processes with memory, we can consider the dynamics of fixed assets (or capital stock), where we take into account the memory effects [169,170]. Let us assume that the retirement of capital occurs with a constant retirement rate of $0 < b < 1$, where the parameter $b > 0$ can also be interpreted as a coefficient of disposal of fixed assets. Let us assume that the investment is equal to $I = const$ monetary units. In the standard model the dynamics of fixed assets (or capital stock) without memory and lag, the rate of change of the fixed assets is equal to the difference between investments and disposal of fixed assets. Let us denote the fixed assets (or capital stock) at time $t \geq 0$ by $K(t)$. The fractional generalization of this standard model, which is proposed in [169], describes the dynamics of the fixed assets the fixed assets (the capital stock) with power-law memory by the fractional differential equation:

$$\Big(D^{\alpha}_{C,0+} K\Big)(t) = I - b\, K(t), \tag{184}$$

where $D^{\alpha}_{C,0+}$ is the Caputo derivative [4]. For $\alpha = 1$, Equation (184) takes the form:

$$\frac{dK(t)}{dt} = I - b\, K(t), \tag{185}$$

which is equation of the standard dynamic model of fixed assets [104], p. 82, without memory and lag.

Equation (185) describes the relaxation to the equilibrium state $K = I/b$. The solution of fractional differential Equation (184) can written in the form of Equations (169) and (175), where $\lambda = -b < 0$ and $C = I = const$. This solution describes a generalized relaxation processes since $\lambda < 0$. Note that asymptotic behavior of the solutions cannot be represented in the form of Equations (172), (176), and (177) since the asymptotic expressions for cases $\lambda < 0$ and $\lambda > 0$ have different forms. For economic interpretation of solution (175) with $\lambda < 0$, we can use the fractional relaxation-oscillation phenomenon that was proposed by Francesco Mainardi [171] in 1996. The detailed description of this phenomenon is given in the works [172–174]. Some aspects of this interpretation are described in section 2.4 of [168], where the concept of the "warranted" characteristic times of processes with memory (the "warranted" relaxations times of processes with memory if $\lambda < 0$) has been proposed. Note that, in contrast to the growth (amplification) processes ($\lambda > 0$), for relaxation processes ($\lambda < 0$) it is necessary to consider two types of "warranted" characteristic times describing oscillations and damping (for details, see [168]).

Note that generalization of the fractional relaxation-oscillation phenomenon for distributed order is described in [175–177].

7. Conclusions

In this paper we formulated some principles and rules that are important for constructing fractional generalizations of standard dynamic models that are described by differential equations of integer order. These rules emphasize the importance of taking into account the non-standard properties of fractional derivatives of non-integer order. The violation of the standard form of the chain rule, the semi-group rule for orders of derivatives, the product (Leibniz) rule, the semi-group property of dynamic maps should be considered as the most important part of the mathematical tools designed to describe non-locality and memory.

The proposed principles of fractional generalization are illustrated by examples from economics. We note that a brief review of the history of applications of fractional calculus in modern mathematical economics and economic theory is proposed in [149].

We also think that these principles are general and can be applied to construct fractional generalizations of standard models in mechanics, physics, biology, and other sciences. It is hoped that various works will soon appear in which these principles will be illustrated with examples from natural and social sciences.

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mathematics

Article

Growth Equation of the General Fractional Calculus

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Abstract: We consider the Cauchy problem $(\mathbb{D}_{(k)}u)(t) = \lambda u(t)$, $u(0) = 1$, where $\mathbb{D}_{(k)}$ is the general convolutional derivative introduced in the paper (A. N. Kochubei, Integral Equations Oper. Theory **71** (2011), 583–600), $\lambda > 0$. The solution is a generalization of the function $t \mapsto E_\alpha(\lambda t^\alpha)$, where $0 < \alpha < 1$, E_α is the Mittag–Leffler function. The asymptotics of this solution, as $t \to \infty$, are studied.

Keywords: generalized fractional derivatives; growth equation; Mittag–Leffler function

1. Introduction

In several models of the dynamics of complex systems, the time evolution for observed quantities has exponential asymptotics of two possible types. In the simplest cases, these asymptotics are related with the solutions to the equations

$$u'(t) = zu(t), \quad t > 0; \quad u(0) = 1,$$

where we will consider positive and negative z separately. For $z < 0$ (the relaxation equation), the solution decays to zero. In particular models such as, e.g., Glauber stochastic dynamics in the continuum, this corresponds to an exponential convergence to an equilibrium; see [1]. The case $z > 0$ may also appear in applications. We can mention the contact model in the continuum where for the mortality below a critical value, the density of the population will grow exponentially fast [2,3], as well as models of economic growth.

On the other hand, the observed behavior of specific physical and biological systems show an emergence of other time asymptotics that may be far from exponential decay or growth. An attempt to obtain other relaxation characteristics is related with a use of generalized time derivatives in dynamical equations (see [4,5]). In this way, we may produce a wide spectrum of possible asymptotics to reflect a demand coming from applications [6].

The general fractional calculus introduced in [7] is based on a version of the fractional derivative, the differential-convolution operator

$$(\mathbb{D}_{(k)}u)(t) = \frac{d}{dt}\int_0^t k(t-\tau)u(\tau)\,d\tau - k(t)u(0),$$

where k is a non-negative locally integrable function satisfying additional assumptions, under which

(A) the Cauchy problem

$$(\mathbb{D}_{(k)}u)(t) = -\lambda u(t), \quad t > 0; \quad u(0) = 1, \tag{1}$$

where $\lambda > 0$, has a unique solution that is completely monotone;

(B) the Cauchy problem

$$(\mathbb{D}_{(k)}w)(t,x) = \Delta w(t,x), \quad t>0,\ x\in\mathbb{R}^n; \quad w(0,x) = w_0(x),$$

is solvable (under appropriate conditions for w_0) and possesses a fundamental solution, a kernel with the property of a probability density.

A class of functions k, for which (A) and (B) hold, was found in [7] and is described below. The simplest example is

$$k(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad t>0, \tag{2}$$

where $0<\alpha<1$, and for this case, $\mathbb{D}_{(k)}$ is the Caputo–Djrbashian fractional derivative $\mathbb{D}^{(\alpha)}$. Another subclass is the one of distributed order derivatives; see [8] for the details.

Note that for the case where k has the form (2), the solution of (1) is $u(t) = E_\alpha(-\lambda t^\alpha)$, where E_α is the Mittag–Leffler function; see [9], Lemma 2.23 (page 98). This solution has a slow decay at infinity due to the asymptotic property of the Mittag–Leffler function; see [10]. Note that using particular classes of fractional derivatives, we observe several specific asymptotics for the solution of Equation (1) with $\lambda>0$. Some results in this direction were already obtained in [8,11,12]. A more detailed analysis of this problem will be performed in a forthcoming paper.

In this paper, we consider the Cauchy problem with the opposite sign in the right-hand side, that is

$$(\mathbb{D}_{(k)}u)(t) = \lambda u(t), \quad t>0; \quad u(0) = 1; \tag{3}$$

as before, $\lambda>0$. In the case of (2), we have $u(t) = E_\alpha(\lambda t^\alpha)$ (see [9], Lemma 2.23 (page 98)), and due to the well-known asymptotics of E_α [10], this is a function of exponential growth. The existence and uniqueness of an absolutely continuous solution of (3) follows from the results of [13] dealing with more general nonlinear equations. Here, we study the asymptotic behavior of the solution of (3). Functions of this kind can be useful for fractional macroeconomic models with long dynamic memory; see [14] and references therein. Let us explain this in a little greater detail.

In modern macroeconomics, the most important are so-called growth models, which in the mathematical sense are reduced (for linear models) to the equation $u'(t) = \lambda u(t) + f(t)$ with $\lambda>0$. In economics, an important role is played by processes with a distributed lag, starting with Phillips' works [15] (see also [16]), and long memory, starting with Granger's work [17] (see also [18]).

If we assume the presence of effects of distributed lag (time delay) or fading memory in economic processes, then the fractional generalization of the linear classical growth models can be described by the fractional differential equation $D^\alpha u(t) = \lambda u(t) + f(t)$ with $\lambda>0, \alpha>0$. The fractional generalizations of well-known economics models were first proposed for the Caputo–Djrbashian fractional derivative D^α. Solving the problem in a more general case will allow us to describe accurately the conditions on the operator kernels (the memory functions), under which equations for models of economic growth with memory have solutions.

In general fractional calculus, which was proposed in [7] (see also [4]), the case $\lambda>0$ is not considered. The growth equation was considered in [8] for the special case of a distributed order derivative, where it was proved that a smooth solution exists and is monotone increasing.

In this article, we propose correct mathematical statements for growth models with memory in more general cases, for the general fractional derivative $\mathbb{D}_{(k)}$ with respect to the time variable. Their application can be useful for mathematical economics for the description of processes with long memory and distributed lag.

Note that the technique used below was developed initially in [12] for use in the study of intermittency in fractional models of statistical mechanics.

2. Preliminaries

Our conditions regarding the function k will be formulated in terms of its Laplace transform

$$\mathcal{K}(p) = \int_0^\infty e^{-pt} k(t)\, dt. \tag{4}$$

Denote $\Phi(p) = p\mathcal{K}(p)$.

We make the following assumptions leading to (A) and (B) (see [7]).

(*) The Laplace transform (4) exists for all positive numbers p. The function $\mathcal{K}$ belongs to the Stieltjes class $\mathcal{S}$, and

$$\mathcal{K}(p) \to \infty, \text{ as } p \to 0; \quad \mathcal{K}(p) \to 0, \text{ as } p \to \infty; \tag{5}$$

$$p\mathcal{K}(p) \to 0, \text{ as } p \to 0; \quad p\mathcal{K}(p) \to \infty, \text{ as } p \to \infty. \tag{6}$$

Recall that the Stieltjes class consists of the functions ψ admitting the integral representation

$$\psi(z) = \frac{a}{z} + b + \int_0^\infty \frac{1}{z+t} \sigma(dt),$$

where $a, b \geqslant 0$, σ is a Borel measure on $[0, \infty)$, such that

$$\int_0^\infty (1+t)^{-1} \sigma(dt) < \infty. \tag{7}$$

For a detailed exposition of the theory of Stieltjes functions including properties of the measure σ, see [19], and especially Chapters 2 and 6.

In particular, for the Stieltjes function $\mathcal{K}$, the limit conditions (5) and (6) imply the representation

$$\mathcal{K}(p) = \int_0^\infty \frac{1}{z+t} \sigma(dt). \tag{8}$$

We can also write [7] that

$$k(s) = \int_0^\infty e^{-ts}\, \sigma(dt), \quad 0 < s < \infty.$$

The function Φ belongs to the class $\mathcal{CBF}$ of complete Bernstein functions, a subclass of the class $\mathcal{BF}$ of Bernstein functions. Recall that a function $f : (0, \infty) \to \mathbb{R}$ is called a Bernstein function if $f \in C^\infty$, $f(z) \geqslant 0$ for all $z > 0$, and

$$(-1)^{n-1} f^{(n)}(z) \geqslant 0 \quad \text{for all } n \geqslant 1, z > 0,$$

so that the derivative of f is completely monotone. A function f belongs to $\mathcal{CBF}$ if it has an analytic continuation to the cut complex plane $\mathbb{C} \backslash (-\infty, 0]$ such that $\operatorname{Im} z \cdot \operatorname{Im} f(z) \geqslant 0$ and there exists the real limit

$$f(0+) = \lim_{(0,\infty) \ni z \to 0} f(z).$$

Both the classes $\mathcal{BF}$ and $\mathcal{CBF}$ admit equivalent descriptions in terms of integral representations; see [19].

Below, we will need the following inequality for complete Bernstein functions (Proposition 2.4 in [20]), valid, in particular, for the function Φ. For any p outside the negative real semi-axis, we have

$$\sqrt{\frac{1+\cos\varphi}{2}}\Phi(|p|) \leqslant |\Phi(p)| \leqslant \sqrt{\frac{2}{1+\cos\varphi}}\Phi(|p|), \quad \varphi = \arg p. \tag{9}$$

Solutions of the Cauchy problem (3) and a similar problem with the classical first order derivative are connected by the subordination identity (see [7]; for the case of the Caputo–Djrbashian derivative, see [21]), an integral transformation with the kernel $G(s,t)$ constructed as follows.

Consider the function

$$g(s,p) = \mathcal{K}(p)e^{-s\Phi(p)}, \quad s > 0, p > 0. \tag{10}$$

It is proved [7] that g is a Laplace transform in the variable t of the required kernel $G(s,t)$, that is,

$$g(s,p) = \int\limits_0^\infty e^{-pt}G(s,t)\,dt.$$

G is non-negative, and

$$\int\limits_0^\infty G(s,t)\,ds = 1 \quad \text{for each } t.$$

3. Cauchy Problem for the Growth Equation

Let us consider the Cauchy problem (3). If $u_\lambda(t)$ is its solution, whose Laplace transform $\widetilde{u_\lambda}(p)$ exists for some p, then it follows from properties of the Laplace transform [22] that

$$\Phi(p)\widetilde{u_\lambda}(p) - \lambda\widetilde{u_\lambda}(p) = \mathcal{K}(p).$$

Hence,

$$\widetilde{u_\lambda}(p) = \frac{\mathcal{K}(p)}{\Phi(p)-\lambda}, \quad \text{if } \Phi(p) > \lambda. \tag{11}$$

On the other hand, consider the function

$$E(t,\lambda) = \int\limits_0^\infty e^{\lambda s}G(s,t)\,ds, \quad t > 0. \tag{12}$$

The existence of the integral in (12) for almost all $t > 0$ is, by the Fubini–Tonelli theorem, a consequence of the absolute convergence of the repeated integral

$$\int\limits_0^\infty e^{\lambda s}\,ds \int\limits_0^\infty e^{-pt}G(s,t)\,dt = \int\limits_0^\infty e^{\lambda s}g(s,\lambda)\,ds = \frac{\mathcal{K}(p)}{\Phi(p)-\lambda},$$

where $p > 0$ is such that $\Phi(p) > \lambda$.

The above calculation shows that $E(t,\lambda) = u_\lambda(t)$, the solution of (3), and the identity (12) provides an integral representation of this solution.

A more detailed analysis of its properties is based on the analytic properties of the Stieltjes function $\mathcal{K}$, or equivalently, of the complete Bernstein function Φ; in particular, we use the representation

$$\Phi(p) = \int\limits_0^\infty \frac{p}{p+t}\,\sigma(dt), \tag{13}$$

which follows from (7). The measure σ satisfies (8).

Since Φ is a Bernstein function, its derivative Φ' is completely monotone. By our assumptions, Φ is not a constant function, so that Φ' is not the identical zero. It follows from Bernstein's description of completely monotone functions that $\Phi'(p) \neq 0$ for any $p > 0$ (see Remark 1.5 in [19]). Therefore, Φ is strictly monotone, and for each $z > 0$, there exists a unique $p_0 = p_0(z) > 0$ such that $\Phi(p_0) = z$. The inequality $\Phi(p) > z$ is equivalent to the inequality $p > p_0(z)$. Since Φ, as a complete Bernstein function, preserves the open upper and lower half-planes (in fact, this follows from (13)), we have $\Phi(p) \neq z$ for any nonreal p.

It is proved in [12] that the function $p_0(z), z > 0$ is strictly superadditive, that is

$$p_0(x+y) > p_0(x) + p_0(y) \quad \text{for any } x, y > 0.$$

Proposition 1. *The solution $u_\lambda(t)$ of the Cauchy problem (3) admits a holomorphic continuation in the variable t to a sector $\Sigma_v = \{re^{i\theta} : r > 0, -v < \theta < v\}$, $0 < v < \frac{\pi}{2}$, and*

$$\sup_{t \in \Sigma_v} \left| e^{-p_0 t} u_\lambda(t) \right| < \infty, \quad p_0 = p_0(\lambda). \tag{14}$$

Proof. It follows from (11) and (13) that the Laplace transform $\widetilde{u_\lambda}(p)$ is holomorphic in p on any sector $p_0 + \Sigma_{\rho + \frac{\pi}{2}}$, $0 < \rho < \frac{\pi}{2}$. In addition,

$$\sup_{p \in p_0 + \Sigma_{\rho + \frac{\pi}{2}}} |(p - p_0)\widetilde{u_\lambda}(p)| < \infty. \tag{15}$$

Now the assertion is implied by (15) and the duality theorem for holomorphic continuations of a function and its Laplace transform; see Theorem 2.6.1 in [23]. $\square$

Now we are ready to formulate and prove our main result.

Theorem 1. *Let the assumptions $(*)$ hold, and in addition,*

$$\int_1^\infty \frac{ds}{s\Phi(s)} < \infty. \tag{16}$$

Then

$$u_\lambda(t) = \frac{\lambda}{\Phi'(p_0(\lambda))p_0(\lambda)} e^{p_0(\lambda)t} + o(e^{p_0(\lambda)t}), \quad t \to \infty. \tag{17}$$

Proof. The representation (11) can be written as

$$\widetilde{u_\lambda}(p) = \frac{1}{p}\left(1 + \frac{\lambda}{\Phi(p) - \lambda}\right).$$

This implies the representation of u_λ as $u_\lambda(t) = 1 + B_\lambda(t)$, where B_λ has the Laplace transform

$$\widetilde{B_\lambda}(p) = \frac{\lambda}{p} \cdot \frac{1}{\Phi(p) - \lambda},$$

for such p that $\Phi(p) > \lambda$.

Using the inequality (9), we find that $|\Phi(p)| \geqslant \frac{1}{\sqrt{2}}\Phi(|p|)$ on any vertical line $\{p = \gamma + i\tau, \tau \in \mathbb{R}\}$ where $\gamma > p_0$. By our assumption (16), $\widetilde{B_\lambda}$ is absolutely integrable on such a line. In addition, it follows from (6) that $\widetilde{B_\lambda}(p) \to 0$, as $p \to \infty$ in the half-plane $\operatorname{Re} p > p_0$. These properties make it possible (see Theorem 28.2 in [22]) to write the inversion formula

$$u_\lambda(t) = 1 + \frac{\lambda}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} \frac{dp}{p(\Phi(p) - \lambda)}, \quad \gamma > p_0.$$

Denote

$$V(t) = 1 + \frac{\lambda}{2\pi i} \int_{r - i\infty}^{r + i\infty} e^{pt} \frac{dp}{p(\Phi(p) - \lambda)},$$

where $0 < r < p_0$. Then

$$|V(t)| \leqslant 1 + Ce^{rt} \left| \int_{-\infty}^{\infty} e^{i\tau t} \frac{d\tau}{(r + i\tau)(\Phi(r + i\tau) - \lambda)} \right| = o(e^{rt}), \quad t \to \infty, \tag{18}$$

by virtue of (9), (16), and the Riemann–Lebesgue theorem.

On the other hand, we may write

$$u_\lambda(t) - V(t) = \frac{\lambda}{2\pi i} \left(\int_{\Lambda_+} + \int_{\Lambda_0} + \int_{\Lambda_-} \right) e^{pt} \frac{dp}{p(\Phi(p) - \lambda)},$$

where the contour Λ_+ consists of the vertical rays $\{\operatorname{Re} p = r, \operatorname{Im} p \geqslant R\}$, $\{\operatorname{Re} p = \gamma, \operatorname{Im} p \geqslant R\}$, and the horizontal segment $\{r \leqslant \operatorname{Re} p \leqslant \gamma, \operatorname{Im} p = R\}$ $(R > 0)$, Λ_- is a mirror reflection of Λ_+ with respect to the real axis, Λ_0 is the finite rectangle consisting of the vertical segments $\{\operatorname{Re} p = r, |\operatorname{Im} p| \leqslant R\}$, $\{\operatorname{Re} p = \gamma, |\operatorname{Im} p| \leqslant R\}$, and the horizontal segments $\{r \leqslant \operatorname{Re} p \leqslant \gamma, \operatorname{Im} p = \pm R\}$.

We have

$$\int_{\Lambda_+} e^{pt} \frac{dp}{p(\Phi(p) - \lambda)} = 0,$$

due to the Cauchy theorem, absolute integrability of the integrand on the vertical rays (see (16)) and the estimate

$$\left| \int_{\Pi_h} e^{pt} \frac{dp}{p(\Phi(p) - \lambda)} \right| \leqslant Ch^{-1} \to 0, \quad h \to \infty,$$

where $\Pi_h = \{r \leqslant \operatorname{Re} p \leqslant \gamma, \operatorname{Im} p = h\}$, $h > R$. In a similar way, we prove that

$$\int_{\Lambda_-} e^{pt} \frac{dp}{p(\Phi(p) - \lambda)} = 0.$$

Due to the inequality $\Phi'(p_0) \neq 0$, there exists a complex neighborhood W of the point $\lambda = \Phi(p_0)$, in which Φ has a single-valued holomorphic inverse function $p = \psi(w)$, so that $\Phi(\psi(w)) = w$ and $p_0 = \psi(\lambda)$. In the above arguments, the numbers r, γ, R were arbitrary. Now we choose R and $\gamma - r$ so small that that the curvilinear rectangle $\Phi(\Lambda_0)$ lies inside W. Making the change of variables $p = \psi(w)$ and using the Cauchy formula, we find that

$$\frac{\lambda}{2\pi i}\int\limits_{\Lambda_0} e^{pt}\frac{dp}{p(\Phi(p)-\lambda)} = \frac{\lambda}{2\pi i}\int\limits_{\Phi(\Lambda_0)} e^{\psi(w)t}\frac{1}{\Phi'(\psi(w))\psi(w)}\cdot\frac{dw}{w-\lambda} = \frac{\lambda}{\Phi'(\psi(\lambda))\psi(\lambda)}e^{\psi(\lambda)t}$$
$$= \frac{\lambda}{\Phi'(p_0(\lambda))p_0(\lambda)}e^{p_0(\lambda)t}.$$

Together with (18), this implies the required asymptotic relation (17). □

Example 1. *(1) In the case (2) of the Caputo–Djrbashian fractional derivative of order $0 < \alpha < 1$, we have $u_\lambda(t) = E_\alpha(\lambda t^\alpha)$, $\Phi(p) = p^\alpha$, and the condition (16) is satisfied. Here the asymptotics (17) coincide with the one given by the principal term of the asymptotic expansion of the Mittag–Leffler function. The above proof is different from the classical proof of the latter (see [10]).*

(2) Let us consider the case of a distributed order derivative with a weight function μ, that is,

$$\mathbb{D}^{(\mu)}u(t) = \int\limits_0^1 (\mathbb{D}^{(\alpha)})u(t)\mu(\alpha)\,d\alpha.$$

Suppose that $\mu \in C^2[0,1]$, $\mu(1) \neq 0$. In this case [8],

$$k(s) = \int\limits_0^1 \frac{s^{-\alpha}}{\Gamma(1-\alpha)}\mu(\alpha)\,d\alpha, \quad \Phi(p) = \int\limits_0^1 p^\alpha \mu(\alpha)\,d\alpha,$$

and under the above assumptions,

$$\Phi(p) = \frac{\mu(1)p}{\log p} + O\left(p|\log p|^{-2}\right), \quad p \to \infty.$$

Condition (16) is satisfied, and our asymptotic result (17) is applicable in this case.

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Applications of the Fractional Diffusion Equation to Option Pricing and Risk Calculations

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Abstract: In this article, we first provide a survey of the exponential option pricing models and show that in the framework of the risk-neutral approach, they are governed by the space-fractional diffusion equation. Then, we introduce a more general class of models based on the space-time-fractional diffusion equation and recall some recent results in this field concerning the European option pricing and the risk-neutral parameter. We proceed with an extension of these results to the class of exotic options. In particular, we show that the call and put prices can be expressed in the form of simple power series in terms of the log-forward moneyness and the risk-neutral parameter. Finally, we provide the closed-form formulas for the first and second order risk sensitivities and study the dependencies of the portfolio hedging and profit-and-loss calculations upon the model parameters.

Keywords: fractional diffusion equation; fundamental solution; option pricing; risk sensitivities; portfolio hedging

1. Introduction

Fractional Calculus (FC) is nearly as old as conventional calculus. Many prominent mathematicians including Leibniz, Fourier, Laplace, Liouville, Riemann, Weyl, and Riesz suggested their own definitions of the fractional integrals and derivatives and studied their properties. Whereas the mathematical theory of FC was nearly completed a long time ago (see, e.g., [1] or the recent reference books [2,3]), only a few applications of FC outside mathematics were known until recently. During the last three decades, the situation changed completely, and currently, the majority of FC publications is devoted to modeling of a broad class of systems and processes using either the FC operators or the so-called fractional ordinary or partial differential equations. In particular, the FC models were successfully employed in physics [4,5], control theory [6], as well as in engineering, life, and social sciences [7,8], to mention only a few of the many application areas.

The two probably most prominent and broadly-recognized FC applications are in linear viscoelasticity [9] and for describing anomalous transport processes [10]. In both cases, the FC models in the form of the fractional ODEs (linear viscoelasticity) and the fractional PDEs (anomalous transport processes), respectively, cannot be derived from "first principles". Instead, they are introduced as interpolations between several models formulated in terms of the ODEs or PDEs, respectively. In the case of linear viscoelasticity, the basic FC model interpolates between the Hooke model (elasticity, ODE of the zero order) and the Newton model (viscosity, ODE of the first order). The FC model for the slow anomalous diffusion interpolates between a stationary state (time-independent Poisson equation) and the diffusion equation (first order equation with respect to time), whereas the fast diffusion is modeled by the fractional diffusion-wave PDE that interpolates between the diffusion equation (first order equation with respect to time) and the wave equation (second order equation with respect to time).

Of course, this kind of model needs an additional justification, say, in the form of a better fitting of certain datasets or better predictions compared to ones delivered by the conventional models. One of the advantages of the FC models is that they have some additional parameters (orders of the fractional derivatives) that can be suitably chosen for a set of data at hand. To do this, an inverse problem for the determination of the optimal parameters from the data has to be solved (see, e.g., [11] for an excellent survey of the basic inverse problems for the fractional differential equations).

It is worth mentioning that the anomalous diffusion models have a clear stochastic interpretation and can be formulated in terms of the continuous time random walk processes. The models in the form of the time- and/or space-fractional differential equations follow from these stochastic models for a special choice of the jump probability density functions with infinite first or/and second moments [10,12,13]. In [14], the fundamental solution for a time-fractional diffusion equation with the Caputo fractional derivative of order $\alpha \in (0,1)$ and the spatial Laplace operator was shown to be a probability density function evolving in time. In [15], a space-time fractional diffusion equation with the spatial Riesz–Feller derivative of order $\alpha \in (0,2]$ and skewness θ ($|\theta| \leq \min\{\alpha, 2-\alpha\}$) and the time-fractional Caputo derivative of order $\beta \in (0,2]$ was investigated in detail. In particular, several subordination formulas for the fundamental solutions of this equation with different values of α and β and an extension of their probabilistic interpretation to the ranges $\{0 < \alpha \leq 2\} \cap \{0 < \beta \leq 1\}$ and $\{1 < \beta \leq \alpha \leq 2\}$ were derived in [15].

A close connection of the fractional diffusion equations with the stochastic processes (fractional Brownian motion, Lévy flights, etc.) made them very promising models for different financial applications. In particular, they were already employed in the hot problems of finance (for recent reviews, see, e.g., [16,17]), particularly in financial markets [18–21], macroeconomics [22], mathematical economics [23], and for describing the concept of memory in economics [24] or economic growth [25]. One of the first applications of FC in finance was through the fractional Brownian motion, which enables incorporating long-range auto-correlations, typically observed in finance [26,27], volatility modeling [28], and option pricing [29]. FC has been specifically used in many option pricing models [30–33], also in connection with the jump processes [34] or in pricing of more complicated types of options, as American options [31], double barrier options [35], or currency options [36]. These models have been also investigated by numerical methods [37,38], and some applications to implied volatility have been also discussed [39].

In this paper, we focus on FC applications to option pricing, which is one of the most important tasks of financial mathematics. It is an important tool for market participants who want to hedge their positions and to estimate the value and risks or their portfolios. The first option pricing model was introduced by Black and Scholes [40] in 1973 based on the Gaussian assumptions for the variation of the stock prices' returns. After that, many generalizations and adaptations of this model were derived in order to capture the behavior of financial markets more realistically (let us mention, among others, models based on stochastic volatility [41], regime switching [42], or jumps [43]). Particularly interesting are the approaches based on replacement of the underlying stochastic process by the fractional Brownian motion or Lévy flights that leads to models driven by the fractional diffusion equations. One of the first models of this kind explicitly designed for the purpose of option pricing was introduced by Carr and Wu [44] by replacing the conventional Gaussian noise by a maximally-skewed Lévy-stable process (in other words, by replacing the underlying diffusion equation by a space-fractional diffusion equation). This model is far more realistic than the Black-Scholes model because it incorporates the heavy-tailed distributions and thus allows reproducing complex, but observable behavior in the distribution of prices (such as large drops) and in long-term volatility patterns. In this paper, we focus on a generalization of this model based on the space-time fractional diffusion equation [15] originally introduced in [45] and further investigated in [46–49]. The main advantage of this model is its interpretation: the model parameters play the role of the volatility (general risk level) and the spatial and temporal risk redistributions, and therefore, this model captures the complex market behaviors more accurately compared to the conventional models. In the first part of this paper, we recall the main features and particular cases of the fractional diffusion model, as well as some recent results on European option pricing. Then, we apply this mathematical framework to more exotic types of options, i.e., binary options. The second part of the paper is devoted to the risk sensitives and Profits and Losses (P&L) calculations, which are very important for market practitioners.

The paper is organized as follows. In Section 2, we provide an overview of some recent results on fractional diffusion models and discuss their applications to financial modeling. Section 3 is devoted to the pricing formulas for the European and binary options driven by the space-time fractional diffusion equation. In Section 4, we apply our results to the risk sensitivities and portfolio management. The last section is dedicated to the conclusions.

2. The Fractional Diffusion Model and Option Pricing

We start with some standard financial definitions. A *call* (resp. *put*) option with a *strike* $K > 0$ and a *maturity* $T > 0$ is a financial instrument that gives the holder the right to buy (resp. to sell) a given *quantity* S (an asset, an index, etc.) at an agreed price $\mathcal{P}$ at time $t = T$. We will denote the option *payoff* by $\mathcal{P}(S(T), K)$.

In the case:

$$\mathcal{P}(S,K) \;=\; \max\{S-K,0\} \;:=\; [S-K]^{+} \tag{1}$$

one speaks of an *European call option*. If the option has the payoff (1), but can be exercised at any time $t \in [0,T]$, one speaks of an *American call option*. European and American options are often referred to as the *vanilla* options, while the payoff of the so-called *exotic* options is determined by some more sophisticated algorithms. Say, the payoff of the *binary* or the *digital call options* is given by the formula:

$$\begin{cases} \mathcal{P}(S,K) \;=\; H(S-K) & \text{(cash or nothing)}, \\ \mathcal{P}(S,K) \;=\; S \times H(S-K) & \text{(asset or nothing)}, \end{cases} \tag{2}$$

where H denotes the Heaviside step function. Let us also mention that there are even more complicated options such as Asian or barrier options, which are path-dependent instruments. This means that the payoff $\mathcal{P}(S(T),K)$ of these options depends on all values of the asset price $S(t)$, $t \leq T$, and not just on $S(T)$. In this paper, the focus is mainly on the European and binary options.

Evidently, in order to calculate the options' prices, the dynamics of the underlying asset has to be described and modeled. A common way to do this is interpretation of $S(t)$ as a stochastic process, so that the payoff function $\mathcal{P}(S(T),K)$ becomes a random variable with a certain probability distribution. In the next subsection, we discuss a particular class of stochastic models for $S(t)$, which are very popular among the market practitioners.

2.1. Exponential Market Models

In a wide class of option pricing models, it is assumed that the underlying asset price is described by a stochastic process $\{S(t)\}_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, whose instantaneous variations can be written in local form as follows:

$$\frac{\mathrm{d}S(t)}{S(t)} \;=\; rS(t)\,\mathrm{d}t \,+\, \sigma\,\mathrm{d}X(t), \qquad t \in [0,T], \tag{3}$$

where r denotes the continuous *risk-free interest rate* and σ stands for the *market volatility*. For financial application, the common choices for $X(t)$ are:

- Gaussian process: $X(t) = W(t)$ (standard Brownian motion). In this case, the stock price $S(t)$ is said to follow a *geometric Brownian motion*. This is the basic hypothesis for the Black-Scholes model [40].
- Lévy process: $X(t) = L_{\alpha,\beta}(t)$ (standardized Lévy-stable process with stability $\alpha \in [0,2]$ and skewness or asymmetry $\beta \in [-1,1]$). In this case, one says that $S(t)$ follows an exponential *Lévy-stable process* (see the details in [43]). Exponential Lévy models generalize the Gaussian framework (because $L_{2,\beta}(t) = W(t)$ for any $\beta \in [-1,1]$), while allowing additional realistic features such as the presence of price jumps with non-zero probability. Their relevance in financial modeling has been known since the works of Mandelbrot and Fama [50,51]. For most financial applications, only the so-called *Lévy-Pareto* distributions, i.e., the values $\alpha \in (1,2]$, are relevant (historically, Mandelbrot calibrated $\alpha = 1.7$ for the cotton market).

It is important to note that the stochastic differential Equation (3) is specified under the *risk-neutral* measure $\mathbb{Q}$ (also known as the *martingale measure*), that is the measure under which the discounted market price is a martingale (for more information regarding the risk-neutral option pricing, see [52] or any monograph on financial mathematics):

$$S(t) = e^{-r\tau}\,\mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t], \tag{4}$$

where $\tau := T - t$. The martingale measure is given by the *Esscher transform* [53] of the physical measure $\mathbb{P}$:

$$\frac{\mathrm{d}\mathbb{Q}|_{\mathcal{F}_t}}{\mathrm{d}\mathbb{P}|_{\mathcal{F}_t}} = \frac{e^{X(t)}}{\mathbb{E}^{\mathbb{P}}[e^{X(t)}]} = e^{X(t)+\mu t}, \tag{5}$$

where the parameter μ is defined as the value of the negative cumulant-generating function evaluated at the point 1:

$$\mu := -\log \mathbb{E}^{\mathbb{P}}\left[e^{X(1)}\right]. \tag{6}$$

In particular, if the process $X(t)$ admits a probability distribution (or density) $g(x,t)$, then μ reads:

$$\mu = -\log \int_{-\infty}^{+\infty} e^x g(x,1)\,\mathrm{d}x \tag{7}$$

and therefore, the existence of a risk-neutral measure is linked to the existence of the two-sided Laplace transform of the density. As we will see later, for exponential Lévy models, the two-sided Laplace transform diverges with the only exception in the case when the asymmetry parameter is maximal, that is when the probability distribution has exponential decay on the positive real semi-axis and polynomial decay (heavy tail) on the negative real semi-axis.

With the above notations, the solution to (3) is the exponential process:

$$S(t) = S(0)e^{(r+\mu)t + \sigma X(t)} \tag{8}$$

and the price of an option with strike K, maturity T, and payoff $\mathcal{P}$ is equal to the present value of its expected payoff:

$$C(S,K,r,\mu,\tau) = e^{-r\tau}\,\mathbb{E}^{\mathbb{Q}}\left[\mathcal{P}(S(T),K)\right]. \tag{9}$$

The expectation (9) can be computed by integrating all possible realizations of the terminal payoff over the martingale measure: if $X(t)$ admits a density $g(x,t)$, then (9) reads:

$$C(S,K,r,\mu,\tau) = e^{-r\tau} \int_{-\infty}^{+\infty} \mathcal{P}(Se^{(r+\mu)\tau+x},K)\,g(x,\tau)\,\mathrm{d}x. \tag{10}$$

2.2. Generalizing Exponential Market Models: The Fractional Diffusion Model

2.2.1. Setup of the Model

By the *space-time fractional diffusion model*, we mean the following Cauchy problem:

$$\begin{cases} \left({}^{*}_{0}\mathcal{D}^{\gamma}_{t} + \mu_{\alpha,\theta,\gamma}\, {}^{\theta}\mathcal{D}^{\alpha}_{x}\right) g(x,t) \;=\; 0, \qquad x \in \mathbb{R},\, t > 0, \\ g(\pm\infty,t) \;=\; 0, \\ g(x,0) \;=\; f_0(x), \\ \dfrac{\partial g}{\partial t}(x,0) \;=\; f_1(x), \end{cases} \tag{11}$$

where the parameters α and γ are restricted as follows: $\alpha \in (0,2]$, $\gamma \in (0,\alpha]$. The asymmetry parameter θ belongs to the so-called *Feller-Takayasu diamond* $|\theta| \leq \min\{\alpha, 2-\alpha\}$. ${}^{*}_{0}\mathcal{D}^{\gamma}_{t}$ denotes the *Caputo fractional derivative*, which is defined as:

$${}^{*}_{t_0}\mathcal{D}^{\nu}_{t} f(t) = \frac{1}{\Gamma(\lceil\nu\rceil - \nu)} \int_{t_0}^{t} \frac{f^{\lceil\nu\rceil}(\tau)}{(t-\tau)^{\nu+1-\lceil\nu\rceil}} \mathrm{d}\tau, \tag{12}$$

and ${}^{\theta}\mathcal{D}^{\alpha}_{x}$ denotes the *Riesz–Feller fractional derivative*, which is usually defined via its Fourier image:

$$\mathcal{F}[{}^{\theta}\mathcal{D}^{\nu}_{x} f(x)](k) = -{}^{\theta}\psi^{\nu}(k) F[f(x)](k) = -\mu|k|^{\nu} e^{i(\mathrm{sign}k)\theta\pi/2} \mathcal{F}[f(x)](k)\,. \tag{13}$$

By definition, both fractional derivatives become ordinary derivative operators if the orders of the derivatives are natural numbers. The *fundamental solution* or the *Green function* of (11), that is the solution corresponding to the initial values $f_0(x) = \delta(x)$ and $f_1(x) = 0$, was derived in [15] in form of a Mellin-Barnes integral (for $x > 0$):

$$g_{\alpha,\theta,\gamma}(x,t) \;=\; \frac{1}{2\pi i} \frac{1}{\alpha x} \int\limits_{c_1 - i\infty}^{c_1 + i\infty} \frac{\Gamma\left(\frac{t_1}{\alpha}\right)\Gamma\left(1-\frac{t_1}{\alpha}\right)\Gamma(1-t_1)}{\Gamma\left(1-\frac{\gamma}{\alpha}t_1\right)\Gamma\left(\frac{\alpha-\theta}{2\alpha}t_1\right)\Gamma\left(1-\frac{\alpha-\theta}{2\alpha}t_1\right)} \left(\frac{x}{(-\mu_{\alpha,\theta,\gamma} t)^{\frac{1}{\alpha}}}\right)^{t_1} \mathrm{d}t_1. \tag{14}$$

The Green function (14) can be extended to the negative values of the argument x due to the symmetry relation:

$$g_{\alpha,\theta,\gamma}(-x,t) \;=\; g_{\alpha,-\theta,\gamma}(x,t). \tag{15}$$

As an extension of the pricing formula for the exponential models (10), we now define the price of an option driven by the fractional diffusion model as follows:

$$C_{\alpha,\theta,\gamma}(S,K,r,\mu_{\alpha,\theta,\gamma},\tau) \;=\; e^{-r\tau} \int\limits_{-\infty}^{+\infty} \mathcal{P}(Se^{(r+\mu_{\alpha,\theta,\gamma})\tau + x}, K)\, g_{\alpha,\theta,\gamma}(x,\tau)\, \mathrm{d}x. \tag{16}$$

Similarly to Equation (7), the risk-neutral parameter now reads:

$$\mu_{\alpha,\theta,\gamma} \;=\; -\log \int\limits_{-\infty}^{+\infty} e^{x} g_{\alpha,\theta,\gamma}(x,1)\, \mathrm{d}x. \tag{17}$$

It follows from (17) that the Green function has to admit an exponential decay on the positive semi-axis to ensure the existence of the risk-neutral parameter. This is the case as soon as the *maximal negative asymmetry* (or *skewness*) holds (see the next section for calculations of $\mu_{\alpha,\gamma,\theta}$ in the space- and time-fractional cases):

$$\theta \;=\; \alpha - 2, \qquad \alpha > 1. \tag{18}$$

In what follows, we will denote the corresponding Green functions, the risk neutral parameters, and the call option prices as follows:

$$g_{\alpha,\gamma} := g_{\alpha,\alpha-2,\gamma}\,, \quad \mu_{\alpha,\gamma} := \mu_{\alpha,\alpha-2,\gamma}\,, \quad C_{\alpha,\gamma} := C_{\alpha,\alpha-2,\gamma}. \tag{19}$$

2.2.2. Financial Interpretation of the Parameters

The fractional diffusion models incorporate two new degrees of freedom (the order of the spatial derivative α and the order of the time derivative γ) that act as the *risk redistribution parameters*. More precisely, while the parameter σ in the model (3) represents the volatility of the returns of the underlying asset and, as such, has an equal impact on all kinds of options (an increase of σ leads to an increase of the price of both call and put options), any changes of the parameters α and γ do not affect all options in the same direction.

For instance, as observed in [45], if $\gamma < 1$, then the short-term options become more expensive and the long-term ones less expensive compared to the non-fractional case $\gamma = 1$. This situation is observable when the market conditions are far from equilibrium (dramatic price jumps, exceptional events impacting the markets, etc.) and can be interpreted as a manifestation of memory [24]. The parameter γ plays therefore the role of a *temporal redistribution*. In this paper, we will show how it affects the profit and loss of a portfolio.

The parameter α represents a *spatial redistribution* because it controls the heavy-tails of the probability distributions (see the discussion thereafter). Its impact has been extensively discussed in [44], where it was shown to be an excellent candidate for the long-term volatility modeling. Indeed, when the maturity increases, it is known that the volatility smirk does not flatten out as expected if the Gaussian hypothesis would be true. By letting α vary between one and two, the negative slope of the smirk can be controlled. It is flat when $\alpha = 2$ (Gaussian case) and becomes steeper when α decreases, thus generating any observable slope in equity index options markets (the impact of γ on the volatility structure for at-the-money option was discussed in [47]). In this paper, we will also prove that the parameter α governs the *delta-hedging* policy of the portfolios, that is the way of constructing portfolios whose values are independent of the fluctuations in the stock price S_t.

Finally, let us mention that a calibration of the model parameters from the traded options for a one-month trading period was suggested in [45] for the European calls and puts (together and separately). The results of calibration showed that α fluctuated around 1.5 and 1.6 (therefore, rather far from the Gaussian hypothesis, but close to the Mandelbrot estimates) and that γ, although close to one, varied simultaneously with and in the same direction as α. This leads to a relative stability for the *diffusion scaling exponent* $\Omega := \frac{\gamma}{\alpha}$.

2.3. Particular Cases

In this subsection, we discuss some well-known exponential models (3) that are particular cases of the fractional diffusion model (16).

2.3.1. Finite Moment Log Stable model

In the framework of the Finite Moment Log Stable (FMLS) model [44], the stochastic process $X(t)$ driving the stock price dynamic (3) is given by:

$$X(t) = L_{\alpha,-1}(t), \tag{20}$$

where $L_{\alpha,\beta}$ is a standardized Lévy-stable process (see [43]) with *stability* α and *asymmetry* (or *skewness*) β, whose probability distribution is the so-called *stable distribution* $g_{\alpha,\beta}$ [54]. As already mentioned, this model was introduced by Carr and Wu in order to capture the behavior of the volatility smirk (the phenomenon that, for a given maturity, implied volatility is higher for out-of-the-money puts than for out-of-the-money calls): as a function of moneyness, it is widely observed that the smirk does not flatten out for longer observable horizons (i.e., τ greater than two years), and this can only be achieved if one violates the Gaussian hypothesis. The *maximal negative hypothesis* $\beta = -1$ ensures the existence of a risk-neutral parameter: indeed, the two-sided Laplace transform of the stable distribution exists only if $\beta = +1$, with the result [55]:

$$\mathbb{E}^{\mathbb{P}} \left[e^{-px} \right] = e^{-p^\alpha \sec \frac{\pi\alpha}{2}}. \tag{21}$$

Therefore, from a symmetry argument, the risk-neutral parameter exists and is finite only if $\beta = -1$. It follows from (21) that its value is given by the formula:

$$\mu_\alpha := \mu_{\alpha,1} = \frac{\left(\frac{\sigma}{\sqrt{2}}\right)^\alpha}{\cos \frac{\pi\alpha}{2}}, \tag{22}$$

where $\sqrt{2}$ was introduced as a normalization constant to recover the Black-Scholes factor for $\alpha = 2$:

$$\mu_2 := -\frac{\sigma^2}{2}. \tag{23}$$

Moreover, under the condition $\beta = -1$, the probability distribution $g_{\alpha,-1} := g_\alpha$ has a fat tail on the negative real axis as soon as $1 < \alpha < 2$ (that is, it decays as $|x|^{-\alpha}$) and decays exponentially on the positive real axis. This highly-skewed behavior could not be captured by a traditional Brownian motion and by any symmetric distribution. However, it is a very plausible assumption as large drops are far more commonly observed in financial markets than large rises.

The FMLS model is actually a particular case of the fractional diffusion model: indeed, the choice $\beta = -1$ is the probabilistic equivalent of the choice $\theta = \alpha - 2$ (18), and the probability distribution g_α is the Green function associated with the space-fractional diffusion equation ($\gamma = 1$):

$$\frac{\partial g_\alpha}{\partial \tau}(x,\tau) + {\mu_\alpha}^{\alpha-2} \mathcal{D}_x^\alpha \, g_\alpha(x,\tau) = 0. \tag{24}$$

From the pricing Formula (16), the option price in the framework of the FMLS model reads:

$$C_{\alpha,1}(S,K,r,\mu_\alpha,\tau) = e^{-r\tau} \int_{-\infty}^{+\infty} \mathcal{P}(Se^{(r+\mu_\alpha)\tau+x}, K) \, g_\alpha(x,\tau) \, \mathrm{d}x, \tag{25}$$

where the Green function is given by the formula:

$$g_\alpha(x,\tau) = \frac{1}{2\pi i} \frac{1}{\alpha} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma(1-t_1)}{\Gamma(1-\frac{t_1}{\alpha})} \, x^{t_1-1} \, (-\mu_\alpha \tau)^{-\frac{t_1}{\alpha}} \, \mathrm{d}t_1. \tag{26}$$

Recently, an analytic option pricing formula for the FMLS model in the case of European options was derived in [48] in the form of a fast convergent double series. In the present paper, we extend this formula to the case of a time-fractional derivatives in (24), that is for $\gamma \neq 1$, and to other types of options.

2.3.2. Black-Scholes Model

The celebrated Black-Scholes model [40] assumes that the stochastic process driving the exponential model (3) is a standard Brownian motion, i.e., $X(t) = W(t)$. This model turns out to be a particular case of the FMLS model (and therefore, of the fractional diffusion model) because for $\alpha = 2$, the Lévy-stable process $L_{2,\beta}$ degenerates into $W(t)$ for any $\beta \in [-1,1]$. It is well known that the probability distribution of the Wiener process is the Green function associated with the *diffusion equation*:

$$\frac{\partial g}{\partial \tau}(x,\tau) \; - \; \frac{\sigma^2}{2}\frac{\partial^2 g}{\partial x^2}(x,\tau) \; = \; 0, \tag{27}$$

which is known to be the *Gaussian kernel*:

$$g(x,\tau) \; = \; \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\sigma^2\tau}}. \tag{28}$$

As expected, the diffusion Equation (27) is a particular case of the space-fractional diffusion Equation (24) for $\alpha \; = \; 2$ and with the Gaussian risk-neutral parameter (23). Using the pricing Formula (16), the price of the option in the Black-Scholes model reads:

$$C_{2,1}(S,K,r,\sigma,\tau) \; = \; e^{-r\tau} \int\limits_{-\infty}^{+\infty} \mathcal{P}(Se^{(r-\frac{\sigma^2}{2})\tau+x}, K) \, \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\sigma^2\tau}} \, \mathrm{d}x. \tag{29}$$

For $\mathcal{P}(S,K) = [S-K]^+$, some elementary manipulations on the integral (29) lead to the Black-Scholes formula for the European options, which, in our system of notations, reads as follows:

$$\begin{aligned} C^{(E)}_{2,1}(S,K,r,\sigma,\tau) \; &= \; SN\left(\frac{\log\frac{S}{K}+r\tau}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right) - Ke^{-r\tau}N\left(\frac{\log\frac{S}{K}+r\tau}{\sigma\sqrt{\tau}} - \frac{\sigma\sqrt{\tau}}{2}\right) \\ &= \; F\left[e^k N\left(\frac{k}{z}+\frac{z}{2}\right) - N\left(\frac{k}{z}-\frac{z}{2}\right)\right]. \end{aligned} \tag{30}$$

where $N(.)$ is the standard normal cumulative distribution function, and the *forward strike price* F and the *log-forward moneyness* k are defined by the expressions:

$$F \; := \; Ke^{-r\tau}, \qquad k \; := \; \log\frac{S}{F} \; = \; \log\frac{S}{K} + r\tau. \tag{31}$$

3. Pricing Formulas

3.1. Risk-Neutral Parameter

In general, the risk-neutral parameter $\mu_{\alpha,\gamma}$ is not known in the analytic form as soon as $\gamma < 1$. However, it has been shown in [46] that $\mu_{\alpha,\gamma}$ can be expressed in terms of a Mellin-Barnes integral involving the FMLS risk-neutral parameter μ_α (which is known in explicit form):

Proposition 1. *Let* $1 < \alpha \leq 2$ *and* $0 < \gamma < \alpha$. *Then, for any* $c \in (0,1)$, *the formula:*

$$\mu_{\alpha,\gamma} \; = \; -\log\left[\frac{1}{2\pi i}\frac{1}{\alpha}\int\limits_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\frac{1-s}{\alpha})}{\Gamma(\gamma s + 1 - \gamma)} \, \mu_\alpha^{\frac{s-1}{\alpha}} \, \mathrm{d}s\right] \tag{32}$$

holds true.

For the proof, see [46] or [47].

Under some conditions on α and γ, the right-hand side of the integral representation (32) can be expressed as a series:

Theorem 1. *Let* $1 < \alpha \leq 2$. *If* $1 - \frac{1}{\alpha} \; < \; \gamma \; < \alpha$, *then the series representation:*

$$\mu_{\alpha,\gamma} \; = \; -\log \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{\Gamma(1+\alpha n)}{\Gamma(1+\gamma\alpha n)} \, \mu_\alpha^n \tag{33}$$

is valid.

Proof. The Stirling asymptotic formula for the Gamma function (see [56]) leads to the following statement ($a_k, b_k, c_j, d_j \in \mathbb{R}$):

$$\sum_k a_k - \sum_j c_j < 0 \Rightarrow \left| \frac{\Pi_k \Gamma(a_k s + b_k)}{\Pi_j \Gamma(c_j s + d_j)} \right| \overset{|s| \to \infty}{\longrightarrow} 0 \text{ when } \arg s \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right). \tag{34}$$

Applying it to the integrand in the integral representation (32) under the condition $1 - \frac{1}{\alpha} - \gamma < 0$, we can evaluate the integral along the vertical line by closing the contour in the right-half plane because it will not contribute when $|s| \to \infty$. In this region, the function $\Gamma(\frac{1-s}{\alpha})$ is singular every time its argument equals a negative integer $-n$, that is when $s = 1 + \alpha n$, $n \in \mathbb{N}$, with residue $\frac{\alpha(-1)^n}{n!}$ (see the details in [56] or [57]). Therefore, we get the formula:

$$\text{res}_{s=1+\alpha n} \left[\frac{1}{\alpha} \frac{\Gamma(s)\Gamma(\frac{1-s}{\alpha})}{\Gamma(\gamma s + 1 - \gamma)} \mu_\alpha^{\frac{s-1}{\alpha}} \right] = \frac{(-1)^n}{n!} \frac{\Gamma(1+\alpha n)}{\Gamma(1+\alpha\gamma n)} \mu_\alpha^n. \tag{35}$$

Summing all residues and applying the residue theorem complete the proof. □

An important approximation to the risk-neutral series (33) is easily obtained via the Taylor series for $\log(1+u)$:

$$\mu_{\alpha,\gamma} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\gamma\alpha)} \mu_\alpha + O(\mu_\alpha^2). \tag{36}$$

Now, it follows from the reflection formula for the Gamma function that one can re-write the FMLS risk-neutral parameter (22) as follows:

$$\mu_\alpha = \frac{1}{\pi} \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma\left(\frac{1+\alpha}{2}\right) \left(\frac{\sigma}{\sqrt{2}}\right)^\alpha. \tag{37}$$

Plugging the last formula into Formula (36), we have the following result:

Corollary 1. *Let $1 < \alpha < 2$ and $1 - \frac{1}{\alpha} < \gamma < \alpha$. Then, the formula:*

$$\mu_{\alpha,\gamma} = \frac{1}{\pi} \frac{\Gamma(1+\alpha)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma(1+\gamma\alpha)} \left(\frac{\sigma}{\sqrt{2}}\right)^\alpha + O\left(\sigma^{2\alpha}\right) \tag{38}$$

holds true.

In particular, in the case of the fractional Black-Scholes model ($\alpha = 2$), we get the expression:

$$\mu_{2,\gamma} = -\frac{\sigma^2}{\Gamma(1+2\gamma)} + O(\sigma^4), \tag{39}$$

which resumes to the well-known Gaussian parameter $-\frac{\sigma^2}{2}$ when $\gamma = 1$.

3.2. European Options

Let us denote by $C_{\alpha,\gamma}^{(E)}(S, K, r, \mu_{\alpha,\gamma}, \tau)$ the price of the European call option in the fractional diffusion model with the payoff $\mathcal{P}^{(E)}(S, K) = [S - K]^+$.

Proposition 2. *Let P be the polyhedron $P := \{(t_1,t_2) \in \mathbb{C}^2\,,\, Re(t_2 - t_1) > 1\,,\, 0 < Re(t_2) < 1\}$. Then, for any vector $\underline{c} = (c_1, c_2) \in P$,*

$$C^{(E)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau) \;=\; \frac{Ke^{-r\tau}}{\alpha} \int\limits_{\underline{c}+i\mathbb{R}^2} (-1)^{-t_2} \frac{\Gamma(t_2)\Gamma(1-t_2)\Gamma(-1-t_1+t_2)}{\Gamma(1-\frac{\gamma}{\alpha}t_1)} (-k-\mu_{\alpha,\gamma}\tau)^{1+t_1-t_2} (-\mu_{\alpha,\gamma}\tau^{\gamma})^{-\frac{t_1}{\alpha}} \frac{dt_1}{2i\pi} \wedge \frac{dt_2}{2i\pi}. \tag{40}$$

Proof. First, we mention that under the maximal negative asymmetry hypothesis and on the positive real axis, the Green function (14) has the form:

$$g_{\alpha,\gamma}(x,t) \;=\; \frac{1}{\alpha x} \int\limits_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma(1-t_1)}{\Gamma\left(1-\frac{\gamma}{\alpha}t_1\right)} \left(\frac{x}{(-\mu_{\alpha,\gamma}t)^{\frac{1}{\alpha}}} \right)^{t_1} \frac{dt_1}{2i\pi}, \qquad 0 < c_1 < 1. \tag{41}$$

On the other hand, using the notations (31), we can rewrite the payoff function in the form:

$$\mathcal{P}^{(E)}(Se^{(r+\mu_{\alpha,\gamma})\tau+x}, K) \;=\; K\left[e^{k+\mu_{\alpha,\gamma}\tau+x} - 1\right]^{+} \tag{42}$$

then integrate by parts over the variable x, and get the expression:

$$C^{(E)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau) \;=\; -\frac{Ke^{-r\tau}}{\alpha} \int\limits_{c_1-i\infty}^{c_1+i\infty} \int\limits_{-(k+\mu_{\alpha,\gamma}\tau)}^{+\infty} e^{(k+\mu_{\alpha,\gamma}\tau+x)} x_1^{t}\, dx\, \frac{\Gamma(1-t_1)}{t_1\Gamma\left(1-\frac{\gamma}{\alpha}t_1\right)} (-\mu_{\alpha,\gamma}\tau^{\gamma})^{-\frac{t_1}{\alpha}} \frac{dt_1}{2i\pi}. \tag{43}$$

The Mellin-Barnes representation for the exponential term (see [58] or any other monograph on integral transforms) reads:

$$e^{k+\mu_{\alpha,\gamma}\tau+x} \;=\; \int\limits_{c_2-i\infty}^{c_2+i\infty} (-1)^{-t_2}\Gamma(t_2)\,(k+\mu_{\alpha,\gamma}\tau+x)^{-t_2} \frac{dt_2}{2i\pi}, \qquad c_2 > 0. \tag{44}$$

Inserting Formulas (41) and (44) into Formula (43) for the call price leads to the Beta integral:

$$\int\limits_{-(k+\mu_{\alpha,\gamma}\tau)}^{+\infty} (k+\mu_{\alpha,\gamma}\tau+x)^{-t_2}\, x^{t_1}\, dx \;=\; \frac{\Gamma(1-t_2)\Gamma(-1-t_1+t_2)}{\Gamma(-t_1)} (-k-\mu_{\alpha,\gamma}\tau)^{1+t_1-t_2}. \tag{45}$$

The relation $t_1\Gamma(-t_1) = -\Gamma(1-t_1)$ along with some elementary simplifications yields the integral representation (40). □

Let us now represent the Mellin-Barnes double integral (40) in terms of a double series by means of residue summation.

Theorem 2 (Pricing formula: European call)**.** *Let $1 < \alpha \leq 2$ and $1 - \frac{1}{\alpha} < \gamma \leq \alpha$. Then, under the maximal negative asymmetry hypothesis ($\theta = \alpha - 2$), the price of the European call driven by the space-time fractional diffusion equation is as follows:*

$$C^{(E)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau) \;=\; \frac{Ke^{-r\tau}}{\alpha} \sum_{\substack{n=0\\m=1}}^{\infty} \frac{1}{n!\Gamma(1+\gamma\frac{m-n}{\alpha})} \,(k\,+\,\mu_{\alpha,\gamma}\tau)^{n}\,(-\mu_{\alpha,\gamma}\tau^{\gamma})^{\frac{m-n}{\alpha}}. \tag{46}$$

Proof. Let ω denote the differential form in the integral at the right-hand side of (40). If we perform the change of the variables:

$$\begin{cases} u_1 := -1 - t_1 + t_2, \\ u_2 := t_2 \end{cases} \tag{47}$$

then ω reads:

$$\omega = (-1)^{-u_2} \frac{\Gamma(u_1)\Gamma(u_2)\Gamma(1-u_2)}{\Gamma(1-\gamma\frac{-1-u_1+u_2}{\alpha})} (-k - \mu_{\alpha,\gamma}\tau)^{-u_1} (-\mu_{\alpha,\gamma}\tau^\gamma)^{-\frac{-1-u_1+u_2}{\alpha}} \frac{\mathrm{d}u_1}{2i\pi} \wedge \frac{\mathrm{d}u_2}{2i\pi}. \tag{48}$$

In the region $\{Re(u_1) < 0, Re(u_2) < 0\}$, ω has a simple pole at every point $(u_1, u_2) = (-n, -m)$, $n, m \in \mathbb{N}$ with the residue:

$$\mathrm{Res}_{(u_1=-n, u_2=-m)}\, \omega = \frac{1}{n!\Gamma(1+\gamma\frac{1+m-n}{\alpha})} (k + \mu_{\alpha,\gamma}\tau)^n (-\mu_{\alpha,\gamma}\tau^\gamma)^{\frac{1+m-n}{\alpha}}. \tag{49}$$

Performing the change of indexation $m \to m+1$ and summing all residues yields the double sum (46). The fact that one can close a contour in this region of $\mathbb{C}^2$ is a consequence of a two-dimensional generalization of the Stirling Formula (34) (see [46,47] for technical details). □

The pricing Formula (46) is a simple and efficient way for the calculation of the pricing of the European call options. The convergence of the partial sums in the double series (46) is very fast, and therefore, only a few terms are needed to obtain an excellent level of precision (see numerical applications and convergence tests in [46,47]). As a particular case of the pricing Formula (46) with $\gamma = 1$, we recover the pricing formula for the FMLS model that was established in [48]:

$$C^{(E)}_{\alpha,1}(S,K,r,\mu_\alpha,\tau) = \frac{Ke^{-r\tau}}{\alpha} \sum_{\substack{n=0\\m=1}}^{\infty} \frac{1}{n!\Gamma(1+\frac{m-n}{\alpha})} (k + \mu_\alpha\tau)^n (-\mu_\alpha\tau)^{\frac{m-n}{\alpha}}. \tag{50}$$

If we set $\alpha = 2$ in the last formula, then we obtain the series expansion for the Black-Scholes European call:

$$C^{(E)}_{2,1}(S,K,r,\sigma,\tau) = \frac{Ke^{-r\tau}}{\alpha} \sum_{\substack{n=0\\m=1}}^{\infty} \frac{1}{n!\Gamma(1+\frac{m-n}{2})} \left(k - Z^2\right)^n Z^{m-n}, \tag{51}$$

where $Z := \frac{\sigma\sqrt{\tau}}{\sqrt{2}}$. In [59], using the change of variables $j = m+n$ and the properties of the Gamma function, it was proven that (51) can be ordered in odd powers of Z:

$$C^{(E)}_{2,1}(S,K,r,\sigma,\tau) = \frac{1}{2}(S - Ke^{-r\tau}) + \frac{Ke^{-r\tau}}{2} \sum_{\substack{j\geq 0\\n\leq 2j}} Z^{2j+1} \frac{(-1)^n}{n!\Gamma(\frac{3}{2}+j-n)} \left(1 - \frac{k}{Z^2}\right)^n. \tag{52}$$

A particularly interesting situation occurs when the asset is *At-The-Money (ATM) forward*, that is when $S = Ke^{-r\tau}$ or equivalently with our notations $k = 0$ in Equation (46). In this case, we get the following result:

Corollary 2 (At-the-money price: European call)**.** *For $S = Ke^{-r\tau}$, the price of the European call option driven by the space-time fractional diffusion equation is given by the formula:*

$$C^{(E,ATM)}_{\alpha,\gamma}(S,\mu_{\alpha,\gamma},\tau) = \frac{S}{\alpha}\sum_{\substack{n=0\\m=1}}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\gamma\frac{m-n}{\alpha})}(-\mu_{\alpha,\gamma}\tau^{\gamma})^{\frac{m+(\alpha-1)n}{\alpha}}\tau^{(1-\gamma)n} \tag{53}$$

$$= \frac{S}{\alpha}\left[\frac{(-\mu_{\alpha,\gamma})^{\frac{1}{\alpha}}\tau^{\frac{\gamma}{\alpha}}}{\Gamma(1+\frac{\gamma}{\alpha})} + \mu_{\gamma,\alpha}\tau + \frac{(-\mu_{\alpha,\gamma})^{\frac{2}{\alpha}}\tau^{\frac{2\gamma}{\alpha}}}{\Gamma(1+\frac{2\gamma}{\alpha})} + O\left((-\mu_{\alpha,\gamma})^{1+\frac{1}{\alpha}}\tau^{1+\frac{\gamma}{\alpha}}\right)\right].$$

As a particular case of (53), the ATM price in the FMLS model reads:

$$C^{(E,ATM)}_{\alpha,1}(S,\mu_{\alpha},\tau) = \frac{S}{\alpha}\sum_{\substack{n=0\\m=1}}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\frac{m-n}{\alpha})}(-\mu\tau)^{\frac{m+(\alpha-1)n}{\alpha}} \tag{54}$$

as was already established in [48]. When $\alpha = 2$, we get:

$$C^{(E,ATM)}_{2,1}(S,\sigma,\tau) \;=\; \frac{S}{2}\left[\frac{1}{\Gamma(\frac{3}{2})}\frac{\sigma\sqrt{\tau}}{\sqrt{2}} + O((\sigma\sqrt{\tau})^3)\right] \;=\; \frac{1}{\sqrt{2\pi}}S\sigma\sqrt{\tau} + O((\sigma\sqrt{\tau})^3)\,. \tag{55}$$

As $\frac{1}{\sqrt{2\pi}} \simeq 0.4$, we have thus recovered the well-known Brenner–Subrahmanyam approximation for the European Black-Scholes call (which was first introduced in [60]):

$$C^{(E,ATM)}_{2,1}(S,\sigma,\tau) \;\simeq\; 0.4S\sigma\sqrt{\tau}\,. \tag{56}$$

3.3. Binary (or Digital) Options

3.3.1. Cash-or-Nothing

Let us denote by $C^{(C/N)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau)$ the price of the binary cash-or-nothing option in the fractional diffusion model with the payoff $\mathcal{P}^{(C/N)}(S,K) = H(S-K)$.

Proposition 3. *For any $c_1 > 0$, the formula:*

$$C^{(C/N)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau) \;=\; \frac{e^{-r\tau}}{\alpha}\int_{c_1-i\infty}^{c_1+i\infty}\frac{\Gamma(-t_1)}{\Gamma(1-\frac{\gamma}{\alpha}t_1)}(-k-\mu_{\alpha,\gamma}\tau)^{t_1}(-\mu_{\alpha,\gamma}\tau^{\gamma})^{-\frac{t_1}{\alpha}}\frac{\mathrm{d}t_1}{2i\pi} \tag{57}$$

holds true.

Proof. Inserting the terminal payoff $\mathcal{P}^{(C/N)}(Se^{(r+\mu_{\alpha,\gamma})\tau+x},K)$ into the call price (16) yields:

$$C^{(C/N)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau) \;=\; \frac{e^{-r\tau}}{\alpha}\int_{c_1-i\infty}^{c_1+i\infty}\int_{-(k+\mu_{\alpha,\gamma}\tau)}^{+\infty} x^{t_1-1}\mathrm{d}x\frac{\Gamma(1-t_1)}{\Gamma(1-\frac{\gamma}{\alpha}t_1)}(-\mu_{\alpha,\gamma}\tau^{\gamma})^{-\frac{t_1}{\alpha}}\frac{\mathrm{d}t_1}{2i\pi}. \tag{58}$$

Performing the integration with respect to the x-variable and using the relation $-t_1\Gamma(1-t_1) = \Gamma(-t_1)$ yield the representation (57). □

Theorem 3 (Pricing formula: cash or nothing call)**.** *Let $1 < \alpha \leq 2$ and $1 - \frac{1}{\alpha} < \gamma \leq \alpha$. Then, under the maximal negative asymmetry hypothesis ($\theta = \alpha - 2$), the price of the cash-or-nothing call option driven by the space-time fractional diffusion equation is given by the formula:*

$$C_{\alpha,\gamma}^{(C/N)}(S,K,r,\mu_{\alpha,\gamma},\tau) \,=\, \frac{e^{-r\tau}}{\alpha} \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(1-\frac{\gamma}{\alpha}n)} \, (k + \mu_{\alpha,\gamma}\tau)^n \, (-\mu_{\alpha,\gamma}\tau^{\gamma})^{-\frac{n}{\alpha}} \,. \tag{59}$$

Proof. As $-1 + \frac{\gamma}{\alpha} \leq 0$, it follows from Formula (34) that the line integral in (57) can be expressed as a sum of the residues induced by the poles of the $\Gamma(-t_1)$ function. These poles are located at the points $t = +n$, and the associated residues are as follows:

$$\frac{(-1)^n}{n!\Gamma(1-\frac{\gamma}{\alpha}n)} \, (-k - \mu_{\alpha,\gamma}\tau)^n \, (-\mu_{\alpha,\gamma}\tau^{\gamma})^{-\frac{n}{\alpha}}. \tag{60}$$

Summing up all residues completes the proof. □

In the particular case of the Black-Scholes model ($\alpha = 2, \gamma = 1$), the series (59) has only the odd terms $n = 2p + 1$ because of the divergence of the denominator at negative integers. Using the known values of the Gamma function at the half-integers, we then obtain the formula

$$\begin{aligned} C_{2,1}^{(C/N)}(S,K,r,\sigma,\tau) \,&=\, \frac{e^{-r\tau}}{\alpha} \left[1 + \sqrt{\frac{2}{\pi}} \sum_{p=0}^{+\infty} \frac{(-1)^p}{2^p p!(2p+1)} \left(\frac{k}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau} \right)^{2p+1} \right] \\ &=\, e^{-r\tau} N \left(\frac{k}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau} \right). \end{aligned} \tag{61}$$

Let us now consider the cash-or-nothing put option $P_{\alpha,\gamma}^{(C/N)}(S,K,r,\mu_{\alpha,\gamma},\tau)$.

Proposition 4. *For any $c_1 \in (-\alpha, 0)$, the integral representation:*

$$\begin{aligned} &P_{\alpha,\gamma}^{(C/N)}(S,K,r,\mu_{\alpha,\gamma},\tau) \,= \\ &\qquad \frac{e^{-r\tau}}{\alpha} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma(\frac{t_1}{\alpha})\Gamma(1-\frac{t_1}{\alpha})\Gamma(-t_1)}{\Gamma\left(1-\frac{\gamma}{\alpha}t_1\right)\Gamma(\frac{\alpha-1}{\alpha}t_1)\Gamma(1-\frac{\alpha-1}{\alpha}t_1)} \, (k+\mu_{\alpha,\gamma}\tau)^{t_1} (-\mu_{\alpha,\gamma}\tau^{\gamma})^{-\frac{t_1}{\alpha}} \, \frac{\mathrm{d}t_1}{2i\pi} \end{aligned} \tag{62}$$

holds true.

Proof. The proof is similar to the one given for the proposition 3. The only difference is that one has to replace the payoff by $\mathcal{P}^{(C/N)}(S,K) = H(K-S)$ and to consider the Green function on the negative real axis. Using the symmetry property (15), it reads as follows:

$$g_{\alpha,\gamma}(x,t) \,=\, \frac{1}{\alpha x} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma(\frac{t_1}{\alpha})\Gamma(1-\frac{t_1}{\alpha})\Gamma(1-t_1)}{\Gamma\left(1-\frac{\gamma}{\alpha}t_1\right)\Gamma(\frac{\alpha-1}{\alpha}t_1)\Gamma(1-\frac{\alpha-1}{\alpha}t_1)} \left(\frac{x}{(-\mu_{\alpha,\gamma}t)^{\frac{1}{\alpha}}} \right)^{t_1} \frac{\mathrm{d}t_1}{2i\pi}. \tag{63}$$

□

The integral for the put price (62) needs more efforts than the integral for the call price (57). Indeed, it possesses two distinct series of poles in the positive half-plane:

- The poles of $\Gamma(1 - \frac{t_1}{\alpha})$ at the points $t_1 = \alpha(1+n), n \in \mathbb{N}$, whose residues are given by the formula:

$$\frac{(-1)^n\Gamma(-\alpha(1+n))}{n!\Gamma(1-\gamma(1+n))\Gamma((\alpha-1)(1+n))\Gamma(1-(\alpha-1)(1+n))} (k+\mu_{\alpha,\gamma}\tau)^{\alpha(1+n)} (-\mu_{\alpha,\gamma}\tau^{\gamma})^{-(1+n)}. \tag{64}$$

- The poles of $\Gamma(-t_1)$ at the points $t_1 = n, n \in \mathbb{N}$, whose residues are given by the formula:

$$\frac{(-1)^n \Gamma(\frac{n}{\alpha})\Gamma(1-\frac{n}{\alpha})}{n!\Gamma(1-\frac{\gamma}{\alpha}n)\Gamma(\frac{\alpha-1}{\alpha}n)\Gamma(1-\frac{\alpha-1}{\alpha}n)}(k+\mu_{\alpha,\gamma}\tau)^n(-\mu_{\alpha,\gamma}\tau^{\gamma})^{-\frac{n}{\alpha}}. \tag{65}$$

However, in the non-time-fractional case, it is possible to derive a simple representation for the put price.

Theorem 4 (Pricing formula: space fractional cash or nothing put)**.** *Let $1 < \alpha \leq 2$. Then, under the maximal negative asymmetry hypothesis ($\theta = \alpha - 2$), the price of the cash-or-nothing call option driven by the space fractional diffusion equation is as follows:*

$$P^{(C/N)}_{\alpha,1}(S,K,r,\mu_\alpha,\tau) \;=\; \frac{e^{-r\tau}}{\alpha}\left[(\alpha-1) \;-\; \sum_{n=1}^{\infty} \frac{1}{n!\Gamma(1-\frac{n}{\alpha})}\;(k\;+\;\mu_\alpha\tau)^n\;(-\mu_\alpha\tau)^{-\frac{n}{\alpha}}\right]. \tag{66}$$

Proof. In the space-fractional case, the parameter γ is equal to one. Then, we have the following simplifications:

- In the denominator of (64), $\Gamma(1-\gamma(1+n))$ reduces to $\Gamma(-n)$, which is infinite for any $n \in \mathbb{N}$. Therefore, (64) is equal to null for any $n \in \mathbb{N}$.
- (65) simplifies to the form:

$$\frac{(-1)^n\Gamma(\frac{n}{\alpha})}{n!\Gamma(\frac{\alpha-1}{\alpha}n)\Gamma(1-\frac{\alpha-1}{\alpha}n)}(k+\mu_\alpha\tau)^n(-\mu_\alpha\tau)^{-\frac{n}{\alpha}}. \tag{67}$$

  Both the numerator and the denominator of (67) are singular for $n = 0$, but their quotient is not singular:

$$\frac{\Gamma(\frac{n}{\alpha})}{\Gamma(\frac{\alpha-1}{\alpha}n)} \underset{n\to 0}{\sim} \frac{\frac{\alpha}{n}}{\frac{\alpha}{(\alpha-1)n}} = \alpha - 1. \tag{68}$$

- Finally, using the functional relation $\Gamma(1-n+\frac{n}{\alpha}) = (-1)^{n-1}\frac{\Gamma(\frac{n}{\alpha})}{(1-\frac{n}{\alpha})(2-\frac{n}{\alpha})\dots(n-1-\frac{n}{\alpha})}$, we can simplify Expression (67):

$$\frac{(-1)^n\Gamma(\frac{n}{\alpha})}{n!\Gamma(\frac{\alpha-1}{\alpha}n)\Gamma(1-\frac{\alpha-1}{\alpha}n)}(k+\mu_\alpha\tau)^n(-\mu_\alpha\tau)^{-\frac{n}{\alpha}} \;=\; -\frac{1}{n!\Gamma(1-\frac{n}{\alpha})}\;(k\;+\;\mu_\alpha\tau)^n\;(-\mu_\alpha\tau)^{-\frac{n}{\alpha}} \tag{69}$$

and the proof is complete. □

Note that in the space fractional case, the call Formula (59) can be written in the form:

$$C^{(C/N)}_{\alpha,1}(S,K,r,\mu_\alpha,\tau) \;=\; \frac{e^{-r\tau}}{\alpha}\left[1\;+\;\sum_{n=1}^{\infty}\frac{1}{n!\Gamma(1-\frac{\gamma}{\alpha}n)}\;(k\;+\;\mu_\alpha\tau)^n\;(-\mu_\alpha\tau)^{-\frac{n}{\alpha}}\right] \tag{70}$$

and therefore, the sum of the space fractional call and put options is given by the formula:

$$C^{(C/N)}_{\alpha,1}(S,K,r,\mu_\alpha,\tau)\;+\;P^{(C/N)}_{\alpha,1}(S,K,r,\mu_\alpha,\tau)\;=\;\frac{e^{-r\tau}}{\alpha}\,[1\;+\;(\alpha-1)]\;=\;e^{-r\tau}. \tag{71}$$

The relation (71) is an example of a *call-put parity relation*. The fact, that it is independent of α is not a surprise since, as already mentioned in the previous section, the space fractional model is equivalent to the FLMS model for which the option prices admit a risk-neutral representation:

$$C^{(C/N)}_{\alpha,1}(S,K,r,\mu_\alpha,\tau) + P^{(C/N)}_{\alpha,1}(S,K,r,\mu_\alpha,\tau) = e^{-r\tau}\,\mathbb{E}^{\mathbb{Q}}[H(S-K)+H(K-S)] \tag{72}$$
$$= e^{-r\tau}\,\mathbb{E}^{\mathbb{Q}}[1] \tag{73}$$
$$= e^{-r\tau}. \tag{74}$$

Of course, it does not hold true for $\gamma \neq 1$ because no probabilistic interpretation exists in this case. Finally, we note that it follows from the Formulas (71) and (61) that in the Black-Scholes case ($\alpha = 2$, $\gamma = 1$), we have the relation:

$$P^{(C/N)}_{2,1}(S,K,r,\sigma,\tau) = e^{-r\tau}\left[1 - N\left(\frac{k}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau}\right)\right] \tag{75}$$
$$= e^{-r\tau}\,N\left(-\frac{k}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}\right) \tag{76}$$

as predicted by the conventional Black-Scholes theory.

3.3.2. Asset-or-Nothing

Let us denote by $C^{(A/N)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau)$ the price of the binary cash-or-nothing option in the fractional diffusion model with the payoff $\mathcal{P}^{(A/N)}(S,K) = S \times H(S-K)$.

Proposition 5. *Let P be the polyhedron $P := \{(t_1,t_2) \in \mathbb{C}^2\,,\, Re(t_2 - t_1) > 1\,,\, 0 < Re(t_2) < 1\}$. Then, for any vector $\underline{c} = (c_1, c_2) \in P$, the integral representation:*

$$C^{(A/N)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau) =$$
$$\frac{Ke^{-r\tau}}{\alpha} \int\limits_{\underline{c}+i\mathbb{R}^2} (-1)^{-t_2}\, \frac{\Gamma(t_2)\Gamma(1-t_2)\Gamma(-t_1+t_2)}{\Gamma(1-\frac{\gamma}{\alpha}t_1)}\, (-k-\mu_{\alpha,\gamma}\tau)^{t_1-t_2} (-\mu_{\alpha,\gamma}\tau^\gamma)^{-\frac{t_1}{\alpha}}\, \frac{\mathrm{d}t_1}{2i\pi} \wedge \frac{\mathrm{d}t_2}{2i\pi} \tag{77}$$

holds true.

Proof. To prove the formula, we replace the payoff function in the option price Formula (16) by:

$$\mathcal{P}^{(A/N)}(Se^{(r+\mu_{\alpha,\gamma})\tau+x}, K) = S\,e^{(r+\mu_{\alpha,\gamma})\tau+x} H(Se^{(r+\mu_{\alpha,\gamma})\tau+x} - K) \tag{78}$$
$$= K\,e^{k+\mu_{\alpha,\gamma}\tau+x}\,\mathbf{1}_{\{x \geq -k-\mu_{\alpha,\gamma}\tau\}} \tag{79}$$

and then proceed exactly as in the proof of Proposition 2. □

Theorem 5 (Pricing formula: asset or nothing call)**.** *Let $1 < \alpha \leq 2$ and $1 - \frac{1}{\alpha} < \gamma \leq \alpha$. Then, under the maximal negative asymmetry hypothesis ($\theta = \alpha - 2$), the price of the asset-or-nothing call option driven by the space-time fractional diffusion equation is given by the formula:*

$$C^{(A/N)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau) = \frac{Ke^{-r\tau}}{\alpha} \sum_{\substack{n=0\\m=0}}^{\infty} \frac{1}{n!\Gamma(1+\gamma\frac{m-n}{\alpha})} \,(k + \mu_{\alpha,\gamma}\tau)^n\,(-\mu_{\alpha,\gamma}\tau^\gamma)^{\frac{m-n}{\alpha}}\,. \tag{80}$$

Proof. The series (80) is obtained by summing the residues associated with the poles of the functions $\Gamma(t_2)$ and $\Gamma(-t_1+t_2)$ exactly in the same manner as in the proof of Theorem 2. □

As a consequence of Theorems 2, 3 and 5, the European call can be represented as a difference between an asset-or-nothing call and a cash-or-nothing call:

$$C^{(E)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau) \;=\; C^{(A/N)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau) \;-\; K\,C^{(C/N)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau). \tag{81}$$

4. Risk Sensitivities and Portfolio Hedging

In quantitative finance, the risk sensitivities are also known as the "Greeks". They quantify the dependence of an option on the market parameters such as the asset (spot) price or volatility and are essential tools for portfolio management. In this section, we derive for them some efficient representations and study how the orders of the time- and space-fractional derivatives impact the hedging policies.

4.1. First Order Sensitivity (Delta)

4.1.1. European Call

From the definition of k, we have the relation $\frac{\partial k}{\partial S} = \frac{1}{S}$. Thus, by differentiation of (46) with respect to S and re-arranging the terms, we obtain the formula:

$$\begin{aligned}\Delta_{C^{(E)}_{\alpha,\gamma}}(S,K,r,\mu_{\alpha,\gamma},\tau) \;&:=\; \frac{\partial C^{(E)}_{\alpha,\gamma}}{\partial S}(S,K,r,\mu_{\alpha,\gamma},\tau) \\ &=\; \frac{e^{-k}}{\alpha}\sum_{\substack{n=0\\m=0}}^{\infty}\frac{1}{n!\Gamma(1+\gamma\frac{m-n}{\alpha})}\,(k\;+\;\mu_{\gamma,\alpha}\tau)^n\,(-\mu_{\gamma,\alpha}\tau^{\gamma})^{\frac{m-n}{\alpha}},\end{aligned} \tag{82}$$

where we have used the definition (31) for the moneyness to deduce the relation $\frac{Ke^{-r\tau}}{S} = e^{-k}$. When the asset is ATM forward ($S = Ke^{-r\tau}$, and therefore, $k=0$), Formula (82) can be simplified:

$$\begin{aligned}\Delta_{C^{(E,ATM)}_{\alpha,\gamma}}(S,K,r,\mu_{\alpha,\gamma},\tau) \;&=\; \frac{1}{\alpha}\sum_{\substack{n=0\\m=0}}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\gamma\frac{m-n}{\alpha})}\,(-\mu_{\alpha,\gamma})^{\frac{(\alpha-1)n+m}{\alpha}}\,\tau^{\frac{(\alpha-\gamma)n+\gamma m}{\alpha}} \\ &=\; \frac{1}{\alpha}\left[1\;+\;\frac{(-\mu_{\alpha,\gamma})^{\frac{1}{\alpha}}\tau^{\frac{\gamma}{\alpha}}}{\Gamma(1+\frac{\gamma}{\alpha})}\;-\;\frac{(-\mu_{\alpha,\gamma})^{1-\frac{1}{\alpha}}\tau^{1-\frac{\gamma}{\alpha}}}{\Gamma(1-\frac{\gamma}{\alpha})}\;+\;O\left(-\mu_{\alpha,\gamma}\tau\right)\right].\end{aligned} \tag{83}$$

Equation (83) shows that the Delta of at-the-money calls is driven by $\frac{1}{\alpha}$ (see also Figure 1), which generalizes the well-known feature of the Black-Scholes model, where at-the-money call options have a Delta of $\frac{1}{2}$. In particular, (83) demonstrates that, at first order, only the space fractional parameter α influences the delta-hedging policy of a portfolio. Indeed, it is sufficient to be long of one unit of the asset S and short of α units of an European call to offset the impact of the variations of S:

$$\Pi \;:=\; S \;-\; \alpha C^{(E,ATM)}_{\alpha,\gamma}(S,K,r,\mu_{\alpha,\gamma},\tau) \;\Longrightarrow\; \left.\frac{\partial \Pi}{\partial S}\right|_{S=S_0} \;=\; 0, \tag{84}$$

where $S_0 = Ke^{-r\tau}$ is the ATM forward price.

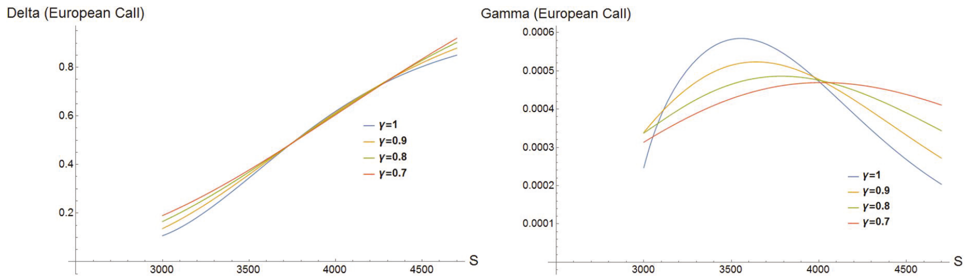

Figure 1. In the left figure, we plot the evolution of the Delta of an European call option as a function of S for various values of γ. We observe that, as expected from (83), the At-The-Money (ATM) call price is approximately equal to $\frac{1}{\alpha}$ for any γ, and moreover, the Delta value is the same on a wide range of prices around S_0, independently of γ. The situation is very different on the right figure (plot of the Gamma of the European call option (93)), where sensitivities vary greatly in dependence of γ. To produce the figures, the following values of parameters were used: $K = 4000$, $r = 1\%$, $\sigma = 20\%$, $\tau = 1Y$, $\alpha = 2$.

4.1.2. Cash-or-Nothing Call

Differentiation of Expression (59) with respect to S leads to the formula:

$$\Delta_{C^{(C/N)}_{\alpha,\gamma}}(S,K,r,\mu_{\alpha,\gamma},\tau) \;=\; \frac{e^{-k}}{\alpha K}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(1-\frac{\gamma}{\alpha}(n+1))}\,(k\,+\,\mu_{\alpha,\gamma}\tau)^n\,(-\mu_{\alpha,\gamma}\tau^\gamma)^{-\frac{n+1}{\alpha}}. \tag{85}$$

In the at-the-money forward situation, we then get:

$$\begin{aligned}\Delta_{C^{(C/N,ATM)}_{\alpha,\gamma}}(S,K,r,\mu_{\alpha,\gamma},\tau) \;&=\; \frac{(-\mu_{\alpha,\gamma}\tau^\gamma)^{-\frac{1}{\alpha}}}{\alpha K}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1-\frac{\gamma}{\alpha}(n+1))}\,(-\mu_{\alpha,\gamma})^{\frac{\alpha-1}{\alpha}n}\,\tau^{\frac{\alpha-\gamma}{\alpha}n}\\ &=\; \frac{(-\mu_{\alpha,\gamma}\tau^\gamma)^{-\frac{1}{\alpha}}}{\alpha K}\left[\frac{1}{\Gamma(1-\frac{\gamma}{\alpha})} - \frac{(-\mu_{\alpha,\gamma})^{1-\frac{1}{\alpha}}\,\tau^{1-\frac{\gamma}{\alpha}}}{\Gamma(1-\frac{2\gamma}{\alpha})} + O\left((-\mu_{\alpha,\gamma})^{2-\frac{2}{\alpha}}\,\tau^{2-\frac{2\gamma}{\alpha}}\right)\right].\end{aligned} \tag{86}$$

This formula can be simplified when $\gamma \to \alpha$ (the so-called *neutral diffusion*). Using the known asymptotic behavior of the Gamma function at the point 0, we then get:

$$\Delta_{C^{(C/N,ATM)}_{\alpha,\gamma}}(S,K,r,\mu_{\alpha,\gamma},\tau) \underset{\gamma\to\alpha}{\sim} \frac{\alpha-\gamma}{\alpha^2}\,\frac{(-\mu_{\alpha,\gamma}\tau^\gamma)^{-\frac{1}{\alpha}}}{K}. \tag{87}$$

Another prominent particular case of Formula (86) is for $\alpha = 2$ and $\gamma = 1$ (the Black-Scholes model). In this case, (86) reduces to the form:

$$\Delta_{C^{(C/N,ATM)}_{2,1}}(S,K,r,\sigma,\tau) \;=\; \frac{1}{K\sqrt{2\pi}}\frac{1}{\sigma\sqrt{\tau}} \,+\, O(\sigma\sqrt{\tau}). \tag{88}$$

4.1.3. Cash-or-Nothing Put

In the space-fractional case ($\gamma = 1$), it follows from the parity relation (71) that the cash-or-nothing call and put options have opposite Deltas:

$$\Delta_{P^{(C/N)}_{\alpha}}(S,K,r,\mu_\alpha,\tau) \;=\; -\,\Delta_{C^{(C/N)}_{\alpha}}(S,K,r,\mu_\alpha,\tau) \tag{89}$$

$$= \;-\,\frac{e^{-k}}{\alpha K}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(1-\frac{\gamma}{\alpha}(n+1))}\,(k\,+\,\mu_{\alpha,\gamma}\tau)^n\,(-\mu_{\alpha,\gamma}\tau^\gamma)^{-\frac{n+1}{\alpha}}. \tag{90}$$

4.1.4. Asset-or-Nothing Call

Differentiation of the relation (81) with respect to S along with Formulas (82) and (85) lead to the following expression for the Delta of the asset-or-nothing call:

$$\Delta_{C_{\alpha,\gamma}^{(A/N)}}(S,K,r,\mu_{\alpha,\gamma},\tau) = \Delta_{C_{\alpha,\gamma}^{(E)}}(S,K,r,\mu_{\alpha,\gamma},\tau) + K\Delta_{C_{\alpha,\gamma}^{(C/N)}}(S,K,r,\mu_{\alpha,\gamma},\tau) \tag{91}$$

$$= \frac{e^{-k}}{\alpha}\sum_{\substack{n=0\\m=0}}^{\infty}\frac{1}{n!\Gamma(1+\gamma\frac{m-n-1}{\alpha})}\,(k+\mu_{\gamma,\alpha}\tau)^n\,(-\mu_{\gamma,\alpha}\tau^\gamma)^{\frac{m-n-1}{\alpha}}. \tag{92}$$

4.2. Second Order Sensitivity (Gamma, Dollar Gamma)

By the definition of k, $\frac{\partial e^{-k}}{\partial S} = \frac{\partial k}{\partial S}e^{-k} = \frac{e^{-k}}{S}$. Thus, differentiation of (82) with respect to S along with a re-arrangement of the terms give the formula:

$$\Gamma_{C_{\alpha,\gamma}^{(E)}}(S,K,r,\mu_{\alpha,\gamma},\tau) := \frac{\partial^2 C_{\alpha,\gamma}^{(E)}}{\partial S^2}(S,K,r,\mu_{\alpha,\gamma},\tau)$$

$$= \frac{e^{-k}}{\alpha S}\sum_{\substack{n=0\\m=0}}^{\infty}\left[\frac{(-\mu_{\gamma,\alpha}\tau^\gamma)^{\frac{m-n-1}{\alpha}}}{\Gamma(1+\gamma\frac{m-n-1}{\alpha})} - \frac{(-\mu_{\gamma,\alpha}\tau^\gamma)^{\frac{m-n}{\alpha}}}{\Gamma(1+\gamma\frac{m-n}{\alpha})}\right]\frac{(k+\mu_{\gamma,\alpha}\tau)^n}{n!}. \tag{93}$$

In the ATM forward situation, (93) is reduced to the form:

$$\Gamma_{C_{\alpha,\gamma}^{(E,ATM)}}(S,K,r,\mu_{\alpha,\gamma},\tau) =$$

$$\frac{1}{\alpha S}\sum_{\substack{n=0\\m=0}}^{\infty}\frac{(-1)^n}{n!}\left[\frac{(-\mu_{\alpha,\gamma})^{\frac{(\alpha-1)n+m-1}{\alpha}}\tau^{\frac{(\alpha-\gamma)n+\gamma(m-1)}{\alpha}}}{\Gamma(1+\gamma\frac{m-n-1}{\alpha})} - \frac{(-\mu_{\alpha,\gamma})^{\frac{(\alpha-1)n+m}{\alpha}}\tau^{\frac{(\alpha-\gamma)n+\gamma m}{\alpha}}}{\Gamma(1+\gamma\frac{m-n}{\alpha})}\right]. \tag{94}$$

The leading term in the expression at the right-hand side of (94) is the following one:

$$\widetilde{\Gamma}_{C_{\alpha,\gamma}^{(E,ATM)}}(S,\mu_{\alpha,\gamma},\tau) = \frac{1}{\alpha S}\frac{(-\mu_{\alpha,\gamma}\tau^\gamma)^{-\frac{1}{\alpha}}}{\Gamma(1-\frac{\gamma}{\alpha})}. \tag{95}$$

Let us assume that we are long of one European call option and that we have delta-hedged our portfolio when $S = S_0$ (according to (84), this is made by being short of $\frac{1}{\alpha}$ units of the asset S). We can then employ the Taylor formula to approximate the value of the portfolio when S varies:

$$C_{\alpha,\gamma}^{(E)}(S,K,r,\mu_{\alpha,\gamma},\tau) - C_{\alpha,\gamma}^{(E,ATM)}(S_0,K,r,\mu_{\alpha,\gamma},\tau) = \frac{1}{2}\widetilde{\Gamma}_{C_{\alpha,\gamma}^{(E,ATM)}}(S_0,\sigma,\tau)^2(S-S_0)^2 + O(S-S_0)^3. \tag{96}$$

The left-hand side of (96) is the *Profit and Loss*, or *P& L* of the portfolio around the money. As we have delta-hedged our position, this P&L is therefore essentially driven by the option's convexity, and one speaks of *Gamma P& L*. For $\alpha = 2$, we can use the approximation (39) of the risk-neutral parameter and its Taylor expansion with respect to γ to obtain the following easy-to-use P& L formulas (γ_E denotes the Euler–Mascheroni constant):

- If $\gamma \to 0$:

$$P\&L(\gamma\to 0) = \frac{1-3\gamma_E\gamma}{8\sigma^2\tau^\gamma}\left(\frac{S-S_0}{S_0}\right)^2 + O\left(\gamma^2(S-S_0)^3\right). \tag{97}$$

- If $\gamma \to 1$:

$$P\&L(\gamma\to 1) = \frac{1-(-3+3\gamma_E+\log 4)(\gamma-1)}{4\pi\sigma^2\tau^\gamma}\left(\frac{S-S_0}{S_0}\right)^2 + O\left((\gamma-1)^2(S-S_0)^3\right). \tag{98}$$

- If $\gamma \to 2$:

$$P\&L(\gamma \to 2) \;=\; \frac{3(\gamma-2)^2}{4\sigma^2\tau^\gamma}\left(\frac{S-S_0}{S_0}\right)^2 + O\left((\gamma-2)^3(S-S_0)^3\right). \tag{99}$$

The term $\left(\frac{S-S_0}{S_0}\right)^2$ is the square of the underlying asset's returns and, as such, is interpreted as the realized variance. Its multiplicative factor is often called the *dollar Gamma (\$Γ)*. By definition, with our notations, it reads:

$$\$\Gamma \;=\; \frac{1}{2}S_0^2\widetilde{\Gamma}_{C_{\alpha,\gamma}^{(E,ATM)}}(S_0,\mu_{\alpha,\gamma},\tau)^2. \tag{100}$$

The Gamma P&L can therefore be written down in the form:

$$P\&L \;=\; \$\Gamma \,\times\, \text{realized variance}. \tag{101}$$

From Equations (97)–(99), we obtain the following particular values for the dollar Gamma:

$$\begin{cases} \gamma \;=\; 0: & \$\Gamma \;=\; \dfrac{1}{8\sigma^2} + O(\gamma), \\ \gamma \;=\; 1: & \$\Gamma \;=\; \dfrac{1}{4\pi\sigma^2\tau} \quad \text{(the Black-Scholes Dollar Gamma)}, \\ \gamma \;=2: & \$\Gamma \;=\; 0. \end{cases} \tag{102}$$

We can observe that, for any γ, the graph of the dollar Gamma remains positive and therefore works in favor of the long call position. In other words, the P&L of the call option will outperform the one of the hedge, because the option price is a convex function of the asset price. Thus, this well-known feature of the conventional Black-Scholes theory is preserved in the framework of the time-fractional diffusion model. The presence of the parameter γ, however, significantly affects the impact of option's maturity to the Gamma P&L. As seen in (102), the Black-Scholes dollar Gamma is equal to $\frac{1}{4\pi\sigma^2\tau}$ and therefore is maximal for short-term options (i.e., for small τ). However, when γ tends to zero, then $\$\Gamma$ tends to $\frac{1}{8\sigma^2}$ and becomes independent of τ. Similarly, when $\gamma \to 2$, the option will no longer outperform the hedge, and the P&L of the portfolio will remain flat, whatever the option's maturity. This phenomenon illustrates the temporal redistribution induced by the time-fractional parameter γ. Finally, Figure 2 demonstrates that the Gamma P&L is a decreasing function of γ as soon as $\tau \geq \frac{2}{\pi}$, and there is an inflection at the point $\gamma = 1$:

$$\left.\frac{\partial^2(\$\Gamma)}{\partial\gamma^2}\right|_{\gamma=1} \;=\; 0. \tag{103}$$

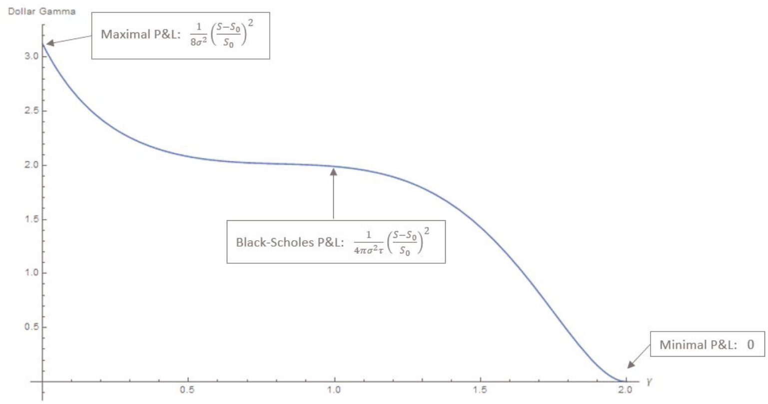

Figure 2. The dollar Gamma for the time-fractional diffusion model ($\alpha = 2, \gamma \in (0, 2]$) maximizes the P&L when $\gamma \to 0$ and offsets the impact of maturity τ at extremal values ($\gamma = 0$, $\gamma = 2$). The graph of the Dollar Gamma possesses an inflection point at the point $\gamma = 1$ (the Black-Scholes model). To produce the figure, the following values of parameters were used: $\sigma = 20\%$, $\tau = 1Y$.

5. Conclusions

In this article, we presented a theory of option pricing based on the fractional diffusion equation and showed that it constituted a generalization of the well-known class of the exponential market models. In particular, we extended the pricing formulas that were previously established for the vanilla options to a basic class of exotic options, computed the related risk sensitivities, and applied the results to delta-hedging and P&L calculations.

The pricing formulas were derived in the form of fast converging series of powers of the log-forward moneyness and of the risk-neutral parameter that generalizes the volatility parameter to the non-Gaussian case and also admits a convenient series representation. These series can be easily used for calculations in practice without the help of any sophisticated numerical tools. Moreover, they clearly exhibit the impact of the model parameters to risks and hedging: the order α of the space derivative governs the delta-hedging of the portfolio, while the order γ of the time derivative drives the P&L of this portfolio.

Other important problems include the development and analysis of similar analytic tools for other types of commonly-traded payoffs (especially for the path-dependent options) and other types of financial derivatives. We also are going to apply these models to the real market data. In order to verify the effectiveness of our approach, we first need to determine the reliable estimations of the model parameters and then to compare the numerical results produced by the model with the market data. Another interesting problem concerns relaxing of the maximal asymmetry hypothesis. However, this would imply divergence of the risk-neutral expectations, and thus, some suitable re-normalization procedures such as, say, the cut-offs of the probability densities, would be probably needed to make progress in the solution of this problem.

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mathematics

Article

Econophysics and Fractional Calculus: Einstein's Evolution Equation, the Fractal Market Hypothesis, Trend Analysis and Future Price Prediction

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Abstract: This paper examines a range of results that can be derived from Einstein's evolution equation focusing on the effect of introducing a Lévy distribution into the evolution equation. In this context, we examine the derivation (derived exclusively from the evolution equation) of the classical and fractional diffusion equations, the classical and generalised Kolmogorov–Feller equations, the evolution of self-affine stochastic fields through the fractional diffusion equation, the fractional Poisson equation (for the time independent case), and, a derivation of the Lyapunov exponent and volatility. In this way, we provide a collection of results (which includes the derivation of certain fractional partial differential equations) that are fundamental to the stochastic modelling associated with elastic scattering problems obtained under a unifying theme, i.e., Einstein's evolution equation. This includes an analysis of stochastic fields governed by a symmetric (zero-mean) Gaussian distribution, a Lévy distribution characterised by the Lévy index $\gamma \in [0, 2]$ and the derivation of two impulse response functions for each case. The relationship between non-Gaussian distributions and fractional calculus is examined and applications to financial forecasting under the fractal market hypothesis considered, the reader being provided with example software functions (written in MATLAB) so that the results presented may be reproduced and/or further investigated.

Keywords: Einstein's evolution equation; Kolmogorov–Feller equation; diffusion equation; fractional diffusion equation; self-affine stochastic fields; random market hypothesis; efficient market hypothesis; fractal market hypothesis; financial time series analysis; evolutionary computing.

1. Introduction

We study one of the principal field equations in statistical mechanics, namely, Einstein's evolution equation (EEE or E^3). This is done in order to derive mathematical models and thereby specific financial indices in a unified manner, an approach which includes the use of fractional calculus.

E^3 models the random motion and (elastic) interactions of a canonical ensemble of particles. It provides a description for the time evolution of the spatial density field that represents the concentration of such particles in a macroscopic sense. In an n-dimensional space, each particle is taken to be undergoing a random walk in which the direction that a particle "propagates" after a "scattering event" (in which energy and momentum are conserved) is random together with the length of propagation. The scattering angle θ is taken to be conform to a distribution of angles $\Pr[\theta(\mathbf{r})],\ \mathbf{r} \in \mathbb{R}^n$ and the (free) propagation length is taken to conform to some distribution of lengths $\Pr[L(\mathbf{r})]$ whose mean value defines the mean free path (MFP). This was the basis for Albert Einstein's original study of Brownian motion in 1905 [1], albeit for the one-dimensional case.

In addition to the work of Josiah Gibbs, the evolution equation that Einstein derived is one the foundations of statistical mechanics [2,3]. The approach can, for example, be applied equally well to modelling the diffusion of light propagating through a complex of scatterers. In this case the light is taken to be a ray-field where each ray (reflected from one particle to another) has a random path length and scattering angle.

1.1. Focus and Context

The focus of this paper is to derive a range of equations and metrics via an n-dimensional version of E^3 in order to demonstrate an inherent connectivity and association in a unified sense. These equations include the classical diffusion equation, the classical and generalised Kolmogorov–Feller equations and the evolution of self-affine stochastic fields through the fractional diffusion equation. The fractional form of these equations is shown to be a direct consequence of introducing non-Gaussian distributions as "governors" for the statistical characteristics under which random processes occur, subject to the condition that all such processes involve independent elastic interactions.

For certain non-Gaussian models such as Lévy processes, this leads naturally to the use of fractional calculus to develop solutions to the evolution equation as studied in this paper. Further, it is shown that such solutions are fundamental to the application of the fractal market hypothesis [4] for analysing financial time series and thereby in developing trading strategies based on this hypothesis. This approach represents an *Econophysics* methodology in which a fundamental model used to describe stochastic processes, originally developed in the study of Brownian motion, is used to solve problems in economics. In this paper, following developments published previously by Blackledge et al. (e.g., [5–14]), it is shown that this approach is inclusive of the application of fractional calculus.

1.2. Structure and Organisation

The structure of the paper is as follows. Section 2 provides a brief overview of the principal mathematical results used in this paper including basic definitions and notation. This section also includes a short introduction to fractional calculus, specifically some of the conventional definitions of a fractional integral and a fractional derivative. Section 3 presents E^3 upon which all the results derived in this paper are ultimately dependent, thereby providing a unifying framework for the work reported as discussed in Section 4, which provides a brief introduction to financial time series analysis in the context of E^3.

Two equations, that are a conditional representation of E^3, are considered in Section 5, namely the Classical Kolmogorov–Feller and the Generalised Kolmogorov–Feller equations which are studied later on in the paper, specifically in Section 14. In the context of E^3, Section 6 provides statements on the random walk hypothesis and the efficient and fractal market hypotheses coupled with a brief history associated with the development of such hypotheses for interpreting and analysing financial time series. As discussed in Section 3, E^3 is predicated on a model for the probability density function of a stochastic system using a continuous random walk model and Section 7 therefore introduces density functions whose basic properties are important to appreciate in the context of the work reported here. Sections 8 and 9 study the derivation from the E^3 of two metrics, namely, the Lyapunov exponent and the volatility, respectively. These metrics are then combined into a Lyapunov-to-volatility ratio (LVR) to develop a trend analysis algorithm which is presented in Section 10, the idea being to provide an indicator that flags when a financial time series changes its trending behaviour. This is based on a change in the polarity of the LVR and it is shown, for example, that in order to obtain suitable accuracies appropriate for algorithmic trading, both pre- (of the financial signal) and post-filtering (of the LVR) is required. This is quantified in Section 10 using a back-testing strategy. In addition to being bi-polar, the amplitude of the LVR has values that reflect periods of relative stability in the dynamic behaviour of a financial signal and in Section 11, a method is proposed to exploit this indication and provide short term predictions on future prices using the principles of evolution computing (EC). In this paper, EC is implemented using an online resource and applications package called 'Eureqa'.

The remaining sections of the paper deal with the classical and fractional diffusion equations, both of which are derived from E^3 in Sections 12 and 13 using Gaussian and non-Gaussian (Lévy) distributions, respectively. In the latter case, and, using the principles of fractional calculus established in Section 2, a time series model is developed that depends upon the Lévy index. Section 14 then provides a complementary approach to deriving similar results using the Generalised Kolmogorov–Feller equation and an orthonormal memory function which yields the same scaling properties compounded in the impulse response function. The application of this index for financial trend analysis is provided in Section 15, illustrating that the Lyapunov exponent and the Lévy index have similar predictive power providing the data is pre- and post-filtered. Section 16 provides a review and discussion of the results presented followed by a general conclusion and some open questions to direct future research.

1.3. Original Contributions

Judging from the open literature, and, to the best of the authors' knowledge, the approach taken in this paper is original as are the numerical results presented. In regard to the latter case, an effort has been made by the authors to integrate important numerical functions with the derivation of certain important metrics associated with the theoretical models used and the mathematical analysis presented. These functions are given in Appendix A and their aim is to provide the reader with the opportunity to reproduce the results presented (the online data sources being referenced throughout) and investigate their performance for different financial data.

2. Mathematical Preliminaries

In this section, we provide a short overview of some of the mathematical results that are of importance to the material developed in this paper, specifically the short introduction to fractional calculus provided in Section 2.3.

2.1. Fourier Transformation and the Convolution Integral

The mathematical models developed in this paper rely on the properties of the Fourier transform coupled with the convolution and correlation integrals in n-dimensions. For a square integrable function $f(\mathbf{r}) \in L^2(\mathbb{R}^n) : \mathbb{C} \to \mathbb{C}$, we define the Fourier and inverse Fourier transforms in the "non-unitary form" as

$$F(\mathbf{k}) = \mathcal{F}_n[f(\mathbf{r})] \equiv \int_{-\infty}^{\infty} f(\mathbf{r}) \exp(-i\mathbf{k}\cdot\mathbf{r}) d^n\mathbf{r}$$

and

$$f(\mathbf{r}) = \mathcal{F}_n^{-1}[F(\mathbf{k})] \equiv \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} F(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r}) d^n\mathbf{k}$$

respectively. Here, $\mathbf{r}$ is the n-dimensional spatial vector where $r \equiv \mid \mathbf{r} \mid = (r_1^2 + r_2^2 + ... + r_n^2)^{\frac{1}{2}}$. Similarly, $\mathbf{k}$ is the spatial frequency vector where $k \equiv \mid \mathbf{k} \mid = 2\pi/\lambda$ for wavelength λ and $\mathbf{k}\cdot\mathbf{r} = k_1 r_1 + k_2 r_2 + ... + k_n r_n$. These integral transforms define a Fourier transform pair which, in this paper, we write using the notation

$$F(\mathbf{k}) \leftrightarrow f(\mathbf{r}).$$

We define the (n-dimensional) Dirac delta function as

$$\delta^n(\mathbf{r}) = \mathcal{F}_n^{-1}[1] \equiv \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \exp(i\mathbf{k}\cdot\mathbf{r}) d^n\mathbf{k} \tag{1}$$

where, with $\otimes$ denoting the convolution integral,

$$f(\mathbf{r}) = \delta^n(\mathbf{r}) \otimes f(\mathbf{r}),$$

the convolution of two functions $f(\mathbf{r})$ and $g(\mathbf{r})$ being given by

$$s(\mathbf{r}) = g(\mathbf{r}) \otimes f(\mathbf{r}) \equiv \int_{-\infty}^{\infty} g(\mathbf{r}-\mathbf{s}) f(\mathbf{s}) d^n\mathbf{s}$$

and their correlation by

$$s(\mathbf{r}) = g(\mathbf{r}) \odot f(\mathbf{r}) \equiv \int_{-\infty}^{\infty} g(\mathbf{r}+\mathbf{s}) f(\mathbf{s}) d^n\mathbf{s}$$

where $[s(\mathbf{r}), g(\mathbf{r}), f(\mathbf{r})] \in L^2(\mathbb{R}^n) : \mathbb{C} \to \mathbb{C}$. Note that the dimension associated with the integral operators $\otimes$ and $\odot$ is taken to be inferred from the dimension of the functions to which these operators are applied. In addition, note that, strictly speaking, the Fourier transform is taken over a Schwartz tempered distributional space, and, in this context, the following theorems are fundamental:

(i) Convolution Theorem

$$g(\mathbf{r}) \otimes f(\mathbf{r}) \leftrightarrow G(\mathbf{k})F(\mathbf{k})$$

where $G(\mathbf{k}) \leftrightarrow g(\mathbf{r})$ and $F(\mathbf{k}) \leftrightarrow f(\mathbf{r})$.

(ii) Correlation Theorem

$$g(\mathbf{r}) \odot f(\mathbf{r}) \leftrightarrow G^*(\mathbf{k})F(\mathbf{k})$$

(iii) Product Theorem

$$g(\mathbf{r})f(\mathbf{r}) \leftrightarrow \frac{1}{(2\pi)^n} G(\mathbf{k}) \otimes F(\mathbf{k})$$

We note the following Fourier transform relationships [15]

$$| x |^{\alpha} \leftrightarrow -\frac{2 \sin\left(\frac{\pi\alpha}{2}\right) \Gamma(1+\alpha)}{| k |^{1+\alpha}} \tag{2}$$

$$\frac{1}{(\mp ix)^{\alpha}} \leftrightarrow \frac{2\pi}{\Gamma(\alpha)} \frac{H(\pm k)}{(\pm k)^{1-\alpha}}, \ 0 < \alpha < 1, \tag{3}$$

where $H(k)$ is the Heaviside step function

$$H(k) = \begin{cases} 1, & k \geq 0; \\ 0, & k < 0. \end{cases} \ \Rightarrow \ \frac{d}{dx} H(k) = \delta(x)$$

and for $\mathbf{r} \in \mathbb{R}^n$

$$\frac{1}{| \mathbf{r} |^{\alpha}} \leftrightarrow \frac{(2\pi)^n}{c_{n,\alpha}} \frac{1}{| \mathbf{k} |^{n-\alpha}}, \ 0 < \mathrm{Re}[\alpha] < n; \ \ c_{n,\alpha} = \pi^{\frac{n}{2}} 2^{\alpha} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \tag{4}$$

where Γ is the Gamma function,

$$\Gamma(z) = \int\limits_0^{\infty} x^{z-1} \exp(-x) dx, \ \ \mathrm{Re}[z] > 0.$$

2.2. *The p- and Uniform-Norm*

We define the p-norm as

$$\|f(\mathbf{r})\|_p \equiv \left(\int\limits_{\mathbb{R}^n} | f(\mathbf{r}) |^p \, d^n\mathbf{r} \right)^{\frac{1}{p}}, \ 1 \leq p \leq \infty$$

with the uniform norm being given by

$$\|f(\mathbf{r})\|_{\infty} = \sup\{| f(\mathbf{r}) |, \mathbf{r} \in \mathbb{R}^n\}$$

and principal properties

$$\|f(\mathbf{r}) + g(\mathbf{r})\|_p \leq \|f(\mathbf{r})\|_p + \|g(\mathbf{r})\|_p, \ \ \|f(\mathbf{r})g(\mathbf{r})\|_p \leq \|f(\mathbf{r})\|_p \|g(\mathbf{r})\|_p$$

and

$$\|f(\mathbf{r}) \otimes g(\mathbf{r})\|_p \leq \|f(\mathbf{r})\|_p \|g(\mathbf{r})\|_p.$$

2.3. *Fractional Integrals and Differentials*

Since, for $n = 0, 1, 2, ...$,

$$\frac{d^{\pm n}}{dx^{\pm n}} f(x) \leftrightarrow (ik)^{\pm n} F(k)$$

we can, in principal, generalise this result to the case when n is non-integer. Thus, suppose we wish to fractionally integrate the differential equation

$$\frac{d^{\alpha}}{dx^{\alpha}} f(x) = g(x), \ 0 < \alpha < 1$$

to obtain a solution for $f(x)$ in terms of $g(x)$. Fourier transforming,

$$F(k) = \frac{G(k)}{(ik)^{\alpha}}, \ F(k) \leftrightarrow f(x), \ G(k) \leftrightarrow g(x)$$

and thus, using the convolution theorem, we can write

$$f(x) = h(x) \otimes g(x), \; h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(ikx)}{(ik)^{\alpha}} dk = \frac{H(x)}{\Gamma(\alpha)} \frac{1}{x^{1-\alpha}}$$

using Relationship (3).

This important result is easily derived by expressing the inverse Fourier transform in terms of a Bromwich integral so that, with $p = ik$, we can write $h(x)$ in terms of the inverse Laplace transform

$$h(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\exp(px)}{p^{\alpha}} dp.$$

Generalising the Laplace transform of the function x^n (for positive integer n) given by

$$\int_{0}^{\infty} x^n \exp(-px) dx = \frac{n!}{p^{1+n}} \;\Rightarrow\; \int_{0}^{\infty} x^{n-1} \exp(-px) dx = \frac{\Gamma(n)}{p^n}, \; \Gamma(n) = (n-1)!$$

to

$$\int_{0}^{\infty} x^{\alpha-1} \exp(-px) dx = \frac{\Gamma(\alpha)}{p^{\alpha}} \Rightarrow \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} \frac{\exp(-px)}{x^{1-\alpha}} dx = \frac{1}{p^{\alpha}}$$

it is then clear that

$$\frac{1}{(ik)^{\alpha}} = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} \frac{H(x)}{x^{1-\alpha}} \exp(-ikx) dx.$$

This expression for $f(x)$ in terms of the convolution $h(x) \otimes g(x)$ is the basic fractional integral known as the Riemann–Liouville integral which, specifying the limits of integration, takes the form

$${}_aD_x^{-\alpha} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f(y)}{(x-y)^{1-\alpha}} dy \tag{5}$$

thereby expressing the integral in terms of an inverse differential operator $D^{-\alpha}$ over the limits a and x. This allows us to express a fractional differential denoted by the operator ${}_aD_x^{\alpha} \equiv d^{\alpha}/dx^{\alpha}$ in terms of a fractional integral by noting that

$$\begin{aligned} {}_aD_x^{\alpha} f(x) &= D_x^1 \, {}_aD_x^{\alpha-1} f(x) = D_x^1 \, {}_aD_x^{-(1-\alpha)} f(x) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{a}^{x} \frac{f(y)}{(x-y)^{\alpha}} dy = \frac{-\alpha}{\Gamma(1-\alpha)} \int_{a}^{x} \frac{f(y)}{(x-y)^{1+\alpha}} dy \end{aligned} \tag{6}$$

When α is a negative value and noting that $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$,

$${}_aD_x^{-\alpha} f(x) = \frac{\alpha}{\Gamma(1+\alpha)} \int_{a}^{x} \frac{f(y)}{(x-y)^{1-\alpha}} dy = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f(y)}{(x-y)^{1-\alpha}} dy$$

thereby recovering the expression for a fractional integral given by Equation (5). Thus, combining the results, we can write

$${}_aD_x^{\pm\alpha} f(x) = \frac{1}{\Gamma(\mp\alpha)} \int_{a}^{x} \frac{f(y)}{(x-y)^{1\pm\alpha}} dy. \tag{7}$$

Since, for some scaling value λ (and with $z = \lambda y$),

$$\frac{1}{\Gamma(\mp\alpha)}\int_a^x \frac{f(\lambda y)}{(x-y)^{1\pm\alpha}}dy = \frac{1}{\Gamma(\mp\alpha)}\int_a^x \frac{f(z)}{[x-(z/\lambda)]^{1\pm\alpha}}d(z/\lambda) = \frac{\lambda^{\pm\alpha}}{\Gamma(\mp\alpha)}\int_a^x \frac{f(z)}{(\lambda x - z)^{1\pm\alpha}}dz,$$

this operator has the self-affine scaling characteristic

$$ {}_aD_x^{\pm\alpha} f(\lambda x) = \lambda^{\pm\alpha}{}_aD_x^{\pm\alpha} f(x). \tag{8}$$

Another related approach to defining a fractional differential is through application of the delta function. For $\mathbf{r} \in \mathbb{R}^1$

$$f(x) = \delta(x) \otimes f(x) \;\text{ and }\; f^{(n)}(x) = \delta^{(n)}(x) \otimes f(x) \;\text{ where }\; f^{(n)}(x) \equiv \frac{d^n}{dx^n}f(x)$$

Generalising this result to the non-integer case, we write

$$f^{(\alpha)}(x) = \delta^{(\alpha)}(x) \otimes f(x),$$

where, from Equation (1),

$$\delta^{(\alpha)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (ik)^\alpha \exp(ikx)dk.$$

We can then write

$$\begin{aligned}\delta^{(\alpha)}(x) &= \frac{d}{dx}\delta^{(\alpha-1)}(x) = \frac{d}{dx}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(ik)^{1-\alpha}}\exp(ikx)dk \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\frac{H(x)}{x^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{\delta(x)}{x^\alpha} - \frac{\alpha}{\Gamma(1-\alpha)}\frac{H(x)}{x^{1+\alpha}}.\end{aligned} \tag{9}$$

A further definition of a fractional differential can be obtained using the sign function $\text{sgn}(x)$ where

$$\text{sgn}(x) = \frac{x}{\mid x \mid} = \begin{cases} +1, & x > 0; \\ -1, & x < 0; \\ 0, & x = 0. \end{cases}$$

and

$$\text{sgn}(x) \leftrightarrow \frac{2}{ik},$$

when we can write

$$f^{(\alpha)}(x) = \frac{1}{2}f(x) \otimes \text{sgn}^{(1+\alpha)}(x), \; \forall\alpha.$$

This result becomes clear if we note that

$$\frac{1}{2}f(x) \otimes \text{sgn}^{(1+\alpha)}(x) = \frac{1}{2}f(x) \otimes \text{sgn}^{(1+\alpha)}(x) \otimes \delta(x) = \frac{1}{2}f(x) \otimes \text{sgn}(x) \otimes \delta^{(1+\alpha)}(x)$$

and therefore that

$$\frac{1}{2}f(x) \otimes \text{sgn}(x) \otimes \delta^{(1+\alpha)}(x) \leftrightarrow \frac{1}{2}F(k)\frac{2}{ik}(ik)^{1+\alpha} = (ik)^\alpha F(k).$$

Defining a fractional differential and integral in terms of the operators ${}_aD_x^\alpha$ and ${}_aD_x^{-\alpha}$, respectively, is based on a generalisation of the Fourier transform under differentiation and integration, respectively. Traditional (integer) calculus goes hand-in-hand with a geometrical interpretation of the associated operations, starting with a differential defining the gradient of a function at a point (at least for a piecewise continuous function). With fractional calculus, generalisations of this type do not easily lend themselves to a geometrical interpretation. However, geometric and physical interpretation of fractional derivatives have been developed (e.g., [16,17]) including the connectivity between fractional calculus and fractal geometry [18] which is based on the scaling relationship compounded in Equation (8). An important characteristic of these interpretations that is relevant to the remit of this paper, is that the operator ${}_aD_x^{-\alpha}$ operating on a stochastic function is characterised by this scaling property, a property that yields self-affine stochastic fields or random scaling fractals.

As discussed later on in this paper, many financial signals can be classified as random scaling fractal signals (with a fractal dimension $D \in [1, 2]$). This is the basis for the fractal market hypothesis in mathematical economics and hence, the applications of fractional calculus. Note however, that the "process" of generalising the Fourier transform used above for defining fractional differentials and integrals is just one such generalisation that can be applied. Thus, the operators defined by Equations (5) and (6), for example, are not unique and there are many definitions and generalisations of a fractional derivative that have been developed [19] and continue to be so [20].

Although there are, in principle, an unlimited number of definitions that may be "designed" to define a fractional derivative, there is a common theme to all of them which is that they are expressed in terms of a convolution. For example, the Caputo fractional derivative is given by

$${}_aD_x^{-\alpha} f(x) = \int\limits_a^x K_\alpha(x-y) f^{(n)}(y) dy \quad \text{where} \quad K_\alpha(x-y) = \frac{(x-y)^{n-\alpha-1}}{\Gamma(n-\alpha)},$$

which is easily formulated via application of the inverse Fourier transform given that if

$$f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \exp(ikx) F(k) dk$$

and

$$D^{-\alpha} f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\exp(ikx)}{(ik)^\alpha} F(k) dk = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{(ik)^n \exp(ikx)}{(ik)^{n+\alpha}} F(k) dk$$

then from Relationship (3),

$$D^{-\alpha} f(x) = \frac{1}{2\pi} \frac{d^n}{dx^n} \int\limits_{-\infty}^{\infty} \frac{\exp(ikx)}{(ik)^{n+\alpha}} F(k) dk = \frac{d^n}{dx^n} \frac{H(x)}{\Gamma(n+\alpha)} \frac{1}{x^{1-(n+\alpha)}} \otimes f(x)$$

and hence

$$D^{\alpha} f(x) = \frac{H(x)}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \frac{1}{x^{1-(n-\alpha)}} \otimes f(x) = \frac{H(x)}{\Gamma(n-\alpha)} \frac{1}{x^{1-(n-\alpha)}} \otimes f^{(n)}(x).$$

The results considered here are fundamental to the implementation of fractional calculus in econophysics (and physics in general) as they are predicated on the Fourier transform which arguably plays the most pivotal role of all in so many aspects of physics and especially in the analysis and processing of signals, e.g., [21,22], including financial signals.

Irrespective of the non-unique definition of a fractional derivative, there is one fundamental difference between a classical and a fractional derivative which is characterised by Equation (7), for example. A nth order derivative of a piecewise continuous function $f(x)$ can be defined at a single point on x at x_0 say, and is independent of any other values of $f(x)$ for $x < x_0$ or $x > x_0$. However, given that a fractional derivative involves the convolution of the function $f(x)$ with $1/x^{1+\alpha}$, for example, its value at a point x_0 depends on prior values of $f(x)$ for $x < x_0$. Thus the value of a fractional derivative of $f(x)$ depends on its "history" and thus, unlike an integer derivative, a fractional derivative therefore incurs "memory". This "memory effect" is another way of approaching the analysis of financial signals using fractional calculus as financial signals are influenced by the memory of past financial conditions, albeit within a stochastic context. This is a key element to the analysis of financial signals using fractional calculus and a fundamental component to applications of the fractal market hypothesis (as discussed later on in this paper).

3. Einstein's Evolution Equation

Let $p(\mathbf{r})$ denote a probability density function (PDF) where

$$\int_{-\infty}^{\infty} p(\mathbf{r})d^n\mathbf{r} = 1,$$

which characterises with the position of particles in an n-dimensional space $\mathbf{r} \in \mathbb{R}^n$ where, at any instant in time t, the particles exist as a result of some "random walk" generated by a sequence of "elastic scattering" processes (with other like particles in the same n-dimensional space) that have occurred over some period of time $< t$. Further, let $u(\mathbf{r}, t)$ denote the density function associated with a canonical assemble of particles all undergoing the same random walk process (i.e., the number of particles per unit space, e.g., per unit volume for $n = 3$).

Consider the initial condition where we have an infinitely small concentration of such particles at a time $t = 0$ located at the origin $\mathbf{r} = \mathbf{0}$. The density function at $t = 0$ is then given by $u(\mathbf{r}, 0) = \delta^n(\mathbf{r})$ where $\delta^n(\mathbf{r})$ is the n-dimensional Dirac delta function. At some short time later $t = \tau << 1$, it can be expected that the density function will be determined by the PDF governing the distribution of particles after a (short duration) random walk. Thus we can write

$$u(\mathbf{r}, \tau) = p(\mathbf{r}) \otimes u(\mathbf{r}, 0) = p(\mathbf{r}) \otimes \delta^n(\mathbf{r}) = p(\mathbf{r}),$$

where $\otimes$ denotes the convolution integral over all $\mathbf{r}$. The PDF $p(\mathbf{r})$ therefore represents the response (in a statistical sense) associated with a short time random walk process, and, in this context, can be considered to be a statistical impulse response function (IRF). Thus for any time t, the density field at some later time $t + \tau$ will be given by

$$u(\mathbf{r}, t + \tau) = p(\mathbf{r}) \otimes u(\mathbf{r}, t). \tag{10}$$

For any instant in time t, Equation (10) shows that the spatial behaviour of the density field at some future time τ is given by the convolution of the density of particles at a previous time with the PDF of the system that governs its "statistical evolution". In this sense, $p(\mathbf{r})$ is analogous to the IRF of a linear stationary system when, for an initial condition $u_0(\mathbf{r}) \equiv u(\mathbf{r}, t = 0)$, say,

$$u(\mathbf{r}, t) = g(r, t) \otimes u_0(\mathbf{r}, t)$$

where $g(r, t)$ is the characteristic Green's function of the system. However, in this case $u(\mathbf{r}, t)$ denotes a deterministic function associated with the behaviour of a deterministic system, whereas in Equation (10), $u(\mathbf{r}, t)$ is the density function associated with the evolution of a statistical system. This "system" is taken to be stationary in a statistical sense because it is assumed that $p(\mathbf{r})$ does not vary in time and the time evolution model given by Equation (10) is referred to as being "Ergodic". Further, we note that if the PDF is symmetric, then $p(\mathbf{r}) \equiv p(r)$.

Equation (10) is Einstein's evolution equation (E^3). It is a "master equation" for elastic scattering processes in statistical mechanics and is an example of a continuous time random walk model. On the basis of Equation (10), one can derive a variety of stochastic field equations as shall be shown later on in this paper.

In regard to the continuous time random walk model given by Equation (10), $p(\mathbf{r})$ is the PDF for the displacement $\mathbf{r}$ of a particle's position over time interval τ. The equivalent discrete time random walk model, Equation (10) takes the form

$$u(\mathbf{r}_m, t_n + \tau) = p(\mathbf{r}_m) \otimes u(\mathbf{r}_m, t_n)$$

where $\mathbf{r}_m$ and t_n are discrete vector and scalar arrays, respectively, and, $\otimes$ denotes the convolution sum. In this case, τ is fixed time step and, in the context of the work reported in this paper, may be considered to be a time-unit for financial markets, i.e., a minute, hour, day or week associated with a price value $u(t_n)$.

For a source function $s(\mathbf{r}, t)$ (a source density), which may be a stochastic function, the evolution equation is

$$u(\mathbf{r}, t + \tau) = p(\mathbf{r}) \otimes u(\mathbf{r}, t) + s(\mathbf{r}, t). \tag{11}$$

This equation describes the evolution of of the density function $u(\mathbf{r}, t)$ when the initial particle concentration is replenished in space and/or time and can be extended further to include a decay factor over time when it is required to consider an evolution equation of the type (for decay rate factor R)

$$u(\mathbf{r}, t + \tau) = p(\mathbf{r}) \otimes u(\mathbf{r}, t) + s(\mathbf{r}, t) - Ru(\mathbf{r}, t) \tag{12}$$

The financial time series models and metrics that are considered in this paper are all derived from Equation (11) and for this reason, in the following section, a short introduction to financial time series analysis is provided. This is necessary for readers to appreciate the focus of the application that is considered in this paper.

4. Financial Time Series Analysis

A financial time series is a discrete set of price values that are most commonly regular samples over a specific time interval (minutes, hours, days, etc.) which depend on the financial price index available (e.g., world-wide indices such as FTSE100, S & P 500, FOREX, etc.). Over longer time intervals, the price index is usually an average of the samples taken over the next smallest time interval. Most financial data is available as a time series and therefore developing mathematical models (both linear and non-linear) of time series data is an essential component underpinning many aspects of mathematical finance leading to algorithms for day-to-day trading, forecasting and econometrics in general.

There are numerous internet resources that provide up-to-date and historical data of different indices over different time scales such as the data available at [23] which is the internet source used to access the data presented in this paper. Similarly, there are numerous "metrics" (also called a financial index) which are the result of processing samples of data over a look-back window of a specified length usually known as the "period". Such metrics range from statistical metrics based on an autoregressive moving average and nonlinear locally non-constant variance models (applicable to volatile financial returns, interest, exchange rates and futures) through to descriptive techniques for various features, such as long term level fluctuations and distributions, short and long memory dependence, directionality and volatility.

Methods of fitting time series models to time series data and their statistical validation determine the application to which they can (or otherwise) be successfully applied to forecasting, systematic trading, fund manager evaluation, hedging and simulation for example. The online resource 'Investopedia' [24] provides descriptions, computational algorithms and examples of the numerous metrics, indices and other parameters that have, and are continuing to be, developed for financial time series analysis.

In this paper, continuous time series models for a financial signal denoted by $u(t)$ are derived exclusively from Equation (11), the associated discrete time series model being denoted by $u_n,\ n = 1, 2, ..., N$ which is taken to describe a digital financial signal consisting of N elements.

5. Einstein's Evolution Equation and the Kolmogorov–Feller Equations

The Classical and Generalised Kolmogorov–Feller Equations can be derived directly from E^3 through application of a Taylor series in time and a memory function (in time), respectively. They are in fact representations of E^3 for the case when $\tau << 1$ and otherwise, respectively, as shall now be shown, both equations being studied later on in this paper.

5.1. The Classical Kolmogorov–Feller Equation

Consider the following Taylor series for the function $u(\mathbf{r}, t + \tau)$ in Equation (10):

$$u(\mathbf{r}, t + \tau) = u(\mathbf{r}, t) + \tau \frac{\partial}{\partial t} u(\mathbf{r}, t) + \frac{\tau^2}{2!} \frac{\partial^2}{\partial t^2} u(\mathbf{r}, t) + ...$$

For $\tau << 1$

$$u(\mathbf{r}, t + \tau) \simeq u(\mathbf{r}, t) + \tau \frac{\partial}{\partial t} u(\mathbf{r}, t)$$

and from Equation (10), we obtain the classical Kolmogorov–Feller equation (CKFE), [25,26]

$$\tau \frac{\partial}{\partial t} u(\mathbf{r}, t) = -u(\mathbf{r}, t) + u(\mathbf{r}, t) \otimes p(r), \tag{13}$$

which is essentially a representation of Equation (10) for $\tau << 1$.

Equation (13) is based on a critical assumption which is that the time evolution of the density field $u(\mathbf{r}, t)$ is influenced only by short term events and that longer term events have no influence on the behaviour of the field at any time t, i.e., the "system" described by Equation (13) has no "memory". This statement is the physical basis upon which the condition $\tau << 1$ is imposed, thereby facilitating the Taylor series expansion of the function $u(\mathbf{r}, t + \tau)$ to first order alone.

5.2. The Generalised Kolmogorov–Feller Equation

Given that Equation (13) is memory invariant, the question arises as to how longer term temporal influences can be modelled, other than by taking an increasingly larger number of terms in the Taylor expansion of $u(\mathbf{r}, t + \tau)$ which is not of practical analytical value, i.e., writing Equation (10) in the form

$$\tau \frac{\partial}{\partial t} u(\mathbf{r}, t) + \frac{\tau^2}{2!} \frac{\partial^2}{\partial t^2} u(\mathbf{r}, t) + ... = -u(\mathbf{r}, t) + u(\mathbf{r}, t) \otimes p(r).$$

The key to solving this problem is to express the infinite series on the left hand side of the equation above in terms of a "memory function" $\mathrm{m}(t)$ and write

$$\tau \mathrm{m}(t) \otimes \frac{\partial}{\partial t} u(\mathbf{r}, t) = -u(\mathbf{r}, t) + u(\mathbf{r}, t) \otimes p(r).$$

This is the generalised Kolmogorov–Feller equation (GKFE) which reduces to the CKFE when $\mathrm{m}(t) = \delta(t)$.

A characteristic time spectrum $M(\omega)$ for $m(t)$ can be obtained by noting that we have, in effect, considered the result

$$u(\mathbf{r}, t + \tau) = u(\mathbf{r}, t) + \tau \frac{\partial}{\partial t} u(\mathbf{r}, t) + \frac{\tau^2}{2!} \frac{\partial^2}{\partial t^2} u(\mathbf{r}, t) + ... = u(\mathbf{r}, t) + \tau \mathrm{m}(t) \otimes \frac{\partial}{\partial t} u(\mathbf{r}, t)$$

so that, after taking the Fourier transform with respect to t, we obtain

$$U(\mathbf{r},\omega)\exp(i\omega\tau) = U(\mathbf{r},\omega) + i\omega\tau U(\mathbf{r},\omega) + \frac{1}{2!}(i\omega\tau)^2 U(\mathbf{r},\omega) + ... = U(\mathbf{r},\omega) + i\omega\tau M(\omega)U(\mathbf{r},\omega)$$

where $U(\mathbf{r},\omega) \leftrightarrow u(\mathbf{r},t)$ and $M(\omega) \leftrightarrow m(t)$, from which it follows that we can write $M(\omega)$ as

$$M(\omega) = \sum_{n=1}^{\infty} \frac{1}{n!}(i\omega\tau)^{n-1} = \frac{\exp(i\omega\tau) - 1}{i\omega\tau}$$

5.3. Orthonormal Memory Functions

For any inverse function or class of inverse functions of the type $n(t)$, say, such that

$$n(t) \otimes m(t) = \delta(t),$$

the GKFE can be written in the form

$$\tau\frac{\partial}{\partial t}u(\mathbf{r},t) = -n(t) \otimes u(\mathbf{r},t) + n(t) \otimes u(\mathbf{r},t) \otimes p(\mathbf{r}), \tag{14}$$

where the GKFE is again recovered when $n(t) = \delta(t)$ given that $\delta(t) \otimes \delta(t) = \delta(t)$. The function $n(t)$ is an orthonormal function of $m(t)$.

6. The Random Walk, the Efficient and the Fractal Market Hypotheses

From Equation (11) we can generate a simple (continuous) financial time series model by integrating over $\mathbf{r}$ to obtain

$$u(t+\tau) = u(t) + s(t), \tag{15}$$

where

$$u(t+\tau) = \int_{-\infty}^{\infty} u(\mathbf{r}, t+\tau)d^n\mathbf{r}, \quad s(t) = \int_{-\infty}^{\infty} s(\mathbf{r},t)d^n\mathbf{r}$$

and, for $p(\mathbf{r}) = \delta^n(\mathbf{r})$,

$$u(t) = \int_{-\infty}^{\infty} [\delta^n(\mathbf{r}) \otimes u(\mathbf{r},t)]d^n\mathbf{r} = \int_{-\infty}^{\infty} u(\mathbf{r},t)d^n\mathbf{r}.$$

If $s(t)$ is taken to be a (bi-polar) stochastic function of time and $u(t)$ is some price value (of some commodity) then Equation (15) describes the case in which a future price at some future time $t+\tau$ is given by the known price at time t plus some random price value $s(t)$. Note that for any value of t, $s(t)$ may be a positive or negative value thereby giving a higher or lower price value at $t+\tau$. The principal point here is that although Equation (15) is the simplest of models for price variation, it can nevertheless be seen to be the result of a spatial integration of E^3 when $p(\mathbf{r}) = \delta^n(\mathbf{r})$. Moreover, it is a model that encompasses some of the earliest questions associated with the dynamics of a free market economy as discussed in the following section.

6.1. The Random Walk Hypothesis

In 1900, Louis Bachelier [27] concluded that the price of a commodity today is the best estimate of its price in the future (at least in the short term). The random behaviour of commodity prices was again noted by Holbrook Working in 1934 [28] in an analysis of time series data. In the 1950s, Maurice Kendall [29] attempted to find periodic cycles in the financial time series of various securities and commodities but did not observe any. Prices appeared to be yesterday's price plus some random change (up or down); he suggested that price changes were independent and that they followed random walks. Thus the first models conceived for price variation were based on the sum of independent random variations often referred to as Brownian motion and quantified in Equation (15). This led to the creation of the random walk hypothesis, and the closely related efficient market hypothesis which states that random price movements indicate a well-functioning or efficient market.

An example of the type of time series that illustrates this effect is given in Figure 1. The figure shows a signal obtained using a zero mean Gaussian random number generated to compute s_n based on the iteration

$$u_{n+1} = u_n + s_n, \ u_1 = 100, \ n = 1, 2, 3, ..., 999.$$

Trivial though this model is, it nevertheless provides remarkably similar signals to those that characterise many financial signals. However, it is an example of a stationary signal in the sense that the scale of random deviations is invariant of time and the trends (up and down) are over similar amplitude and time scales—characteristics that are not properties of financial signals in general, at least over large time scales.

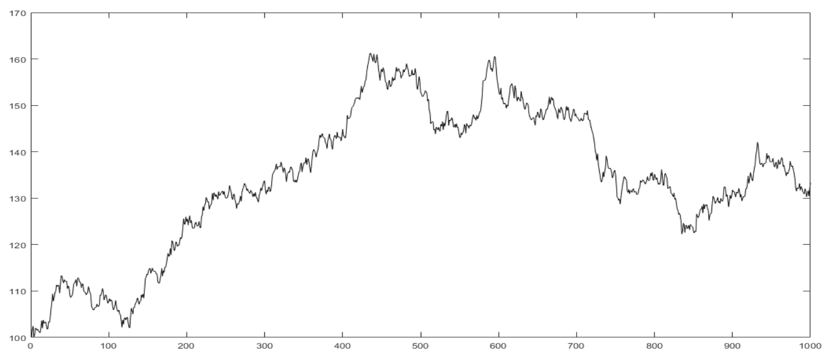

Figure 1. Simulation of a financial signal based on the sum of independent random walks; basis for the Random Walk Hypothesis.

6.2. The Efficient Market Hypothesis

It is often stated that asset prices should follow Gaussian random walks because of the efficient market hypothesis (EMH), e.g., [30–32] (and references therein). The EMH states that the current price of an asset fully reflects all available information relevant to it and that new information is immediately incorporated into the price. Thus, in an efficient market, models for asset pricing are concerned with the arrival of new information which is taken to be independent and random.

The EMH implies independent price increments, but why should they be Gaussian distributed? A Gaussian PDF is chosen because price movements are presumed to be an aggregation of smaller ones and sums of independent random contributions have a Gaussian PDF due to the central limit theorem. This is equivalent to arguing that all financial time series used to construct an “averaged signal” such as the FTSE100 or Dow Jones Industrial Average are statistically independent. Such an argument is not fully justified because it assumes that the reaction of investors to one particular stock market is independent of investors in other stock markets which, in general, will not be the case as each investor may have a common reaction to economic issues that transcend any particular stock. In other words, asset management throughout the markets relies on a high degree of connectivity and the arrival of new information can send "shocks" through the market as people react to it and then to each other’s reactions.

The EMH assumes that there is a rational and unique way to use available information, that all agents possess this knowledge and that any chain reaction produced by a “shock” happens instantaneously. This is clearly not physically possible or financial viable and financial models that are based on such a hypothesis have and will continue to fail.

6.3. The Fractal Market Hypothesis

One of the principal concerns with regard to the EMH relates to the issue of assuming that the markets are Gaussian distributed. This is because it has long been known that financial time series (specifically price changes) do not adhere to a Gaussian distribution and this is arguably the most important of the shortcomings relating to the EMH model (i.e., the failure of the independence and the Gaussian distribution of increments assumption). It is fundamental to the inability for EMH-based analysis such as the Black-Scholes model [33] to explain the characteristics of financial signals such as clustering, flights and failure to explain "boom-bust" events, and, in particular, financial "crashes" leading to recession.

More recently, financial time series have been shown to be random self-affine signals which has led to the related development of the fractal market hypothesis in which price variations are in effect random walks whose statistical distribution of values is similar over different time scales. Ralph Elliott (a professional accountant) first reported on the apparent self-affine properties of financial data in 1938 [34,35]. He was the first to observe that segments of financial time series data of different sizes could be scaled in such a way that they were statistically the same, producing so-called Elliot waves. He proposed that trends in financial prices resulted from investors' predominant psychology and found that swings in mass psychology always seemed to be a manifestation of the same recurring self-affine patterns in financial markets.

A primary goal of an investor is to attempt to obtain information that can provide some confidence in the immediate future of a commodity's price, based on patterns of the past. One of the principal components of this goal is based on the observation that there are "waves within waves" that appear to permeate financial signals when studied in sufficient detail and imagination. It is these repeating self-affine wave patterns that occupy both the financial investor and the financial systems modeller alike and it is clear that although economies have undergone many changes in the last 100 years, the dynamics of market data does not appear to have changed significantly (ignoring scale).

The Elliott wave principal developed in the late 1930s and the fractal market hypothesis developed in the late 1990s provide data consistent models for the interpretation and analysis of financial signals and investment theory. In turn, and, as discussed in this paper, fractal signals and fields can be cast in terms of solutions to certain fractional differential equations for which an understanding of the fractional calculus is a pre-requisite. Hence, the application of fractional calculus is and is likely to continue to have a primary role in mathematical economics.

In this context, and, on the basis of Equation (11), an overview of the contents of this paper and its subject connectivity is quantified in terms of the flow diagram given in Table 1 where the discrete time dependent behaviour of $u(t)$ is taken to represent a digital financial time series $u_n,\ n = 1, 2, ..., N$. This flow diagram highlights the relationship between the E^3 and the applications of fractional calculus in mathematical economics which is a theme of this paper. It is illustrative of the unified approach that has been taken in order to produce a coherent exposition for the development of three fundamental indices that are used to analyse financial signals, namely, the Lyapunov exponent, the volatility and the Lévy index. As shall be studied later on in this paper, these indices are used to undertake a trend analysis which, in turn, provides a confidence criterion for the application of evolutionary computing to predict future prices.

6.4. Principal Properties of Financial Signals

Whatever the hypothesis that is considered in regard to understanding and analysing financial signals, there are some basic characteristics of such signals that are common. These include the following:

- financial signals are stochastic signals;
- they are non-stationary signals;
- their distributions (specifically the price differences) are non-Gaussian;

- they are often characterized by long term historical correlations;
- they have random repeating patterns at different scales—they are statistically self-affine (random fractals);
- they have instabilities at all scales—sometimes referred to a "Lévy flights".

The models, metrics and computation algorithms reported in this paper attempt to take each of the above properties into account while maintaining adherence to E^3 as a unifying theme.

Table 1. Fow diagram illustrating the connectivity between Einstein's Evolution Equation (E^3), two well known financial indices (i.e., the volatility σ and Lyapunov exponent λ) and the classical and fractional diffusion equations both of which can be derived from the evolution equation using the Characteristic Functions (CFs) shown (where c is a constant, k is the spatial frequency and γ is the Lévy index). The flow diagram also illustrates the relationship between the evolution equation and two principal market hypotheses: the efficient market hypothesis and the fractal market hypothesis, the latter hypothesis being a concomitant of the fractional calculus. The asterisk (*) denotes the connection between the Generalised KFE and the introduction of a memory function which allows E^3 to be written in a different form without loss of generality.

Kolmogorov–Feller Equation (KFE)	$\leftarrow$	Taylor Series Analysis	$\rightarrow$	Generalised KFE*
		$\uparrow$		$\downarrow$
Lyapunov Exponent (λ)	$\leftarrow$	E^3		Volatility (σ)
		$\downarrow$		
Gaussian Distribution with CF $\exp(-c \mid k \mid^2)$	$\leftarrow$	Probability Density Function	$\rightarrow$	Lévy Distribution with CF $\exp(-c \mid k \mid^\gamma),\ \gamma \in [0,2]$
$\downarrow$				$\downarrow$
Classical Diffusion Equation				Fractional Diffusion Equation
$\updownarrow$				$\updownarrow$
Classical Calculus	$\leftrightarrow$	Memory Function*	$\leftrightarrow$	Fractional Calculus
$\updownarrow$				$\updownarrow$
Efficient Market Hypothesis				Fractal Market Hypothesis
$\downarrow$				$\downarrow$
Time Series Model	$\rightarrow$	Financial Trend Analysis based on time variations in σ, λ & γ	$\leftarrow$	Time Series Model
		$\downarrow$		
		Evolutionary Computing		
		$\downarrow$		
		Future Price Prediction		

7. Density Function Distributions

Suppose that the one-dimensional density function $u(x,t)$ is ergodic and has a PDF $p(x) \equiv \Pr[u(x,t)] \forall t$ where

$$\int_{-\infty}^{\infty} p(x)dx = 1.$$

If, for all time $t > 0$, the distributions of $u(y,t)$ and $u(z,t)$ are identical, what is the (symmetric) distribution of the density functions in the plane $\mathbf{r} \in \mathbb{R}^2$ and the volume $\mathbf{r} \in \mathbb{R}^3$?

It is clear that the cumulative distribution function of $u(x,t)$ is given by

$$c(x) = \int^{x} p(x)dx,$$

and hence, from the fundamental theorem of calculus

$$p(x) = \frac{d}{dx}c(x).$$

Thus, for $\mathbf{r} \in \mathbb{R}^2$, when $p(x) = p(y)$, the (circularly symmetric) cumulative distribution is (using polar coordinates (r, θ) with $r = \sqrt{x^2 + y^2}$)

$$c(r) = \int p(r) d^2 r = \int_0^{2\pi} \int_0^r p(r) r dr d\theta = 2\pi \int_0^r p(r) r dr$$

and so the PDF $p_2(r)$, say, is given by

$$p_2(r) = 2\pi \frac{d}{dr} \int_0^r p(r) r dr = 2\pi r p(r), \ r \in \mathbb{R}^2.$$

Similarly for $\mathbf{r} \in \mathbb{R}^3$, when $p(x) = p(y) = p(z)$, then for the spherically symmetric case (using spherical polar coordinates (r, θ, ϕ) with $r = \sqrt{x^2 + y^2 + z^2}$),

$$c(r) = \int p(r) d^3 r = \int_0^r \int_{-1}^1 \int_0^{2\pi} p(r) r^2 dr d(\cos\theta) d\phi = 4\pi \int_0^r p(r) r^2 dr$$

so that

$$p_3(r) = 4\pi \frac{d}{dr} \int_0^r p(r) r^2 dr = 4\pi r^2 p(r). \ r \in \mathbb{R}^3.$$

7.1. Gaussian and Rayleigh Distributions

In the case when $u(x,t), u(y,t)$ and $u(z,t)$ are (zero mean) Gaussian distributed and

$$p(x) = \frac{\exp[-x^2/(2\sigma^2)]}{\sqrt{2\pi\sigma^2}}$$

where σ is the standard deviation and when the characteristic function (CF) is given by [36]

$$P(k) = \mathcal{F}_n[p(x)] = \exp(-\sigma^2 k^2/2),$$

then

$$p_2(r) = \frac{r}{\sigma^2} \exp[-r^2/(2\sigma^2)], \ \mathbf{r} \in \mathbb{R}^2,$$

which is a standard Rayleigh distribution with characteristics function [36]

$$P_2(k) = -i\sqrt{2\pi}\sigma k \exp(-\sigma^2 k^2/2).$$

For the three dimensional case

$$p_3(r) = \frac{2}{\pi} \frac{r^2}{\sigma^3} \exp[-r^2/(2\sigma^2)], \ \mathbf{r} \in \mathbb{R}^3,$$

which has the CF [36]

$$P_3(k) = -2(\sigma^2 k^2 - 1) \exp(-\sigma^2 k^2/2).$$

The distributions $p_2(r)$ and $p_3(r)$ represent the random length of the two- and three-vectors respectively.

The case associated with $p_2(r)$ frequently occurs when a random time signal $u(t)$ has a distribution $p(x)$. By computing the Hilbert transform of this signal, we obtain the quadrature component $w(t)$ which has the same distribution as $u(t)$. The analytic signal $s(t)$, is then given by

$$s(t) = u(t) + iw(t) \text{ where } w(t) = \frac{1}{\pi t} \otimes u(t) \tag{16}$$

and the amplitude modulations given by $A(t) = \sqrt{u^2(t) + w^2(t)}$ are therefore $2\pi r p(r)$ distributed.

7.2. Lévy and Associated Distributions

The symmetric Lévy distribution features in material considered later on in this work and is a key to the connectivity between E^3 and the fractional diffusion equation. We therefore take the opportunity at this point in the paper to consider some of the basic definitions and results associated with this distribution. The CF of a (zero-mean) Gaussian distribution can be written as $P(k) = \exp(-c \mid k \mid^2)$ where $c \geq 0$ is a real constant (=$\sigma^2/2$). The Lévy distribution is one whose CF is based on a generalisation of the CF of a Gaussian distribution to

$$P(k) = \exp(-c \mid k \mid^\gamma),\ \gamma \in [0, 2] \tag{17}$$

where γ is the Lévy index. It is then clear that $\gamma = 2$ recovers a Gaussian PDF, $\gamma = 1$ generates a Cauchy distribution given that

$$\exp(-c \mid k \mid) \leftrightarrow \frac{1}{\pi c}\left(\frac{c^2}{c^2 + x^2}\right) \sim \frac{1}{x^2},\ x \to \infty$$

and it is noted that

$$\lim_{\gamma \to 0} \exp(-c \mid k \mid^\gamma) \leftrightarrow \delta(x).$$

For $\gamma \in (0, 2)$ it possible to derive the asymptotic result [37]

$$\exp(-c \mid k \mid^\gamma) \leftrightarrow \frac{1}{\mid x \mid^{1+\gamma}},\ x \to \infty.$$

A simple derivation of this result can be obtained by noting that

$$\begin{aligned} p(x) &= \mathcal{F}_1^{-1}[\exp(-c \mid k \mid^\gamma)] = \mathcal{F}_1^{-1}[1] + \sum_{n=1}^{\infty} (-1)^n \frac{c^n}{n!} \mathcal{F}_1^{-1}[\mid k \mid^{n\gamma}] \\ &= \delta(x) - \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{c^n}{n!} \left[\frac{\sin\left(\frac{\pi n \gamma}{2}\right) \Gamma(1 + n\gamma)}{\mid x \mid^{1+n\gamma}}\right] \sim \frac{1}{\mid x \mid^{1+\gamma}},\ \mid x \mid \to \infty \end{aligned}$$

using Relationship (2). The non-asymptotic Lévy distribution for arbitrary values of γ can easily be evaluated numerically through application of a discrete Fourier transform. Figure 2 shows examples of the Lévy distribution $p(x)$ for different values of γ (with $c = 2$) and associated distributions $xp(x)$ for the same values of γ but for $c = 1/2$. It is noted that the tails of each distribution for $\gamma < 2$ are longer than those for the case when $\gamma = 2$, thereby representing stochastic processes in which rare but extreme events are more likely to occur than with a Gaussian distributed process. These events include Lévy flights which, in financial time series analysis, mark positions in time when the value of a price may increase or decrease in a way that is inconsistent with the statistical signature of the series in a more general sense. An example of this is given in Figure 3 which shows Lévy flights in the complex plane associated with a FTSE 100 signal, the data having been obtained from [23].

Identifying metrics that can flag the positions in time at which a Lévy flight may occur are an important feature of financial trading. The first of these that we consider in this paper is the Lyapunov exponent which, in the context E^3, is discussed in the following section.

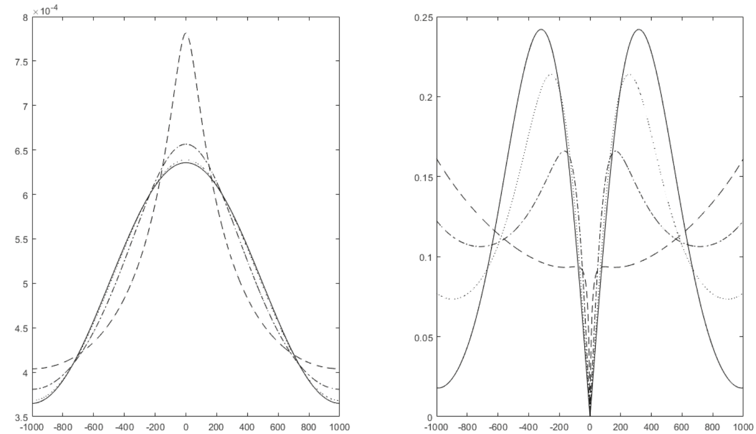

Figure 2. Left: Lévy distributions $p(x)$ for $\gamma = 2$ (solid line), $\gamma = 1.5$ (dotted line), $\gamma = 1$ (dot-dashed line), and $\gamma = 0.5$ (dashed line) for $c = 2$; **Right**: Plots of associated distributions $xp(x)$ for $\gamma = 2$ (solid line), $\gamma = 1.5$ (dotted line), $\gamma = 1$ (dot-dashed line) and $\gamma = 0.5$ (dashed line) for $c = 1/2$.

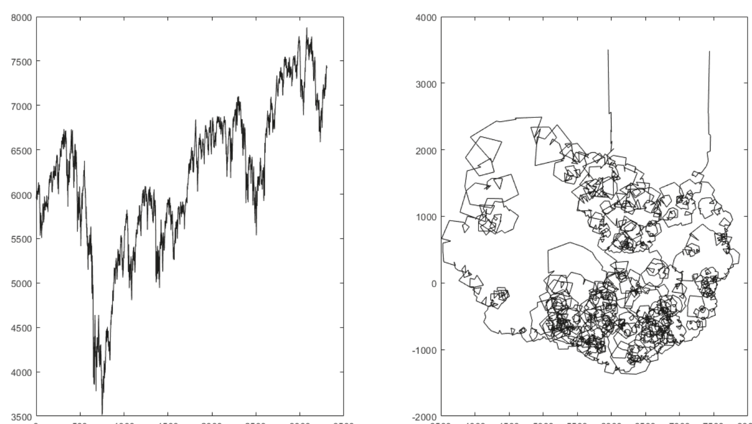

Figure 3. Example of Lévy flights. **Left**: Plot of the FTSE100 daily prices from 14/03/2006 to 12/04/2019—$u(t)$; **Right**: Complex plane plot of $s(t)$ given by Equation (16). Both time functions $u(t)$ and $s(t)$ are taken to be uniformly sampled discrete functions u_n and s_n, respectively; the analytic signal is computed using a fast Fourier transform (FFT) and the algorithm presented in [38].

8. The Lyapunov Exponent

The Lyapunov exponent is a quantity that characterises the rate of separation of infinitesimally close trajectories, a trajectory being a time-ordered set of states of a dynamical system.

Specifically, a trajectory is a sequence $f_n(x)$, $n \in \mathbb{N}$ calculated by the iterated application of a mapping f to an element x of its source. In this section, we illustrate the derivation of this exponent from E^3.

Consider some dynamical system that is modelled by an iterative equation characterised by a function $f(t, x_0)$ which produces a solution at some time t given by $x(t)$ for an initial condition $x_0 \equiv x(t = 0)$. The system is such that it may be stable or unstable depending on the initial condition x_0 and system parameters, i.e., the numerical value of the parameters that characterise the function $f(t, x_0)$. For stability, we can expect the solution $x(t)$ to be characterised by convergence to a specific value (which could be zero) so that as $t \to \infty$, $x(t) = x(t+\tau)$ where $\tau << 1$. If the solution is unstable we can expect $x(t)$ might increase in value with t and/or have some chaotic behaviour where $x(t)$ becomes a chaotic variable of time. In this case, a fundamental "diagnostic" is associated with asking the following question: Is a given system, characterised by the function $f(t, x_0)$, unstable, and, if so, how unstable is it? The answer to this question is compounded in the Lyapunov exponent, whose value is typically taken to be a measure of how sensitive $x(t)$ is to the initial condition x_0. If we denote $\delta x(t)$ to be some change to the solution which depends on a change to the initial condition denoted by δx_0, then this sensitivity is compounded in the following equation

$$\|\delta x(t)\|_p \sim \exp(\lambda t)\|\delta x_0\|_p, \tag{18}$$

where λ is referred to as the (leading) Lyapunov exponent and has the solution

$$\lambda \sim \frac{1}{t} \ln\left[\frac{\|\delta x(t)\|_p}{\|\delta x_0\|_p}\right]. \tag{19}$$

This exponent represents the mean rate of separation of trajectories of the system where the term "trajectory" refers to the time evolution of $x(t)$ subject to the initial condition x_0. Thus, any two trajectories $x(t) = f(t, x_0)$ and $x(t) + \delta x(t) = f(t, x_0 + \delta x_0)$, say, that are close to each other for $t << 1$ and consequently separate exponentially with time, will represent a system defined by function $f(t, x_0)$ that has a large value of λ. On the other hand, if all values of $x(t)$ and $x(t) + \delta x(t)$ converge to the same value in some neighbourhood of time, then $\delta x(t)$ must approach zero, and, from Equation (18), this implies that $\lambda < 0$. Thus, on the basis of Equation (18), a positive value of λ defines a system with chaotic behaviour in time and a negative value of λ characterises a stable system which convergences in time. Moreover, the larger the value of λ becomes the faster the rate of convergence (for $\lambda < 1$) or the "route to chaos" (for $\lambda > 1$), [39–41].

Given the description above as to what the Lyapunov exponent is and what it characterises, we consider a derivation of this exponent within the context of Equation (10) for $\mathbf{r} \in \mathbb{R}^3$ and uniform discretisation in time so that we can write

$$u(\mathbf{r}, t_{n+1}) = p(\mathbf{r}) \otimes u(\mathbf{r}, t_n), \; n = 0, \; 1, \; 2, \; ...., \; N. \tag{20}$$

Suppose that after many time steps, this iteration converges to the function $u(\mathbf{r}, t_\infty)$, say. We can then represent the iteration in the form

$$u(\mathbf{r}, t_n) = u(\mathbf{r}, t_\infty) + \epsilon(\mathbf{r}, t_n), \tag{21}$$

where $\epsilon(\mathbf{r}, t_n)$ denotes the error at any time step n. Convergence to the function $u(\mathbf{r}, t_\infty)$ then occurs if $\epsilon(\mathbf{r}, t_n) \to 0$ as $n \to \infty$. If we now consider a model for the error at each time step given by (for some real constant ε)

$$\epsilon(\mathbf{r}, t_{n+1}) = \varepsilon \exp(\lambda t_n) \tag{22}$$

with $t_n = n\tau$ (where τ defines the time sampling interval) it is clear that we can then write

$$\epsilon(\mathbf{r}, t_{n+1}) = \epsilon(\mathbf{r}, t_n) \exp(\lambda\tau),$$

or, after taking the $p = 1$-norm of both sides,

$$\bar{\epsilon}(t_{n+1}) = \bar{\epsilon}(t_n)\exp(\lambda\tau)$$

where

$$\bar{\epsilon}(t_n) = \|\epsilon(\mathbf{r}, t_n)\|_1.$$

Thus we can consider an expression for λ given by

$$\lambda = \lim_{N\to\infty} \frac{1}{N\tau} \sum_{n=1}^{N} \ln \frac{\bar{\epsilon}(t_{n+1})}{\bar{\epsilon}(t_n)}$$

If λ is negative, then the iterative process is stable since we can expect that for $n >> 1$, $\bar{\epsilon}(t_{n+1})/\bar{\epsilon}(t_n) < 1$ and thus $\ln \bar{\epsilon}(t_{n+1})/\bar{\epsilon}(t_n) < 0$. If λ is positive then the iterative process will diverge, such a criterion for convergence/divergence being dependent on the exponential model given in Equation (22) used to represent the error function at each iteration. This result applies for any time iteration process. However, in the case of Equation (20), we note that, if $u(\mathbf{r}, t_0) = \delta^3(\mathbf{r})$ then $u(\mathbf{r}, t_1) = p(\mathbf{r})$, $u(\mathbf{r}, t_2) = p(\mathbf{r}) \otimes u(\mathbf{r}, t_1) = p(\mathbf{r}) \otimes p(\mathbf{r})$, ... so that, through application of the Central Limit Theorem, we have

$$u(\mathbf{r}, t_\infty) = \prod_{n=1}^{\infty}{}^{\otimes} p_n(\mathbf{r}) \equiv p(\mathbf{r}) \otimes p(\mathbf{r}) \otimes \ \ldots = \text{Gauss}(\mathbf{r})$$

where $\text{Gauss}(\mathbf{r})$ is a normalised three-dimensional Gaussian function such that

$$\int_{-\infty}^{\infty} \text{Gauss}(\mathbf{r}) d^3\mathbf{r} = 1$$

From Equation (21) we can now consider the equation $u(\mathbf{r}, t_n) = \text{Gauss}(\mathbf{r}) + \epsilon(\mathbf{r}, t_n)$, or, after taking $p = 1$ norms, $\bar{u}(t_n) \leq 1 + \bar{\epsilon}(t_n)$ where $\bar{u}(t_n) = \|u(\mathbf{r}, t_n)\|_1$. For a discrete time series $u_n > 0 \forall n$, say, we compute the Lyapunov exponent using the relatively simple formula

$$\lambda = \frac{1}{N\tau} \sum_{n=1}^{N} \ln\left(\frac{u_{n+1}}{u_n}\right). \tag{23}$$

Hence, for a time series which is assumed to be predicated on Equation (10), we can compute the corresponding Lyapunov exponent using Equation (23), albeit, in practice, for a finite array of size N. This includes financial time series data when λ can be computed for a moving look-back window to generate a signal composed of Lyapunov exponents. In this context, the product $N\tau$ merely scales the computed value of the exponent but if $u_{n+1} > u_n$, $\forall n = 1, 2, \ldots, N$ then $\lambda > 0$ and if $u_{n+1} < u_n$, $\forall n = 1, 2, \ldots, N$ then $\lambda < 0$. Hence, irrespective of the scale used, a change of polarity in the value of λ is a signature of a change in the gradient of the time series. For this reason a change in polarity of the Lyapunov exponent can be used to quantify the transition between the growth or decay of a financial series.

An example of this is given in Figure 4 which shows a financial signal (the first 1000 elements of the FTSE 100 prices given in Figure 3)—from 14/03/2006–26/02/2010—which has been normalised for display purposes, i.e., $u_n := u_n/\|u_n\|_\infty$. The associated Lyapunov exponent has been computed using function Lyapunov given in Appendix A.2 and re-scaled for values $\tau = 0.01$ and $N = 32$ according to Equation (23). Note that the first N values in Figure 3 are missing which is due to the window being a look-back window holding data that contributes to the first computation of the exponent at point N.

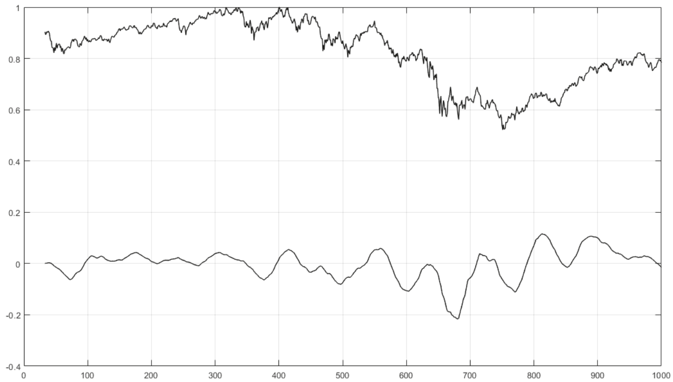

Figure 4. Example of computing the time varying Lyapunov exponent (lower signal) for a financial signal (upper signal after normalisation) through application of Equation (23) for a look-back window of size $N = 32$ and with $\tau = 0.01$.

As with other metrics computed from financial time series, the Lyapunov exponent obtained depends critically on the look-back window that is applied. However, subject to a delay which is proportional to the size of the look-back window used (determined by the time scale of the analysis), the polarity (and continuity thereof) of the signal can be used to estimate the macroscopic trends in a financial time series as illustrated in Figure 4. This is discussed further later on in this paper.

Since we can write Equation (23) in the form

$$\lambda = \frac{1}{N\tau}\sum_{n=0}^{N}(\ln u_{n+1} - \ln u_n)$$

and that

$$\frac{\ln u_{n+1} - \ln u_n}{\tau} \sim \frac{d}{dt}\ln u(t),$$

then, for the continuous case with time series function $u(t),\ t \in [0, T]$, we can write

$$\lambda \sim \frac{1}{T}\int_0^T \frac{d}{dt}\ln u(t)dt = \frac{1}{T}\ln\left[\frac{u(T)}{u(0)}\right]$$

giving a result that is analogous to Equation (19).

9. The Evolution Equation, Volatility and Risk

For a stochastic source term $s(\mathbf{r}, t)$, as given in Equation (11), Equation (14) becomes

$$\tau\frac{\partial}{\partial t}u(\mathbf{r}, t) = -n(t) \otimes u(\mathbf{r}, t) + n(t) \otimes u(\mathbf{r}, t) \otimes p(\mathbf{r}) + n(t) \otimes s(\mathbf{r}, t)$$

Consider the case when $p(\mathbf{r}) = \delta^n(\mathbf{r})$. Integrating over $\mathbf{r} \in \mathbb{R}^n$, we can then write the rate equation

$$d_t u(t) = \sigma[n(t) \otimes s(t)] \ \Rightarrow\ u(t) = \sigma\int^t [n(t) \otimes s(t)]dt = \sigma s(t) \otimes \int^t n(t)dt \ (+\text{constant})$$

where

$$d_t u(t) \equiv \frac{d}{dt}u(t),\ \ u(t) = \int_{-\infty}^{\infty} u(\mathbf{r}, t)d^n\mathbf{r},\ \ s(t) = \int_{-\infty}^{\infty} s(\mathbf{r}, t)d^n\mathbf{r}\ \text{ and }\ \sigma = \frac{1}{\tau}.$$

Suppose we write this equation in the form

$$d_t \ln u(t) \equiv \frac{1}{u(t)} \frac{d}{dt} u(t) = \frac{\sigma}{u(t)} \left[n(t) \otimes s(t) \right], \quad u(t) > 0 \; \forall \; t$$

and consider an iterative solution for $u(t)$ given by

$$d_t \ln u_k(t) = \frac{\sigma}{u_k(t)} \left[n(t) \otimes s(t) \right], \quad k = 0, 1, 2, ...$$

so that the first iterate $u(t) := u_1(t)$ becomes the solution to the rate equation

$$d_t \ln u(t) \sim \sigma f(t) \tag{24}$$

where, for $u_0(t) = 1$,

$$f(t) = n(t) \otimes s(t).$$

Equation (24) then shows that the volatility is a measure of the randomness of $\ln u(t)$ through the convolution of $s(t)$ with the time integration of $n(t)$. If, for example, $\mid s(t) \mid \leq 1$, then, in the term $\sigma s(t)$, σ determines the amplitude of $s(t)$.

Equation (24) does not provide a practically useful formula for σ as it relies on defining the functions $n(t)$ and $s(t)$ when what is ideally required is a definition for σ that relies on knowledge of $u(t)$ alone. To do this we are required to derive a formula for σ in terms of the function $u(t)$ through the elimination $f(t)$ and this requires a condition to be applied. In this context, suppose we assume that $f(t)$ is a phase only function (with unit amplitude) of compact support T and with a bandwidth Ω. This requires that both $s(t)$ and $n(t)$ are phase only functions of the same compact support and bandwidth. In this case $F(\omega) = \exp[i\theta(\omega)]$ where $\theta(\omega)$ is the "Phase Spectrum" and using Parseval's Theorem, we have

$$\int_{-T/2}^{T/2} \mid f(t) \mid^2 dt = \frac{1}{2\pi} \int_{-\Omega/2}^{\Omega/2} \mid F(\omega) \mid^2 d\omega = \frac{\Omega}{2\pi}.$$

Hence, we obtain an expression for the volatility given by

$$\sigma = \sqrt{\frac{2\pi}{\Omega}} \| d_t \ln u(t) \|_2, \quad \| d_t \ln u(t) \|_2 := \left(\int_{-T/2}^{T/2} \mid d_t \ln u(t) \mid^2 dt \right)^{\frac{1}{2}}.$$

For a uniformly sampled discrete time series $u_n, \; n = 1, 2, 3, ..., N$, application of a forward differencing scheme for a time interval Δt when

$$d_t \ln u(t) \to \frac{\ln u_{n+1} - \ln u_n}{\Delta t} = \frac{1}{\Delta t} \ln \left(\frac{u_{n+1}}{u_n} \right)$$

gives

$$\sigma = \sqrt{\frac{2\pi}{\Omega \Delta t}} \left\| \ln \left(\frac{u_{n+1}}{u_n} \right) \right\|_2, \quad \left\| \ln \left(\frac{u_{n+1}}{u_n} \right) \right\|_2 = \left[\sum_{n=1}^{N} \left| \ln \left(\frac{u_{n+1}}{u_n} \right) \right|^2 \right]^{\frac{1}{2}}.$$

The sampling interval Δt of u_n is related to the sampling interval $\Delta \omega$ of the discrete Fourier transform of u_n by the equation

$$\Delta t \Delta \omega = \frac{2\pi}{N}$$

and since the bandwidth of the discrete spectrum of u_n is $N\Delta\omega$ is is clear that $\Delta t\Omega = 2\pi$. Thus we derive a simple formula for the volatility given by

$$\sigma = \left\| \ln \left(\frac{u_{n+1}}{u_n} \right) \right\|_2 . \tag{25}$$

Comparing Equation (25) with Equation (23), we observe similarities in regard to the commonality of the quotient u_{n+1}/u_n and the logarithmic function but where $\lambda < 0$ or $\lambda \geq 0$ and where $\sigma \geq 0 \forall n$.

An example of the short time volatility is given in Figure 5 which shows a financial signal (the first 1000 elements of the FTSE 100 prices given in Figure 3), normalised for display purposes. In this example σ in Equation (25) was computed using function volatility given in Appendix A.3 for $N = 32$.

In financial time series modelling, the volatility is a measure of the noise in the signal. For data that has a stable trend (up or down) the volatility is relatively low, and, in this context, trading is best undertaken working with financial signals that have a low volatility other than options trading where there may have been a "bet" of a move of a certain magnitude. In this sense, the volatility of a signal provides a measure of the risk, a low risk loosely equating to a low volatility. In the derivation of the volatility provided in this section: $\sigma = 1/\tau$ where τ is a coefficient in Equation (14). In this respect, and, in context of the evolution equation, τ is a measure of risk, the greater the value of τ the lower the risk associated with an investment. For short $\delta(t)$ type memory functions the GKFE reduces to the classical Kolmogorov–Feller equation which, in terms of its relationship to the evolution equation requires that $\tau << 1$. Thus low risk requires that a financial time series is characterised by long memory functions, at least in terms of the model compounded in Equation (14)—a result that makes intuitive sense.

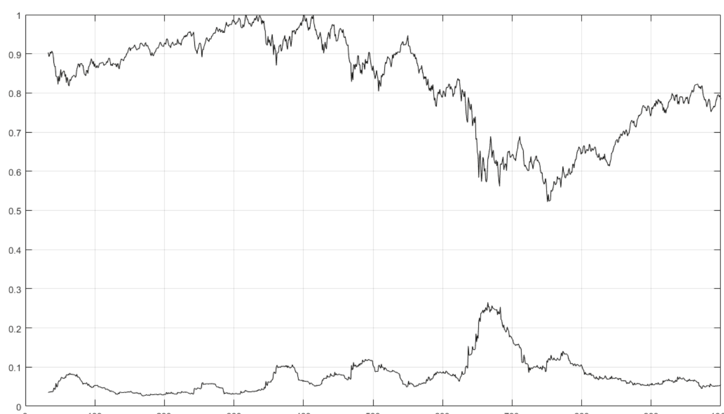

Figure 5. Example of computing the time dependent volatility (lower signal) using Equation (25) for a normalised financial signal (upper signal) with a look-back window of $N = 32$.

10. Trend Analysis Using the Lyapunov Exponent to Volatility Index Ratio

The changes in polarity or "zero-crossings" associated with the Lyapunov exponent (computed on a moving window basis) as discussed in Section 8 provide the positions in time where there is a transition in the type of trend (growth leading to decay and decay leading to growth). The value of the volatility indicates the "stability" of the time series, the temporal characteristics of all indicators being dependent of the size of the window or "period" used. This suggests scaling the Lyapunov exponent with the inverse of the volatility, i.e., computing the quotient

$$\lambda_\sigma = \frac{\lambda}{\sigma} \tag{26}$$

where σ is defined by Equation (25) and λ is defined by Equation (23) with $\tau = 1/N$ thereby making λ_σ scale independent. This index then assesses not only changes in the direction of a trend but the corresponding stability of that trend. This idea has obvious applications to a range of time series but especially in regard to financial time series analysis where forecasting both the type and characteristics of a trend is of fundamental importance, a positive trend with low volatility indicating a good investment horizon, for example. We define λ_σ as the Lyapunov-to-volatility ratio (LVR). Figure 6 shows the time varying LVR of a financial signal (the first 1000 elements of the FTSE 100 prices given in Figure 3 after normalisation) for $N = 32$.

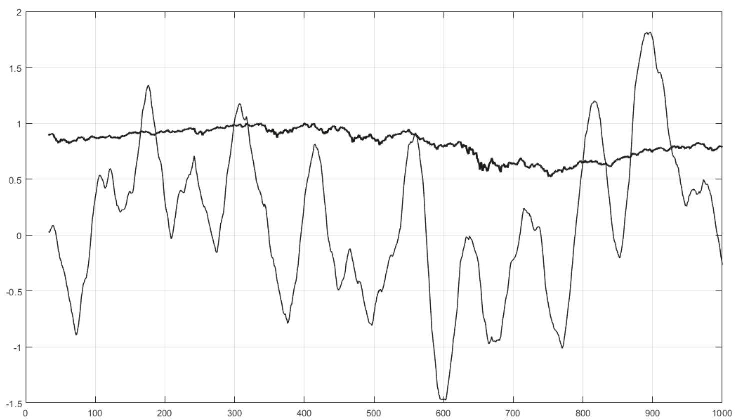

Figure 6. Example of computing the time dependent LVR λ_σ (solid line —) for a normalised financial signal (bold solid line **—**) with a look-back window of $N = 32$.

10.1. Pre- and Post-Filtering

As shall be discussed later, the numerical accuracy of results obtained in predicting a trend and its longevity, is critically dependent on the filtering of both the input data and λ_σ—pre- and post-filtering, respectively.

10.1.1. Pre-Filtering

The positions in time at which the zero crossings are evaluated using Equation (26) depend on the accuracy of the algorithm used to compute λ_σ which in turn, depends on the intrinsic noise associated with the time series data. This can yield errors in the positions at which the zero-crossings are computed especially in regard to changes associated with very short time micro-trends.

In the context of longer term macro-trends, such micro-trends may legitimately be interpreted as noise although, in the context of financial times series analysis, for example, the term "noise" must be understood to reflect legitimate price values. To overcome this effect, u_n is filtered using a moving average filter defined by:

$$u(t) := w(t) \otimes u(t)$$

where

$$w(t) = \begin{cases} 1/W, & \mid t \mid \leq W \\ 0, & \mid t \mid > W \end{cases}$$

and W defines the length of the "moving window". The function given in Appendix A.4 provides a moving average filter for pre-filtering the data u_n using a window of size W.

10.1.2. Post-Filtering

In addition to pre-filtering the time series data, an option for post-filtering the λ_σ is required to further control the dynamic behaviour of this index. We therefore again consider a moving average filter given by

$$\lambda_\sigma(t) := w(t) \otimes \lambda_\sigma(t)$$

where

$$w(t) = \begin{cases} 1/T, & \mid t \mid \leq T \\ 0, & \mid t \mid > T \end{cases}$$

and T defines the length of the "moving window", $T \neq W$.

10.2. Zero-Crossings Analysis

On the basis of the ideas considered in the previous section, the critical points at which a trend forecasting decision is made are the zero crossing points associated with λ_σ. By computing $\lambda_\sigma(t)$ where t is the position in time of the window, identification of the zero crossings denoted by the function $z_c(t)$ involves the follow basic procedure:

$$z_c(t) = \begin{cases} +1, & \lambda_\sigma(t) < 0 \quad \& \quad \lambda_\sigma(t + \varepsilon) \geq 0; \\ -1, & \lambda_\sigma(t) > 0 \quad \& \quad \lambda_\sigma(t + \varepsilon) \leq 0; \\ 0, & \text{otherwise.} \end{cases}$$

where ε is a small perturbation in time. This procedure generates a series of Kronecker delta functions whose polarity determines the position(s) in time at which a trend is expected to be positive or negative. Thus the function $z_c(t)$ identifies the zero crossings associated with the end of an upward trend and the start of a downward trend when $z_c(t) = -1$ and the end of downward trend and the start of an upward trend when $z_c(t) = +1$. This is therefore a "critical indicator" in regard to forecasting the trending behaviour of a time series.

10.3. Back-Testing Evaluator

Back-testing algorithms are designed to "gauge" the accuracy of results in terms of trend predictions, for example, and, are usually, but not exclusively, related to testing a strategy for forecasting the behaviour of a financial time series. They are usually designed to assess the overall accuracy of some trading strategy based on historical data when the future outcomes of such a strategy can be evaluated. In this context, the function given in Appendix A.5 evaluates the performance associated with the zero-crossings analysis discussed in the previous section. This evaluation operates on the basis that the price differences should reflect the interval between the start and end points of a predicted trend if the prediction is correct. Thus, in the case when $z_c(t) > 0$ and the trend is positive, the price difference between this point in the time series and the next point in time series when $z_c(t) < 0$ should be positive, thereby representing a net price gain between the two zero crossings. Similarly, when $z_c(t) < 0$ and the trend is negative, the price difference between this point in the time series and the next point in time series when $z_c(t) > 0$ should be negative, thereby representing a net price loss between the two zero crossings. In those cases where this occurs throughout the duration of the time series considered, the predicted entry and exits points are taken to be correct, or else, they are taken to be incorrect. The accuracy associated with this evaluation is computed as a percentage in terms of successful entries and exits, i.e., going "long" (when an investment might be made because the price of a commodity is increasing) and going "short" (when an investment would be held or sold at the start of a downward trend), respectively.

10.4. Example Results

A function called Backtester is provided in Appendix A.6 which gives the user options on the sizes of the look-back window length W and T and the size L of the data stream that is used from the available input data. This data is provided in the form of a column vector via a .txt file. The function normalises this data so that it can be plotted on a scale that is consistent with the scale of $\lambda_\sigma[n]$. The function provides a plot which shows the evolution of u_n (normalised), $\lambda_\sigma[n]$ and $z_c[n]$ and then evaluates the results using function evaluator as discussed in the previous section. Note that both the Lyapunov exponent and the volatility are evaluated from the original data (and not the normalised data used for the plot) using a look-back window of T.

Figure 7 shows some example results of running Backtester for the first 1000 elements of the FTSE 100 prices given in Figure 3. The three examples provided are for function Backtester(10,10,1000), Backtester(20,10,1000) and Backtester(30,10,1000) for which the combined entry/exit (long/short) accuracy is 36.55% , 64.58% and 72.73%, respectively. From these results it is clear that the accuracy improves significantly with the extent of the pre-filtering that is applied to the time series before computation of the LVR. This is to be expected as pre-filtering reduces the noise associated with the time series prior to the computation of the LVR.

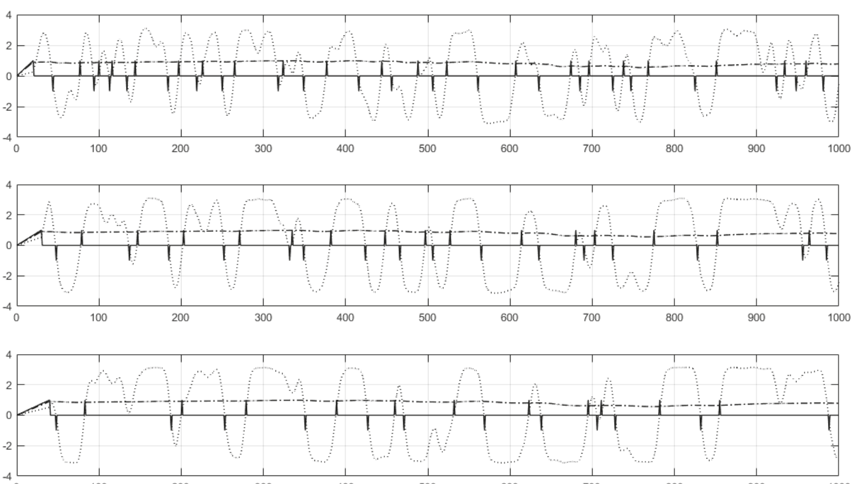

Figure 7. Example graphical outputs from function Backtester. Each plot shows the post-filtered (for $T = 10$) LVR $\lambda_\sigma[n]$ (dotted line), the pre-filtered time series u_n after normalisation (dot-dashed line) and $z_c[n]$ (solid line) which identifies the zero-crossings of $\lambda_\sigma[n]$. The plots provided are for the case when pre-filtering is undertaken for look-back window sizes of $W = 10$ (top), $W = 20$ (centre) and $W = 30$ (lower plot). The financial time series data used in this case is the FTSE100 daily prices from 14/03/2006 to 26/02/2010.

In order to quantify both the pre- and post-filtering effect on the combined accuracy of the long/short predictions, Figure 8 shows a surface (mesh) plot of the combined accuracy as a function of the pre- and post-filtering look-back window sizes W and T, respectively. The maximum value associated with this 'WT-map' is 87.5% which occurs at (W, T) coordinates (40, 10). From Figure 8 it can be seen that the highest combined accuracies (>70%) are obtained for approximate values of $W \in [30, 50]$ and $T \in [10, 20]$. However, it should be noted that WT-maps of this kind are data dependent and will vary with the type of financial time series that is processed and on the non-stationary characteristics that occur over the length of the data series that is chosen (i.e., the input parameter L in function Backtester). Hence, WT-maps of the type given in Figure 8 provide a "signature" for a financial signal from which optimal values of the pre- and post filtering windows can be established. This optimisation is based on finding the smallest values of W and T that will maintain a combined accuracy compatible with an expected return over a given time scale.

As a comparative example, Figure 9 shows the equivalent WT-map for 1000 elements of the Eruo-Dollar (USA) daily (averaged) exchange rates from 29/04/2008 to 27/02/2012 as given in Figure 10. In this case, a maximum value of 83.33% occurs at WT coordinates with minimum values of (38, 12). Although the quantitative details of this WT-map are unique to the data used, in qualitative terms, it is similar to the WT-map given in Figure 8 revealing that greater accuracy is achieved for large values of W relative to T which is intuitively to be expected. Clearly, for any specific financial date series, a WT-map is required to provide an optimal accuracy associated with the trend analysis of that series under the assumption that the stochastic behaviour of the series is stationary, i.e., the financial signal is Ergodic.

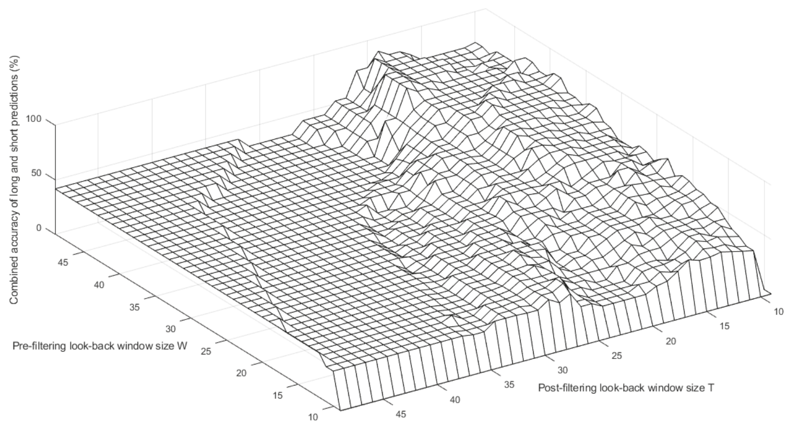

Figure 8. Surface (mesh) plot of the combined long/short predictive accuracy as a function of the pre- and post-filtering look-back window sizes W and T, respectively, for FTSE100 daily prices from 14/03/2006 to 26/02/2010.

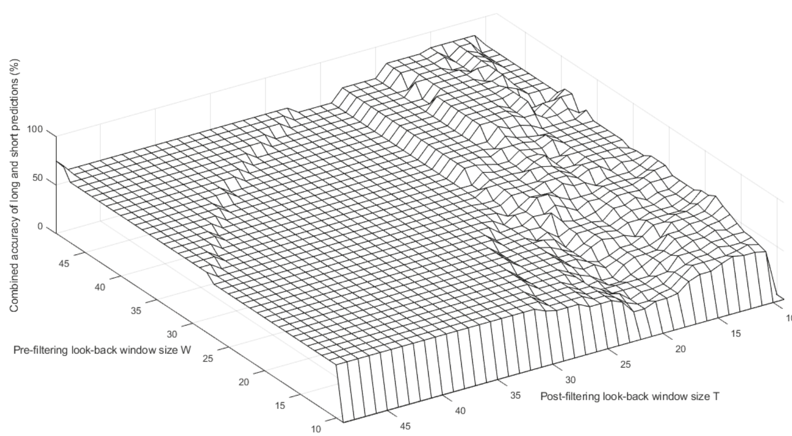

Figure 9. Surface (mesh) plot of the combined long/short predictive accuracy as a function of the pre- and post-filtering look-back window sizes W and T, respectively, for Eruo-Dollar (USA) daily (averaged) exchange rates from 29/04/2008 to 27/02/2012 as given in Figure 10.

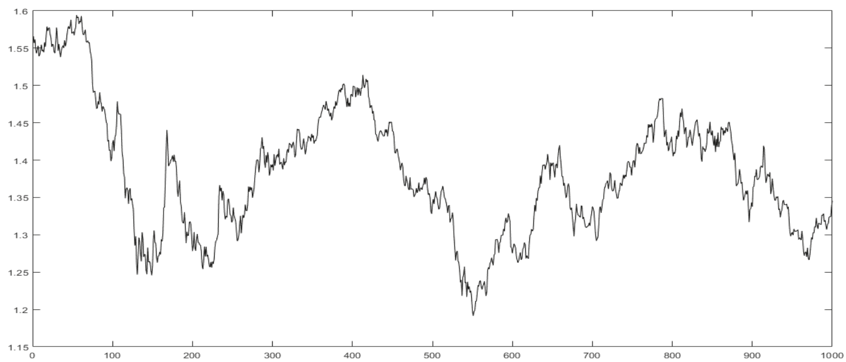

Figure 10. Eruo-Dollar (USA) daily (averaged) exchange rates from 29/04/2008 to 27/02/2012.

11. Price Prediction Using Evolutionary Computing

The trend analysis considered in the previous sections provides evidence of being able to predict a positive or negative trend in a financial signal over a period of time that yields a net positive of negative gradient, respectively, before the trend is reversed. This is based on pre-filtering the financial signal and post-filtering the LVR in which the accuracy achieved is based on the look-back windows of both filters. Such an approach only provides a statement on the expected future trend of a financial signal; it does not provide an estimate of the actual future price.

On the basis of Equation (15) and the random walk hypothesis it represents, it is not possible to determine a future price with 100% accuracy whatever the time scale, given that most financial signals are known to be self-affine stochastic fields which exhibit the same statistical distributions over all time scales. Thus, it is well known and understood (but not always appreciated) that in economics, only an estimate (essentially an informed guess) of a future price is possible. However, in principle, the lower the volatility of the signal, the less likely it is to exhibit large random variation at some future (short) time and hence, the larger the LVR the more likely it is that an estimate of a future price will be a more accurate prediction. In terms of Equation (15), this means that $u(t+\tau) \sim u(t)$ given that $s(t) \sim 0$, i.e., 'tomorrow's price is likely to be close to today's price. This provides the basis for using evolutionary computing to estimate short time price values by using the LVR to flag when the approach can be used effectively, i.e., when the LVR reaches a maximum or minimum above or below a certain threshold, respectively—as illustrated in Figure 7 for a threshold of 2, for example.

11.1. Evolutionary Computing

Evolutionary computing (EC) involves "applying the Darwinian principles of natural selection to algorithmic problem solving" [42] and has its origins in the 1960s with the introduction of "evolutionary programming" [43], "genetic algorithms" [44], and "evolutionary strategies" [45]. Following independent developments in the 1990s these areas merged to form the discipline of genetic programming known today as EC in which a correlation exists between natural evolution and evolution by computational problem solving [46].

In the context of a local environment that has a population striving for survival and to reproduce, with natural evolution, the success (fitness) of each individual is dependent on their environment and how well they meet their goals. Similarly, with a trial-and-error mathematical process, a candidate solution is judged in the context of the problem that it is trying to solve and how well the candidate solves the problem which determines whether or not it is kept as a candidate solution. A common theme in EC is the idea of taking a population of individuals "operating" according to environmental pressures causing natural selection and thereby the growth of a fitter population. Many aspects of EC are stochastic and the starting point of candidate solutions can be either deterministic or stochastic. In either case, the aim is to produce a "solution" that minimises some fitness function.

11.2. Eureqa

Eureqa is an EC tool originally developed by the Cornell Creative Machine Laboratory (Cornell University) and commercialised by Nutonian Inc. (Boston, MA, USA) [47]. The underlying principle is to use genetic programming to generate equations, each of which provides an increasingly better fitness function to model a given dataset. The system iteratively generates a sequence of non-linear functions to describe input (digital) signals which may include stochastic signals [48]. It is a modelling engine predicated on artificial intelligence using evolutionary searches to determine an equation that represents a set of data [49]. The system automatically discovers formulae through evolutionary algorithms requiring no human intervention starting by randomly creating equations via sequences of mathematical building blocks based on a combination of common functions. The content of these formulae is ordered only by a basic syntax (e.g., two addition signs cannot appear one after the other). Beyond this basic syntax, the sequences generated by the program are entirely random.

11.3. Application to Financial Forecasting

With a little data "Eureqa generates fundamental laws of nature" [50]. However, there have been few applications of EC to financial forecasting. This is partly due to the significance of Equation (15) and the basic random walk hypothesis which financial signals adhere to, albeit as self-affine stochastic fields. Thus although EC can be used to generate a non-linear equation for some short time financial signal, no fundamental significance in terms of a "law of nature" can be inferred by such an equation due to the random walk nature of the data that is used. To date, the only "law of nature" that can be used to describe financial signals is that they are statistically self-affine fields to which the fractal market hypothesis is thereby applicable. Nevertheless, EC can be used to provide short time predictions including the performance of equity markets [51] and energy commodities [52], for example. This is done by using EC to generate representative equations for existing prices over a look-back window and can, in principle, be applied successive for a moving (look-back) window especially for time periods where the volatility of the time series is low and future prices can be expected to be random but locally similar to past prices.

11.4. An Example Result

With reference to Figure 7, we consider the daily prices for array values between 870 and 900 (inclusively) which correspond to days 24/08/2009 to 08/09/2009 when the LVR is $\sim$3 and relatively flat. With these 30 price values, Eureqa provides the following formula:

$$\begin{aligned} f(t) &= 5025.73417939762 + 8.96527863946579t^2 + 1.52597679067939 \times 10^{-6}t^6 \\ &+ \cos(8.96527863946579t) - 76.8453768284695t - 0.253321938783733t^3 \\ &- 48.5578781261177\sin(0.96446841878421 + 7.16244232473996t) \end{aligned} \tag{27}$$

obtained after 51,056 generations giving a correlation coefficient of 0.98362871, an R^2 (coefficient of determination) goodness of fit of 0.96684623, a mean absolute error of 16.623419 and a complexity of 51.

Figure 11 shows a comparison of the true price values with the estimates obtained using a discretised version of Equation (27) given by

$$\begin{aligned} f_n &= 5025.73417939762 + 8.96527863946579t_n^2 + 1.52597679067939 \times 10^{-6}t_n^6 \\ &+ \cos(8.96527863946579t_n) - 76.8453768284695t_n - 0.253321938783733t_n^3 \\ &- 48.5578781261177\sin(0.96446841878421 + 7.16244232473996t_n), \ n = 1, 2, ..., N, \ t_n = n \end{aligned} \tag{28}$$

for $N = 30$, and, additionally, for $n = 31, 32$ and 33 thereby providing future price prediction for three days ahead.

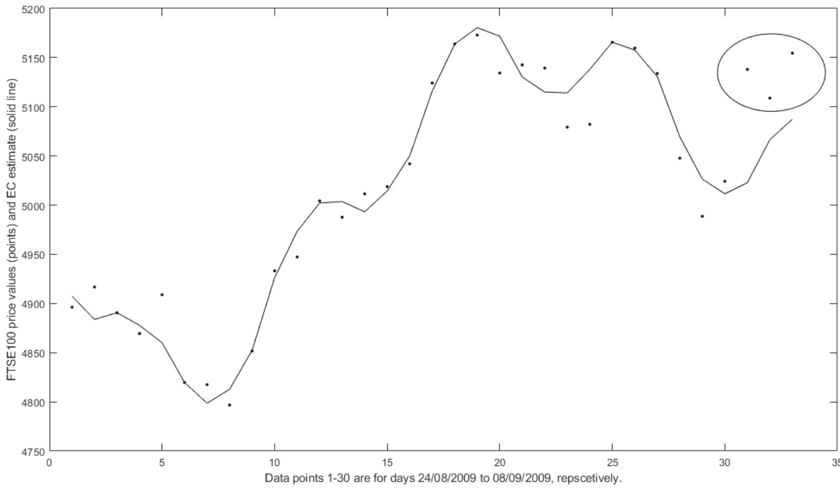

Figure 11. FTSE 100 price values (dots) for 30 days between 24/08/2009 and 08/09/2009 (inclusively) and the estimate (solid line) obtained from the evolutionary computed formula given by Equation (28). The graph compares the actual prices (circled dots) and predicted future values for three days after 08/09/2009 using the same formula.

A comparison of the numerical values for these price predictions is given in Table 2.

Table 2. Table comparing the actual and predicted prices for three days of the FTSE100 using Equation (28).

Day	**09/09/2009**	**10/09/2009**	**11/09/2009**
t_n	31	32	33
Predicted price value f_n	5022.9	5066.6	5087.4
Actual price value u_n	5138.0	5108.9	5154.6

11.5. Discussion

With reference to Figure 11, the local trend in prices before and inclusive of element 30 (i.e., elements 26–30) is downward and so based on the principle of Equation (15) for $s(t) \sim 0$ and application of exponential smoothing for time series forecasting [53], for example, continuation of this trend will lead to inaccurate predictions that are inconsistent with the local increase in prices for elements 31, 32 and 33—the circled dots in Figure 11. However, the equivalent future predictions given by Equation (28) are consistent with the actual values which represent a short time up-ward trend as shown in Table 2. The predictive ability of EC can only be considered for very short future time increments (a look-forward prediction window) but this example result does provide evidence for the success of using EC exercised on a moving look-back window basis.

A quantitative study on the accuracy of this approach in terms of the look-back window and the look-forward (prediction) window relative to the local LVR lies beyond the scope of this work. However, it is to be expected that the success of this approach will be predicated on the size of the amplitude of $\mid \lambda_\sigma \mid$ when the volatility is low. Hence, based on the results given in Figure 7, an EC moving window approach may be used when $\mid \lambda_\sigma \mid \geq 2$ where λ_σ is given by Equation (26). In this context, the LVR not only provides a method of predicting trends (subject to appropriate pre- and post-filtering) based on its change in polarity but also flags when to apply EC to generate future price estimates.

12. Derivations of the Diffusion Equation from the Evolution Equation

So far in this paper, we have developed a predictive indicator that is based on combining the Lyapunov exponent and the volatility into a ratio (the LVR), both parameters having been derived from E^3 and computed on a moving window basis. We have then used the amplitude of this ratio to gauge the likelihood of using EC to successfully predict short term future price values. We have not yet studied the effect of applying specific models for the PDF associated with E^3 which is the subject of later sections. In particular, we show how the classical diffusion equation is a result of considering a Gaussian PDF in E^3 and the non-classical fractional diffusion equation is the result of considering a non-Gaussian PDF, in particular, a Lévy distribution and undertaken using the associated characteristic functions.

In this section we consider three approaches to deriving the classical diffusion equation in order to show the connectivity between this equation and E^3 in terms of applying different conditions and approximations. We start with Einstein's original approach which is independent of the specific PDF but on the condition that the PDF is symmetric.

12.1. Einstein's Derivation for $\mathbf{r} \in \mathbb{R}^1$

In his 1905 paper [1], Einstein considered the one-dimensional case, when $\mathbf{r} \in \mathbb{R}^1$, and where the PDF is taken to be symmetric so that $p(x) = p(-x)$. In this case, Equation (10) can be written as

$$u(x, t+\tau) = \int_{-\infty}^{\infty} p(x-\lambda) u(\lambda, t) d\lambda = \int_{-\infty}^{\infty} p(x+\lambda) u(\lambda, t) d\lambda = \int_{-\infty}^{\infty} p(\lambda) u(x+\lambda, t) d\lambda$$

Taylor expanding $u(x, t)$ to first order in time, and, to second order in space, we then obtain

$$u(x,t) + \tau \frac{\partial}{\partial t} u(x,t) = \int_{-\infty}^{\infty} d\lambda p(\lambda) \left[u(x,t) + \lambda \frac{\partial}{\partial x} u(x,t) + \frac{\lambda^2}{2} \frac{\partial^2}{\partial x^2} u(x,t) \right]$$

$$= u(x,t) \int_{-\infty}^{\infty} p(\lambda) d\lambda + \frac{\partial}{\partial x} u(x,t) \int_{-\infty}^{\infty} \lambda p(\lambda) d\lambda + \frac{\partial^2}{\partial x^2} u(x,t) \int_{-\infty}^{\infty} \frac{\lambda^2}{2} p(\lambda) d\lambda$$

$$= u(x,t) + \frac{\partial^2}{\partial x^2} u(x,t) \int_{-\infty}^{\infty} \frac{\lambda^2}{2} p(\lambda) d\lambda$$

since

$$\int_{-\infty}^{\infty} p(\lambda) d\lambda = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} \lambda p(\lambda) d\lambda = 0.$$

We can thus write the equation

$$\frac{\partial}{\partial t} u(x,t) = D \frac{\partial^2}{\partial x^2} u(x,t) \tag{29}$$

where

$$D = \int_{-\infty}^{\infty} \frac{\lambda^2}{2\tau} p(\lambda) d\lambda. \tag{30}$$

which is the one-dimensional diffusion equation for diffusivity D with dimensions of Length^2/Time. This derivation of the diffusion equation relies on the conditions $\tau << 1$ and $\lambda^2 << 1$ which are required in order to truncate the Taylor series expansion of $u(x, t+\tau)$ in time and $u(x+\lambda, t)$ in space. However, this derivation of the diffusion equation is independent of the PDF (subject to the condition that the PDF is symmetric) which determines the diffusivity D through Equation (30).

12.2. Einstein's Derivation for $\mathbf{r} \in \mathbb{R}^3$

A similar approach can be used to deriving of the diffusion equation for $\mathbf{r} \in \mathbb{R}^3$ as shall now be demonstrated. In this case

$$u(\mathbf{r}, t+\tau) = p(r) \otimes u(\mathbf{r}, t), \ p(r) = p(-r)$$

can be written out in the form

$$u(\mathbf{r}, t+\tau) = \int\limits_{-\infty}^{\infty} u(\mathbf{r}+\lambda, t) p(\lambda) d\lambda$$

where λ is a scalar with dimensions of length and components λ_x, λ_y and λ_z. Expanding $u(\mathbf{r}+\lambda, t)$ in terms of a three-dimensional Taylor series,

$$u(\mathbf{r}+\lambda, t) = u(\mathbf{r}, t) + \lambda_x \frac{\partial u(\mathbf{r}, t)}{\partial x} + \lambda_y \frac{\partial u(\mathbf{r}, t)}{\partial y} + \lambda_z \frac{\partial u(\mathbf{r}, t)}{\partial z} + \frac{\lambda_x^2}{2!} \frac{\partial^2 u(\mathbf{r}, t)}{\partial x^2} + \frac{\lambda_y^2}{2!} \frac{\partial^2 u(\mathbf{r}, t)}{\partial y^2} + \frac{\lambda_z^2}{2!} \frac{\partial^2 u(\mathbf{r}, t)}{\partial z^2}$$
$$+ \lambda_x \lambda_y \frac{\partial^2 u(\mathbf{r}, t)}{\partial x \partial y} + \lambda_x \lambda_z \frac{\partial^2 u(\mathbf{r}, t)}{\partial x \partial z} + \lambda_y \lambda_z \frac{\partial^2 u(\mathbf{r}, t)}{\partial y \partial z} + ...$$

so that, for $\tau << 1$,

$$u(\mathbf{r}, t) + \tau \frac{\partial u(\mathbf{r}, t)}{\partial t} = \int\limits_{-\infty}^{\infty} u(\mathbf{r}, t) p(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} \left(\lambda_x \frac{\partial u(\mathbf{r}, t)}{\partial x} + \lambda_y \frac{\partial u(\mathbf{r}, t)}{\partial y} + \lambda_z \frac{\partial u(\mathbf{r}, t)}{\partial z} \right) p(\lambda) d\lambda$$
$$+ \int\limits_{-\infty}^{\infty} \left(\frac{\lambda_x^2}{2!} \frac{\partial^2 u(\mathbf{r}, t)}{\partial x^2} + \frac{\lambda_y^2}{2!} \frac{\partial^2 u(\mathbf{r}, t)}{\partial y^2} + \frac{\lambda_z^2}{2!} \frac{\partial^2 u(\mathbf{r}, t)}{\partial z^2} \right) p(\lambda) d\lambda$$
$$+ \int\limits_{-\infty}^{\infty} \left(\lambda_x \lambda_y \frac{\partial^2 u(\mathbf{r}, t)}{\partial x \partial y} + \lambda_x \lambda_z \frac{\partial^2 u(\mathbf{r}, t)}{\partial x \partial z} + \lambda_y \lambda_z \frac{\partial^2 u(\mathbf{r}, t)}{\partial y \partial z} \right) p(\lambda) d\lambda$$

We then obtain

$$\tau \frac{\partial}{\partial t} u(\mathbf{r}, t) = \int\limits_{-\infty}^{\infty} \frac{\lambda_x^2}{2} \frac{\partial^2 u(\mathbf{r}, t)}{\partial x^2} p(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} \frac{\lambda_x \lambda_y}{2} \frac{\partial^2 u(\mathbf{r}, t)}{\partial x \partial y} p(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} \frac{\lambda_x \lambda_z}{2} \frac{\partial^2 u(\mathbf{r}, t)}{\partial x \partial z} p(\lambda) d\lambda$$
$$+ \int\limits_{-\infty}^{\infty} \frac{\lambda_y \lambda_x}{2} \frac{\partial^2 u(\mathbf{r}, t)}{\partial y \partial x} p(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} \frac{\lambda_y^2}{2} \frac{\partial^2 u(\mathbf{r}, t)}{\partial y^2} p(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} \frac{\lambda_y \lambda_z}{2} \frac{\partial^2 u(\mathbf{r}, t)}{\partial y \partial z} p(\lambda) d\lambda$$
$$+ \int\limits_{-\infty}^{\infty} \frac{\lambda_z \lambda_x}{2} \frac{\partial^2 u}{\partial z \partial x} p(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} \frac{\lambda_z \lambda_y}{2} \frac{\partial^2 u(\mathbf{r}, t)}{\partial z \partial y} p(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} \frac{\lambda_z^2}{2} \frac{\partial^2 u(\mathbf{r}, t)}{\partial z^2} p(\lambda) d\lambda$$
$$+ \int\limits_{-\infty}^{\infty} \lambda_x \frac{\partial u(\mathbf{r}, t)}{\partial x} p(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} \lambda_y \frac{\partial u(\mathbf{r}, t)}{\partial y} p(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} \lambda_z \frac{\partial u(\mathbf{r}, t)}{\partial z} p(\lambda) d\lambda$$

which can be written as

$$\frac{\partial}{\partial t}u(\mathbf{r},t) = \nabla \cdot \mathbf{D}\nabla u(\mathbf{r},t) + \mathbf{V} \cdot \nabla u(\mathbf{r},t)$$

where $\mathbf{D}$ is the diffusion tensor given by

$$\mathbf{D} = \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix}, \qquad D_{ij} = \int_{-\infty}^{\infty} \frac{\lambda_i \lambda_j}{2\tau} p(\lambda) d\lambda$$

and $\mathbf{V}$ is a flow vector which describes any drift velocity that the particle ensemble may have and is given by

$$\mathbf{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}, \qquad V_i = \int_{-\infty}^{\infty} \frac{\lambda_i}{\tau} p(\lambda) d\lambda.$$

Note that as $\lambda_i \lambda_j = \lambda_j \lambda_i$, the diffusion tensor is diagonally symmetric (i.e., $D_{ij} = D_{ji}$). For isotropic diffusion when $\langle \lambda_i \lambda_j \rangle = 0$ for $i \neq j$ and $\langle \lambda_i \lambda_j \rangle = \langle \lambda^2 \rangle$ for $i = j$ and with a zero drift velocity when $\mathbf{V} = \mathbf{0}$,

$$\frac{\partial}{\partial t}u(\mathbf{r},t) = \nabla \cdot \begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \nabla u(\mathbf{r},t) = D\nabla^2 u(\mathbf{r},t), \ D = \int_{-\infty}^{\infty} \frac{\lambda^2}{2\tau} p(\lambda) d\lambda$$

12.3. PDF Dependent Derivation of the Diffusion Equation

Consider the case when, for $\mathbf{r} \in \mathbb{R}^1$, $p(x)$ is a zero-mean normal (Gaussian) distribution with Standard Deviation σ and Variance σ^2, i.e.,

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

Taylor expansion to first order of Equation (10) followed by application of the convolution theorem yields the Fourier space equation

$$U(k,t) + \tau \frac{\partial}{\partial t} U(k,t) = P(k)U(k,t) \tag{31}$$

where

$$U(k,t) = \int_{-\infty}^{\infty} u(x,t) \exp(-ikx) dx$$

and

$$P(k) = \int_{-\infty}^{\infty} p(x) \exp(-ikx) dx = \exp\left(-\frac{\sigma^2 k^2}{2}\right),$$

$P(k)$ being the Characteristic Function.

Suppose we now consider the case when the variance is small, i.e., $\sigma^2 << 1$. Then

$$P(k) = 1 - \frac{\sigma^2 k^2}{2} + \frac{1}{2!}\left(\frac{\sigma^2 k^2}{2}\right)^2 + ... \sim 1 - \frac{\sigma^2 k^2}{2}, \ \ (\sigma k)^2 << 1$$

and Equation (31) can be written as

$$\frac{\partial}{\partial t} U(k,t) = -U(k,t) \frac{\sigma^2 k^2}{2\tau}$$

through which we again obtain the diffusion equation

$$\frac{\partial}{\partial t}u(x,t) = D\frac{\partial^2}{\partial x^2}u(x,t) \text{ where } D = \frac{\sigma^2}{2\tau}$$

given that

$$-k^2U(k,t) \leftrightarrow \frac{\partial^2}{\partial x^2}u(x,t).$$

In this case, the "key" to the derivation of the diffusion equation is the assumption that the variance of a normal distribution is small and that $\tau << 1$. We note that an identical analysis in the two- and three-dimensional domains yields the the two- and three-dimensional diffusion equation

$$\frac{\partial}{\partial t}u(\mathbf{r},t) = D\nabla^2 u(\mathbf{r},t), \; \mathbf{r} \in \mathbb{R}^n, \; n = 2, \; 3$$

12.4. Generalisation

We can generalise this approach further by writing the evolution equation in Fourier space using the convolution theorem and in expanded form as

$$U(\mathbf{k},t) + \tau\frac{\partial}{\partial t}U(\mathbf{k},t) + \frac{\tau^2}{2!}\frac{\partial^2}{\partial t^2}U(\mathbf{k},t) + ... = U(\mathbf{k},t) - \tau Dk^2U(\mathbf{k},t) + \frac{\tau^2}{2!}D^2k^4U(\mathbf{k},t) - ...$$

so that upon inverse Fourier transformation we have, for $\mathbf{r} \in \mathbb{R}^n, \; n = 1, \; 2, \; 3$

$$\tau\frac{\partial}{\partial t}u(\mathbf{r},t) + \frac{\tau^2}{2!}\frac{\partial^2}{\partial t^2}u(\mathbf{r},t) + ... = \tau D\nabla^2 u(\mathbf{r},t) - \frac{\tau^2}{2!}D^2\nabla^4 + ...$$

Equating terms with the same coefficients in regard to powers of τ, we have (for any positive integer m)

$$\frac{\partial}{\partial t}u(\mathbf{r},t) = D\nabla^2 u(\mathbf{r},t), \; \frac{\partial^2}{\partial t^2}u(\mathbf{r},t) = D^2\nabla^4 u(\mathbf{r},t), \; ..., \; \frac{\partial^m}{\partial t^m}u(\mathbf{r},t) = D^m\nabla^{m+2}u(\mathbf{r},t)$$

Since all such equations can be constructed from the diffusion equation, i.e.,

$$\frac{\partial^2}{\partial t^2}u(\mathbf{r},t) = D\nabla^2\frac{\partial}{\partial t}u(\mathbf{r},t) = D\nabla^2[D\nabla^2 u(\mathbf{r},t)], \; ...$$

this analysis confirms that the diffusion equation is E^3 for the case when the PDF is a Gaussian distribution.

12.5. Green's Function Solution

For the initial condition $u_0(\mathbf{r}) \equiv u(\mathbf{r},t=0), \; \mathbf{r} \in \mathbb{R}^n, \; n = 1,2,3$ and in the infinite domain, the Green's function solution to the homogeneous diffusion equation is [54]

$$u(\mathbf{r},t) = G(\mathbf{r},t) \otimes u_0(\mathbf{r})$$

where $G(\mathbf{r},t)$ is the Green's function given by

$$G(\mathbf{r},t) = \left(\frac{1}{4\pi Dt}\right)^{\frac{n}{2}} \exp\left(-\frac{r^2}{4Dt}\right), \; t > 0; \;\; n = 1,2,3.$$

12.6. The Black–Scholes Model

There is a synergy associated with the diffusion equation and the Black–Scholes model for a call premium which is compounded in the partial differential equation [55]

$$\frac{\partial}{\partial t}c(x,t) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2}c(x,t) + rs\frac{\partial}{\partial s}c(x,t) - rc(x,t) = 0$$

where $c(x,t)$ is the call premium, s is the stock price, σ is the volatility and r is the risk. Subject to specific initial and boundary conditions, this equation can be transformed into the classical diffusion equation through application of a change of variables when it can be written in the form

$$\frac{\partial}{\partial \tau}u(x,\tau) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}u(x,t)$$

which has the same Green's function solution as given in the previous section, for $n = 1$ and initial condition $u(x, t = 0)$. Thus, just as the classical diffusion equation is a manifestation of the PDF associated with E^3 being normal, so the Black-Scholes model may be taken to be predicated on Gaussian processes.

13. The Fractional Diffusion Equation

The fractional diffusion equation (FDE) can be derived by generalising the Gaussian characteristic function $P(k) = \exp\left(-\sigma^2 k^2/2\right)$ to the form

$$P(k) = \exp\left(-c \mid k \mid^\gamma\right)$$

where $\gamma \in [0,2]$ is the Lévy index and c is a constant with dimensions of Length$^\gamma$ as previously discussed in Section 7.2.

Using the Reisz definition of the fractional Laplacian operator ∇^γ, $\mathbf{r} \in \mathbb{R}^n$, namely,

$$\nabla^\gamma \leftrightarrow - \mid \mathbf{k} \mid^\gamma$$

with $D_\gamma = c/\tau$, repetition of the analysis given in Section 12.4 yields the homogeneous FDE

$$\frac{\partial}{\partial t}u(\mathbf{r},t) = D_\gamma \nabla^\gamma u(\mathbf{r},t)$$

where D_γ is the fractional diffusivity with dimensions of Length$^\gamma$/Time, and, for $\mathbf{r} \in \mathbb{R}^3$ with Cartesian coordinates (x,y,z),

$$\nabla^\gamma \equiv \frac{\partial^\gamma}{\partial \mid x \mid^\gamma} + \frac{\partial^\gamma}{\partial \mid y \mid^\gamma} + \frac{\partial^\gamma}{\partial \mid z \mid^\gamma}$$

Thus, we obtain a fundamental connectivity between between Einstein's evolution equation and fractional calculus, i.e., application of the Lévy distribution in Equation (10) yields the FDE.

13.1. Continuity Equation

For the case when $\gamma = 1$, we can use the FDE to construct the transport equation

$$\frac{\partial}{\partial t}u(\mathbf{r},t) + D_1 \hat{\mathbf{n}} \cdot \nabla u(\mathbf{r},t) = 0$$

where $\hat{\mathbf{n}}$ is the unit vector. This is a continuity equation, and, in the context of the evolution equation, illustrates the connectivity between the concept of flux (the flow of an ensemble of particles) and the Cauchy distribution (as discussed in Section 7.2).

13.2. Time-Independent Analysis

If we consider Equation (11), then for the time dependent case the FDE becomes

$$\nabla^{\gamma} u(\mathbf{r}) = s(\mathbf{r}), \ r \in \mathbb{R}^n$$

where $u(\mathbf{r})$ is a stochastic function. Since $\nabla^{\gamma} u(\mathbf{r}) \leftrightarrow - \mid \mathbf{k} \mid^{\gamma}$ we can construct the solution

$$u(\mathbf{r}) = \mathcal{F}_n^{-1} \left[\frac{S(\mathbf{k})}{\mid \mathbf{k} \mid^{\gamma}} \right], \ S(\mathbf{k}) \leftrightarrow s(\mathbf{r})$$

Using Equation (4) and the convolution theorem, we can then write $u(\mathbf{r})$ as

$$u(\mathbf{r}) = \frac{c_{n,\gamma}}{(2\pi)^n} \frac{1}{\mid \mathbf{r} \mid^{n-\gamma}} \otimes s(\mathbf{r}), \ 0 < \mathrm{Re}[\gamma] < n; \ \ c_{n,\gamma} = \pi^{\frac{n}{2}} 2^{n-\gamma} \frac{\Gamma\left(\frac{n-\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)}.$$

This solution for $u(\mathbf{r})$ defines the Riesz potential and has a fundamental scaling property obtained by considering the convolution of the source function for a scaling factor λ when it is simple to show that

$$\frac{c_{n,\gamma}}{(2\pi)^n} \frac{1}{\mid \mathbf{r} \mid^{n-\gamma}} \otimes s(\lambda \mathbf{r}) = \frac{1}{\lambda^{\gamma}} u(\lambda \mathbf{r})$$

and hence $u(\mathbf{r})$ is a scale-invariant stochastic function defined by the relationship

$$\Pr[u(\lambda \mathbf{r})] = \lambda^{\gamma} \Pr[u(\mathbf{r})]. \tag{32}$$

Thus, for a stochastic source, the Riesz potential $u(\mathbf{r})$ is a random scaling self-affine field—a random scaling fractal. In this context, Appendix B develops the relationship between the topological dimension n, the fractal dimension D and the Lévy index γ which is given by

$$D = \frac{3n + 2 - 2\gamma}{2}. \tag{33}$$

Thus, for example, a Mandelbrot surface, which has a fractal dimension $D = 4 - \gamma \in [2, 3]$, can be defined in terms of the solution to the two-dimensional fractional Poisson equation (FPE)

$$\nabla^{\gamma} u(\mathbf{r}) = s(\mathbf{r}), \ r \in \mathbb{R}^2, \ \gamma \in [2, 1]$$

and if $s(\mathbf{r})$ has a white spectrum, i.e., a spectrum whose power spectral density function (PSDF) is a constant, then the PSDF of $u(\mathbf{r})$ is determined by $1 / \mid \mathbf{k} \mid^{4-\gamma}$.

We note that by Taylor expanding Equation (11) for the time-independent case, then in Fourier space, we obtain $U(\mathbf{k}) = P(\mathbf{k}) U(\mathbf{k}) + S(\mathbf{k})$ and with $P(\mathbf{k}) = \exp(-c \mid \mathbf{k} \mid^{\gamma})$ it can be shown that [37]

$$u(\mathbf{r}) = \mathcal{F}_n^{-1} \left[\frac{S(\mathbf{k})}{1 - \exp(-c \mid \mathbf{k} \mid^{\gamma})} \right] \sim \frac{1}{r^{n+\gamma}} \otimes s(\mathbf{r}), \ r \to \infty; \ r \in \mathbb{R}^n$$

This asymptotic result yields a similar inverse power law but with the scaling law,

$$\Pr[u(\mathbf{r})] = \lambda^{\gamma} \Pr[u(\lambda \mathbf{r})],$$

a result that is a characteristic of scale-invariant field theory when the field equations are scale invariant so that for any solution $\phi(\mathbf{r})$, say, of the field equations, there exist other solutions of the form $\lambda^{\Delta} \phi(\mathbf{r})$ for an exponent Δ (not necessarily related to γ).

13.3. Time-Dependent Analysis

We study the FDE with $\mathbf{r} \in \mathbb{R}^1$ for a stochastic source, namely,

$$\left(D_\gamma \frac{\partial^\gamma}{\partial \mid x \mid^\gamma} - \frac{\partial}{\partial t} \right) u(x,t) = -s(x,t) \tag{34}$$

and consider the generic Green's function solution

$$u(x,t) = g(x,t) \otimes s(x,t)$$

where the convolution operation is taken to apply to both x and t. We are then required to compute the Green's function in this case.

13.4. Green's Function for the Fractional Diffusion Equation

We consider an evaluation of the Green's function for the fractional diffusion equation which is defined as the solution to

$$\left(D_\gamma \frac{\partial^\gamma}{\partial \mid x \mid^\gamma} - \frac{\partial}{\partial t} \right) g(\mid x \mid, t) = -\delta(\mid x \mid)\delta(t), \ t \geq 0. \tag{35}$$

Writing the Green's function in terms of the Fourier transformation

$$g(\mid x \mid, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} G(k,\omega) \exp(ik \mid x \mid) \exp(i\omega t) dk d\omega$$

noting that

$$\delta(\mid x \mid) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ik \mid x \mid) dk, \text{ and } \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega t) d\omega$$

and using the Reisz definition of a fractional derivative, Equation (35) becomes

$$[-D_\gamma \mid k \mid^\gamma - (i\omega)] G(k,\omega) = -1$$

which can be written in the factored form

$$[(\mid k \mid^{\gamma/2} + (-i\omega/D_\gamma)^{1/2})(\mid k \mid^{\gamma/2} - (-i\omega/D_\gamma)^{1/2})] G(k,\omega) = \frac{1}{D_\gamma}. \tag{36}$$

It is well known that for the equation

$$\left(D_\gamma \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) g(\mid x \mid, t) = -\delta(\mid x \mid)\delta(t),$$

$$(k^2 - \omega^2/D_\gamma) G(k,\omega) = \frac{1}{D_\gamma} \text{ or } [(k + \omega/\sqrt{D_\gamma})(k - \omega/\sqrt{D_\gamma})] G(k,\omega) = \frac{1}{D_\gamma}$$

and the outgoing Green's function is given by

$$\widetilde{g}(x \mid x_0, \omega) = \frac{i}{2\omega\sqrt{D_\gamma}} \exp[i(\omega/\sqrt{D_\gamma}) \mid x \mid]. \tag{37}$$

Generalising this result for Equation (36), we therefore consider the expression

$$\widetilde{g}(\mid x \mid, \omega) = \frac{i}{2D_\gamma(-i\omega/D_\gamma)^{1/\gamma}} \exp[i(-i\omega/D_\gamma)^{1/\gamma} \mid x \mid], \tag{38}$$

given that when $\gamma = 2$ and $-i\omega/D_\gamma := \omega^2/D_\gamma$, Equation (37) is recovered.

To find the time evolution of the Green's function, we are required to take the inverse Fourier transform of $\widetilde{g}(|\ x\ |, \omega)$, and evaluate the integral

$$g(|\ x\ |, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i}{2D_\gamma(-i\omega/D_\gamma)^{1/\gamma}} \exp[i(-i\omega/D_\gamma)^{1/\gamma}\ |\ x\ |] \exp(i\omega t) d\omega. \tag{39}$$

This can be achieved by writing the exponential function $\exp[i(-i\omega/D_\gamma)^{1/\gamma}\ |\ x\ |]$ as a series which yields the series solution

$$\begin{aligned} g(|\ x\ |, t) &= \frac{i}{2D_\gamma}\mathcal{F}_n^{-1}\left[\frac{1}{(-i\omega/D_\gamma)^{1/\gamma}} + i\ |\ x\ | + \sum_{n=2}^{\infty}\frac{i^n}{n!}(-i\omega/D_\gamma)^{(n-1)/\gamma}\ |\ x\ |^n\right] \\ &= \frac{i}{2D_\gamma}\mathcal{F}_n^{-1}\left[\frac{D_\gamma^{1/\gamma}}{(-1)^{1/\gamma}(i\omega)^{1/\gamma}} + i\ |\ x\ | + \sum_{n=2}^{\infty}\frac{i^n}{n!}\left(\frac{-1}{D_\gamma}\right)^{(n-1)/\gamma}(i\omega)^{(n-1)/\gamma}\ |\ x\ |^n\right] \\ &= \frac{i}{2D_\gamma}\left[\frac{D_\gamma^{1/\gamma}}{(-1)^{1/\gamma}\Gamma(1/\gamma)}\frac{H(t)}{t^{1-1/\gamma}} + i\ |\ x\ |\ \delta(t) + \sum_{n=2}^{\infty}\frac{i^n}{n!}\left(\frac{-1}{D_\gamma}\right)^{(n-1)/\gamma}\ |\ x\ |^n\ \delta^{[(n-1)/\gamma]}(t)\right] \end{aligned} \tag{40}$$

where

$$H(t) = \int_{-\infty}^{t} \delta(s) ds$$

is the Heaviside step function. This result comes from noting that for $0 < \alpha < 1$

$$\frac{1}{\Gamma(\alpha)}\frac{1}{t^{1-\alpha}} \leftrightarrow \frac{1}{(i\omega)^\alpha} \text{ and } (i\omega)^\alpha \leftrightarrow \delta^{(\alpha)}(t) \equiv \frac{d^\alpha}{dt^\alpha}\delta(t),$$

the function $\delta^{[(n-1)/\gamma]}(t)$ being defined in terms of Equation (9).

Note that from Equation (39) when $\gamma = 2$

$$\begin{aligned} g(|\ x\ |, t) &= \frac{1}{2\pi\sqrt{D_2}}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{i\omega}}\exp(-\sqrt{i\omega/D_2}\ |\ x\ |)\exp(i\omega t) d\omega \\ &= \frac{1}{2\sqrt{D_2}}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\exp(-\sqrt{s}\ |\ x\ |)}{\sqrt{s}\sqrt{D_2}}\exp(st) ds \\ &= \frac{1}{2\sqrt{\pi D_2 t}}\exp(-\ |\ x\ |^2/4D_2 t),\ t > 0, \end{aligned} \tag{41}$$

which is the Green's function for the classical diffusion equation where D_2 is the classical diffusivity.

13.5. Asymptotic Solution

From Equation (40), it is clear that we can define the time dependent Green's function for the case when $x \to 0$ as

$$g(0, t) = \frac{c_\gamma H(t)}{t^{1-1/\gamma}},\ c_\gamma = \frac{i}{2D_\gamma^{1-1/\gamma}(-1)^{1/\gamma}\Gamma(1/\gamma)}. \tag{42}$$

The Green's function solution to Equation (34) is then given by

$$u(t) = g(t) \otimes s(t) \tag{43}$$

where $u(t), g(t)$ and $s(t)$ represent the functions $u(0,t), g(0,t)$ and $s(0,t)$. We note that as $x \to 0$, Equation (41) reduces to

$$g(0,t) = \frac{1}{2\sqrt{\pi D_2 t}}, \ t > 0 \tag{44}$$

and that this result is consistent with Equation (42) given that for $\gamma = 2$, $\Gamma(1/2) = \sqrt{\pi}$ and we have

$$g(0,t) = \frac{i}{2\sqrt{D_2}\sqrt{-1}\Gamma(1/2)t^{1/2}} = \frac{1}{2\sqrt{\pi D_2 t}}, \ t > 0.$$

The scaling relationship associated with Equation (43) is given by (c.f. Equation (32))

$$\Pr[u(\lambda x)] = \lambda^{1/\gamma}\Pr[u(x)]$$

and from Equation (33), the relationship between Fractal Dimension D and the Lv́ey index in this case is $D = (5 - 2/\gamma)/2, \in [1,2]; \ \Rightarrow \gamma \in [2/3, 2]$. Figure 12 shows examples of the function $u_\gamma(t)$ for $\gamma = 2/3, 1$ and $3/2$ using the same stochastic source function $s(t)$. Comparing these results with the example given in Figure 3, it is clear that the case of $\gamma \sim 1$ provides a time series that (through visual inspection) better matches that of the financial signal. This is verified through application of regression applied to the data given in Figure 3 which yields $\gamma = 1.1455$ based on assuming that the data has an amplitude spectrum $\mid U(\omega) \mid$ with the following spectral power law:

$$\mid U(\omega) \mid \sim \frac{1}{\mid \omega \mid^{1/\gamma}}, \ \mid \omega \mid > 0 \tag{45}$$

This value of γ is the one associated with the data given in Figure 3 in its entirety, and, like the Lyapunov exponent and the volatility, it can be computed on a moving window basis to obtain a (short) time dependent signature which is explored further in Section 15.

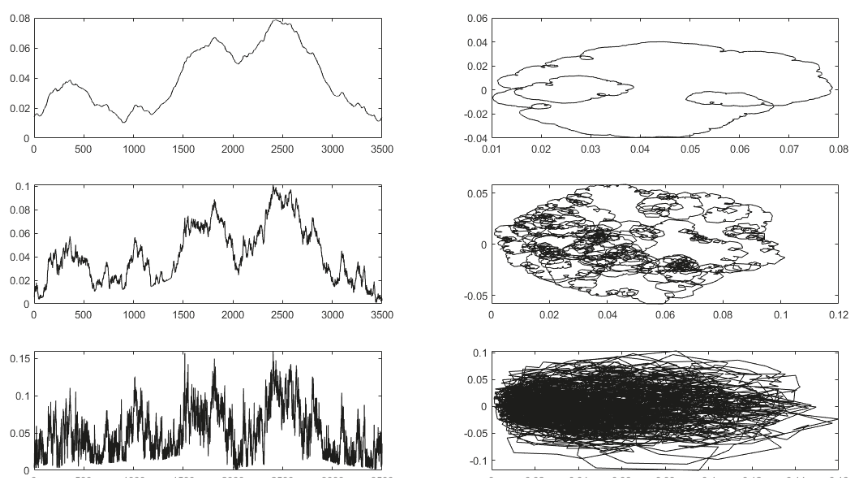

Figure 12. Examples of the function u_n given by the discretised form of Equation (43) for $\gamma = 1/2, 1$, and $3/2$ (left plots, respectively) and the associated complex plane representations obtained using Equation (16)—right plots, respectively.

The regression algorithm used to achieve this result is given in Appendix A.7 and is based on computing the exponent α associated with the power law $U(\omega) \sim\mid \omega \mid^{\alpha}$ using the least squares method (LSM). For a uniformly sampled frequency (or time) series $u_n > 0 \forall n,\ n = 1, 2, ..., N$ α is given by

$$\alpha = \frac{\sum\limits_{n=1}^{N} \ln U_n \sum\limits_{n=1}^{N} \ln n - N \sum\limits_{n=1}^{N} \ln U_n \ln n}{\left(\sum\limits_{n=1}^{N} \ln n\right)^2 - N \sum\limits_{n=1}^{N} \ln n^2}.$$

The Lv́ey index is then related to α by the equation $\gamma = -1/\alpha$. Note that to compute γ using the LSM requires computation of the amplitude spectrum using a discrete (fast) Fourier transform. The data used is that in the positive half-space of the amplitude spectrum with the DC component removed, thereby adhering to the condition $\mid \omega \mid > 0$ in the spectral power law defined by Equation (45). Thus, the LSM is applied for $\mid U_n \mid,\ n = 2, 3, ..., N/2$.

13.6. Discussion: Impulse Response Functions for Classical and Fractional Diffusion

Given Equation (43), it is clear that, in the asymptotic limit $x \to 0$, the difference between classical and fractional diffusion is compounded in the different Green's function given by Equations (42) and (44). Thus, ignoring the scaling parameters in Equations (42) and (44) as well as those of their Fourier transforms, we can compare the asymptotic solutions as follows:

- Classical Diffusion

$$u(t) = \frac{1}{\sqrt{t}} \otimes s(t),\ t > 0\ \Rightarrow\mid U(\omega) \mid= \frac{\mid S(\omega) \mid}{\sqrt{\mid \omega \mid}} \tag{46}$$

- Fractional Diffusion

$$u(t) = \frac{1}{t^{1-1/\gamma}} \otimes s(t),\ t > 0\ \Rightarrow\mid U(\omega) \mid= \frac{\mid S(\omega) \mid}{\mid \omega \mid^{1/\gamma}},\ \gamma \in [1, 2] \Rightarrow D \in [1.5, 2] \tag{47}$$

Unlike classical diffusion, fractional diffusion is characterised by a range of values of the Lévy index. The efficient market hypothesis is predicated on classical diffusion processes, based on E^3 for a Gaussian distribution. The ramifications of this is that the time series model for $u(0, t)$ given by Equation (46) is characterised by the impulse response function (IRF) $1/\sqrt{t}$. By comparison, the fractal market hypothesis is predicated on fractional diffusion processes based on E^3 for a Lévy distribution. The consequence of this is that the time series model for $u(0, t)$ given by Equation (47) is characterised by the IRF $1/t^{1-1/\gamma}$. Since financial signals tend to be non-stationary random fractals, variations in γ as a function of time are informative. However, before we study this, we consider another way to derive what is, in effective, the same basic result but via a different approach, an approach that is also based on E^3 but obtained via the GKFE subject to application of an appropriate memory function. This is discussed in the following section.

14. Solution to the GKFE for an Orthonormal Memory Function

In this section, we show that the temporal power law which characterises Equation (43)—i.e., $1/t^{1-1/\gamma}$—can be obtained from Equation (14) for a specific orthonormal memory function. The purpose of this is to show another route to deriving the power law which is informative in that it is based on the application of a memory function alone and does not involve specific application of the FDE as presented in the previous section. In this case, and, for $\mathbf{r} \in \mathbb{R}^1$, by writing Equation (14) in the form

$$\tau \frac{\partial}{\partial t} u(x, t) + u(x, t) = u(x, t) - n(t) \otimes u(x, t) + n(t) \otimes u(x, t) \otimes p(x)$$

we can construct a Green's function solution is given by

$$u(x,t) = g(t) \otimes u(x,t) - g(t) \otimes n(t) \otimes u(x,t) + g(t) \otimes n(t) \otimes u(x,t) \otimes p(x) \tag{48}$$

where $g(t)$ is the Green's function given by

$$g(t) = \frac{1}{\tau}\exp(-t/\tau),\ t > 0$$

which is the solution to

$$\tau\frac{\partial}{\partial t}g(t) + g(t) = \delta(t).$$

Provided the Laplace transform of the function $n(t)$ exists, we can write this Green's function solution as

$$u(x,t) = h(t) \otimes u(x,t) \otimes p(x) \tag{49}$$

where

$$h(t) \leftrightarrow \frac{\bar{n}(s)}{\tau s + \bar{n}(s)}$$

and $\leftrightarrow$ denotes the Laplace transformation, i.e., the mutual transformation from t-space to s-space. This result is obtained by using the convolution theorems for the Fourier and Laplace transforms, when Equation (14) can be written as

$$\bar{\tilde{u}}(k,s) = \bar{g}(s)\bar{\tilde{u}}(k,s) - \bar{g}(s)\bar{n}(s)\bar{\tilde{u}}(k,s) + \bar{g}(s)\bar{n}(s)\bar{\tilde{u}}(k,s)\tilde{p}(k)$$

where

$$\bar{\tilde{u}}(k,s) = \int_0^\infty \int_{-\infty}^\infty u(x,t)\exp(-ikx)dx\exp(-st)dt,\ \bar{g}(s) = \int_0^\infty g(t)\exp(-st)dt,$$

$$\bar{n}(s) = \int_0^\infty n(t)\exp(-st)dt \text{ and } \tilde{p}(k) = \int_{-\infty}^\infty p(x)\exp(-ikx)dx$$

Thus, noting that $\bar{g}(s) = (1+\tau s)^{-1}$, we can write

$$\bar{\tilde{u}}(k,s) = -\frac{\bar{g}(s)}{1-\bar{g}(s)}\bar{n}(s)\bar{\tilde{u}}(k,s) + \frac{\bar{g}(s)}{1-\bar{g}(s)}\bar{n}(s)\bar{\tilde{u}}(k,s)\tilde{p}(k) = -\frac{\bar{n}(s)}{\tau s}\bar{\tilde{u}}(x,t) + \frac{\bar{n}(s)}{\tau s}\bar{\tilde{u}}(k,s)\tilde{p}(k)$$

leading to the equation

$$\bar{\tilde{u}}(k,s) = \bar{h}(s)\bar{\tilde{u}}(k,s)\tilde{p}(k).$$

Inverse Fourier-Laplace transformation then gives Equation (49).

Equation (49) supports an iterative solution of the form

$$u_{m+1}(x,t) = h(t) \otimes u_m(x,t) \otimes p(x),\ m = 0,1,2,...$$

and we may therefore consider an approximation based on the first iterate, i.e.,

$$u(x,t) = h(t) \otimes u_0(x,t) \otimes p(x)$$

The condition required for this approximation to apply can be obtained as follows: Given that

$$\|u(x,t)\|_1 \leq \|h(t)\|_1\|u_0(x,t)\|_1\|p(x)\|_1 = \|h(t)\|_1\|u_0(x,t)\|_1$$

then

$$\frac{\|u(x,t)\|_1}{\|u_0(x,t)\|_1} \leq \|h(t)\|_1$$

and hence we required that

$$\|h(t)\|_1 << 1. \tag{50}$$

Further, if we consider the case when $u_0(x,t) = \delta(x)s(t)$, then we can write

$$u(t) = \int_{-\infty}^{\infty} u(x,t)dx = h(t) \otimes s(t) \int_{-\infty}^{\infty} p(x)dx = h(t) \otimes s(t).$$

If we now choose a memory function $m(t)$ whose Laplace transform is $s^{\beta-1}$ then the orthonormality property $n(t) \otimes m(t) = \delta(t)$ is satisfied if the Laplace transform of $n(t)$ is $s^{1-\beta}$ given that from the convolution theorem for Laplace transforms $\bar{n}(s)\bar{m}(s) = 1$. In this case

$$\bar{h}(s) = \frac{s^{1-\beta}}{\tau s + s^{1-\beta}} = \frac{1}{1+\tau s^{\beta}} \sim \frac{1}{\tau s^{\beta}},\ \tau >> 1.$$

Since

$$\frac{1}{s^{\beta}} \leftrightarrow \frac{H(t)}{\Gamma(\beta)t^{1-\beta}}$$

we obtain the solution

$$u(t) = h(t) \otimes s(t)$$

where

$$h(t) = \frac{1}{\tau\Gamma(\beta)t^{1-\beta}},\ t > 0.$$

This solution is characterised by Riemann–Liouville (fractional) integral which has self-affine properties, i.e., properties that exhibit "stochastic trending characteristics". In other words, $u(t)$ defines a random scaling fractal function whose impulse response function is $1/t^{1-\beta}$, a result that is, in light of the above analysis, been shown to be a PDF independent first order solution to the GKFE for a memory function

$$m(t) = \frac{1}{\Gamma(1-\beta)t^{\beta}}.$$

In order to comply with Condition (50), we require that

$$\|h(t)\|_1 = \frac{1}{\tau\Gamma(\beta)} \int_0^{\tau} \frac{1}{t^{1-\beta}} = \frac{1}{\beta\Gamma(\beta)\tau^{1-\beta}} << 1,$$

which is satisfied for the case when $\tau >> 1,\ \beta \in [0,1)$.

Clearly, ignoring differences in scaling, compatibility of this solution for $u(t)$ with Equation (43) is obtained when $\beta = 1/\gamma$. Thus, subject to the conditions imposed in each case we have shown that there exist temporal solutions to the FDE and the GKFE that exhibit a fundamental power law of $1/t^{1-1/\gamma}$ for Lévy index γ. In the former case, the solution is predicated on defining the PDF in the evolution equation (a Lévy distribution) whereas in the latter case, the result is independent of the PDF but predicated on the definition of the memory function (with power law $1/t^{\beta}$). In both cases, the solution is characterised by a fractional integral which is self-affine, a property that is fundamental to the analysis and interpretation of financial signals and underpins the fractal market hypothesis.

15. Time Varying Lévy- and α-Indices

As with the other indices considered in this paper, the time dependence of γ for a financial signal can be obtained by computing it over a moving (look-back) window. Figure 13 shows an example of this short time signature. for a financial signal (the first 1000 elements of the FTSE 100 prices given in Figure 3), normalised for display purposes. In this example γ has been computed using the function given in Appendix A.7.

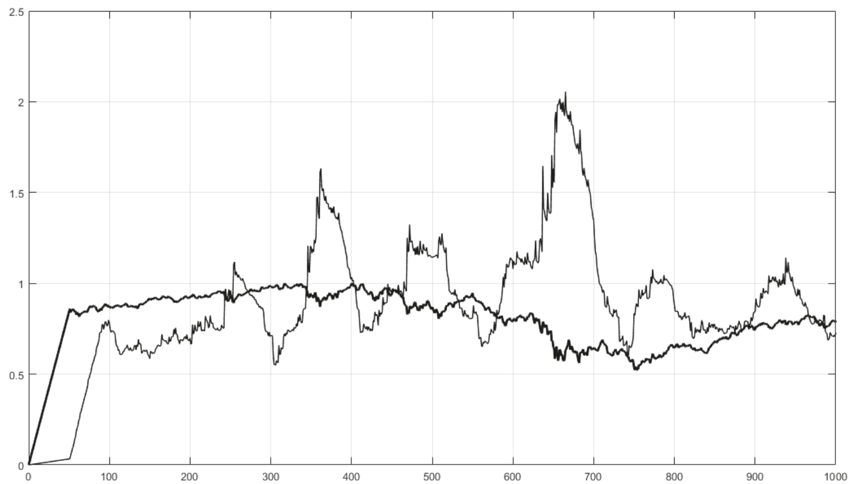

Figure 13. Example of computing the time dependent Lévy index (solid line —) for a normalised financial signal (bold solid line **—**) based on the application of Equation (45) using function Levy with a look-back window of 50 data elements.

This result assumes that the short-time amplitude spectrum adheres to the scaling law $\mid \omega \mid^{-\frac{1}{\gamma}}$, and, strictly within the context of this spectral model, the numerical range of γ is only limited by the original definition of the of a Lévy distribution, i.e., $\gamma \in [0, 2]$ as given in Equation (17). The statement $\gamma \in (1, 2]$ given in Equation (47) is a result of imposing the condition that $0 < 1/\gamma < 1$ in order that the Fourier transform pair relationship given by Equation (3) is satisfied. However, if we arbitrarily consider a modified IRF given by $1/t^{1-1/\gamma}$, $\gamma \in [0, 2]$, then it is clear that we can consider a short time scaling function given by (for $t > 0$)

$$u(t) = t^{\alpha} \quad \text{where} \quad \begin{cases} \alpha > 0, & \gamma < 1; \\ \alpha = 0, & \gamma = 1; \\ \alpha < 0, & \gamma > 1. \end{cases}$$

for the case when $s(t) = \delta(t)$. This result has similar properties to the Lyapunov exponent in terms of providing an 'α-index' that reflects up-ward (for $\alpha > 0$) and down-ward (for $\alpha < 0$) trends. Further, as with the LVR considered in Section 10 and compounded in Equation (26) we can scale the α-index by the inverse of the volatility to produce Alpha-to-volatility ratio (AVR) index given by

$$\alpha_{\sigma} = \frac{\alpha}{\sigma} \tag{51}$$

In practice, the value of the α can easily be computed using the LSM which is compounded in the function Alpha given in Appendix A.8.

Following the same procedure to that discussed in Section 10.4 (specifically Figure 7), Figure 14 shows example results of running Backtester for the first 1000 elements of the FTSE 100 prices given in Figure 3 but for the AVR index $\alpha_\sigma[n]$ instead of $\lambda_\sigma[n]$. The example given is for Backtester (30,10,1000) which yields a combined entry/exit (long/short) accuracy of 60.98%. Note that this results is obtained by replacing the code

```
L(m)=Lyapunov(s,T,1);%Compute the Lyapunov Exponent.
V(m)=Volatility(s,T);%Compute the Volatility.
R(m)=L(m)/V(m);%Compute the Lyapunov to Volatility Ratio (LVR).
```

with

```
A(m)=Alpha(s,T);%Compute the Alpha Index.
V(m)=Volatility(s,T);%Compute the Volatility.
R(m)=A(m)/V(m);%Compute the Alpha-To-Volatility Ratio (AVR).
```

in function Backtester given in Appendix A.6.

Apart from the scale in amplitude, the signature of the ARV is very similar to the LVR (comparing Figures 14 and 7). However, the trend prediction accuracy is relatively low and the computational time greater (due to the repeated application of the LSM) which suggests that the LVR is a more reliable and computationally efficient index. However, this statement must be understood within the context of the limited data that was used and demonstrated for this publication and must be quantified further using WT-maps for a range of financial signals and the functions given in Appendix A, a study that lies beyond the scope of this work.

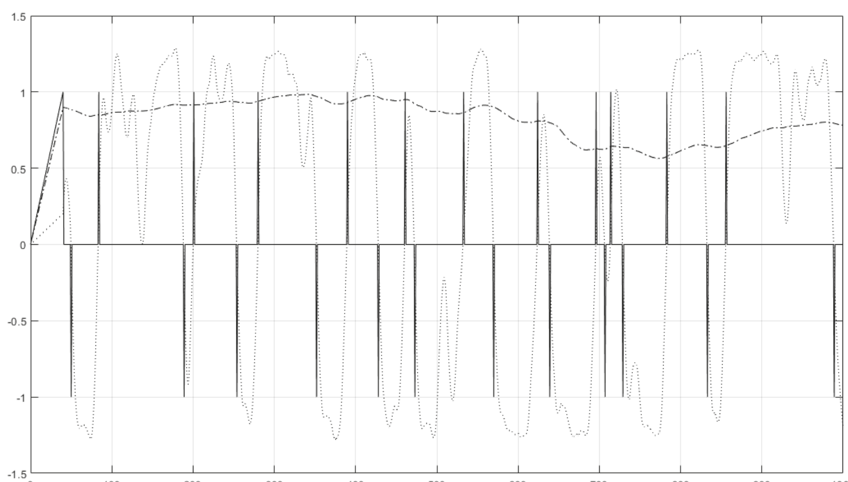

Figure 14. Example graphical output from function Backtester. The plot shows the post-filtered (for $T = 10$) AVR $\alpha_\sigma[n]$ (dotted line), the pre-filtered time series u_n (dot-dashed line) and $z_c[n]$ (solid line) which identifies the zero-crossings of $\alpha_\sigma[n]$. The plot is for the case when pre-filtering is undertaken for a look-back window size of $W = 30$ and post-filtering using a look-back window size of $T = 10$. The financial time series data used in this case is FTSE100 daily prices: 14/03/2006–26/02/2010.

16. Summary, Conclusions and Open Questions

One of the principal themes of this paper has been to develop financial indices that in all cases can been traced back to a fundamental field equation of statistical physics, namely, Einstein's evolution equation—Equation (10). In this context, we have developed expressions for the following financial indices:

- the Lyapunov Exponent;
- the Volatility;
- the Lévy Index.

16.1. Summary

We have explored the ability for the time varying Lyapunov-to-volatility ratio (LVR) to predict the trend of a financial signal in terms of a change in polarity and the period over which that polarity is sustained subject to pre- and post-filtering as discussed in Sections 10.1.1 and 10.1.2, respectively. The filtering processes are critically dependent on the values of the look-back windows that are applied and a quantification of the values required to optimise the predictive power has been explored in Section 10.4 in terms of the WT map. Application of the LVR provides a time signature whose maximum and minimum values correlate with regions of a financial signal that have up-ward and down-ward trends with low volatility, respectively. In Section 11, a short study has been presented to use this result as a criterion for the application of EC to predict short term future prices. In this context, computing the time varying LVR has two primary uses:

- predicting the entry points in time for making, holding or withdrawing an investment;
- assessing the position in time when application of EC can be expected to yield optimally accurate short term price predictions.

It is noted that in regard to the application of EC, the volatility alone can be used as an assessment criterion, low volatilities providing a flag for the use of EC on a moving window basis to update previous price predictions.

While the derivation and the application of the LVR is predicated on the evolution equation (at least, as demonstrated in this paper), it does not rely on the application of fractional calculus which has been a focal issue in regard to the composition of this paper. Thus, the latter half of this paper was devoted to an analysis of fractional calculus with the aim of showing how, in particular, the classical diffusion and fractional diffusion equations are both directly related to the evolution equation and can be derived directly from it, the difference between the two equations being compounded in the PDF that "governs" the spatial distribution of the density field.

We have shown that the classical diffusion equation is predicated on a Gaussian distribution and that the fractional diffusion equation is predicated on a (symmetric) Lévy distribution. In turn, it has been shown that at the spatial origin (i.e., as $x \to 0$), the temporal impulse response functions for these two cases are given by $1/\sqrt{t}$ and $1/t^{1-1/\gamma}$, respectively, functions that underpin the efficient and fractal market hypotheses, respectively. In deriving these functions, we have attempted to show the intrinsic connectivity between the application of Lévy statistics to the evolution equation, the fractional diffusion, the application of fractional calculus for solving this equation and the analysis of the solution leading directly to the description of a stochastic self-affine field—a random scaling fractal signal.

In addition to the theoretical concepts presented in this paper, we have provided a set of numerical algorithms that allows the reader to reproduce the results given. These algorithms are based on the m-code given in Appendix A. They have been designed to give interested readers the facility to study the methods used for the wide variety of financial time series available online and to develop the algorithms further as required. Their development has been based on maintaining consistency with the theoretical analysis derived at the expense of any further and more sophisticated software engineering. Hence, issues such as error checks on input/output data, processing parameters and data/processor compatibility have not been considered.

16.2. Conclusions

The application of fractional calculus in mathematical finance is well known and in this paper we have provided a unified approach to showing that this is the case using Einstein's evolution equation as a fundamental field equation. This approach has the potential for the development of a range of new models for a financial signal by introducing different PDFs in Equation (11) to those that have been considered here, the categorisation of such models for different time series lying beyond the scope of this publication.

The primary results are given in Section 10 which shows that a relatively high accuracy for predicting up-ward and down-ward trends can be obtained, thereby providing the potential for a profitable trading strategy to be implemented. However, it must be noted that the quantitative results given in Section 10 in regard to this statement are strictly applicable only to the data used (i.e., the daily FTSE100 and Euro-USA dollar exchange rate). Application of the algorithms presented must therefore be fully quantified and characterised for any and all specific financial time series data used, "quantification" being compounded in the associated WT map.

The use of EC discussed in Section 11 verifies that short time price prediction can be exercised if the LVR has reached a maximum or minimum threshold in excess of +2 or -2, respectively. However, as pointed out in Section 11, the material presented in this respect has only been introduced to complement the main theme of this paper. Further studies are required to assess the accuracy of EC prediction on a moving window basis in terms of the number of future projected price values which maintain an appropriate forecasting accuracy and the associated look-back window used to generate short time forecasting equations of the type given by Equation (27), for example.

16.3. Open Questions

There are a number of open questions which this paper has raised that are the subject of further investigation. The reader is invited to consider the following examples:

- The specific form of the evolution equation used in this work has been based on Equation (11) and it may be of value to consider the affect of the decay term $-Ru(\mathbf{r}, t)$ given in Equation (12).
- Given that the critical step in deriving the IRF $1/t^{1-1/\gamma}$ (from which γ can be computed) is the asymptotic condition $x \to 0$, what are the consequences of developing a numerical algorithm to compute γ when this condition is negated?
- What is the impact of the LVR and AVR in terms of their possible inclusion into machine learning algorithms that use sets of more conventional financial indices and other statistical metrics for forecasting?

In regard to more generic questions, the following examples may be of interest:

- In regard to E^3, the PDFs considered in this work are the delta function, Gaussian function and Lévy distribution which provide models associated with the random walk, efficient and fractal hypothesis, respectively. An investigation into the models for $u(\mathbf{r}, t)$ and metrics thereof, associated with the application of different PDF (including non-symmetric distributions), is therefore warranted.
- Similarly, what is the effect of introducing different memory functions into the generalised Kolmogorov–Feller equation, i.e., E^3 in all but name, expressed in terms of memory function $m(t)$, and, further, is it possible to develop an inverse solution in which a financial signal $u(t)$ can be used to derive a estimate of $m(t)$ for a known distribution $p(\mathbf{r})$.
- What is the relationship/connectivity (or otherwise) between fractional and Itô calculus in regard to E^3?

16.4. Final Remarks

One of the primary aims of this paper was to realise the connectivity compounded in Table 1, and, in this broader context, to show the relationship between E^3 and fractional calculus through the application of a non-Gaussian distribution, specifically a symmetric Lévy distribution whose characteristic function is a generalisation of the Gaussian function (for a real constant c) $\exp(-c \mid k \mid^2)$ to $\exp(-c \mid k \mid^\gamma)$, $0 < \gamma < 2$. The effect of this has been to show that there is a close relationship between non-Gaussian processes of this type and the self-affine characteristics of stochastic signals modelled in terms of the solution to a fractional differential equation, i.e., the fractional diffusion equation. This approach provides the basis for a more general study that transcends the specific distributions considered in order to derive stochastic models that are a more complete and accurate description associated with the varied properties of financial signals in which the applications of fractional calculus is a central theme.

In terms of the computational methods presented, a primary aim is to classify the WT maps for a range of different financial data in terms of the LVR and AVR and to further quantify the accuracy of these two indices in regard to different data types. The purpose of this is to categorise the type of financial times series that are best suited to the trend analysis proposed in terms of a robust predictive accuracy. In turn, this exercise will inform a quantification of the use of EC for predicting short term prices with the aim of obtain a quantitative relationship between the look-back window used, the number of future prices that can be predicted with a specified accuracy and the amplitude of the LVR and/or ALR for a specific financial signal.

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Abbreviations

The following abbreviations are used in this manuscript:

AVR	Alpha-to-Volatility Ratio
CF	Characteristic Function
CGI	Computer Generated Imagery
DC	Direct Current
EC	Evolutionary Computing
E^3	Einstein's Evolution Equation
FDE	Fractional Diffusion Equation
FFT	Fast Fourier Transform
FMH	Fractal Market Hypothesis
FPE	Fractional Poisson Equation
GKFE	Generalised Kolmogorov–Feller Equation
IRF	Impulse Response Function
KFE	Kolmogorov–Feller Equation
LSM	Least Squares Method
LVR	Lyapunov-to-Volatility Ratio
MFP	Mean Free Path
PDF	Probability Density Function
PSDF	Power Spectral Density Function

Appendix A. Prototype MATLAB Functions

The functions given in this appendix have not been exhaustively tested and no data/parameter error checks or processing anomalies, for example, have been implemented. The functions are provided to give the reader a guide to the basic numerical solutions required to implement the computational procedures discussed in this paper, and, in turn, to help the reader appreciate the theoretical models presented. It is expected that interested readers will use the functions provided as a guide to extending their operational characteristics and software engineer their functionality. Where possible, the notation used for array variables and constants are based on the mathematical notation used in this paper or are acronyms for the function names. The software was developed and implemented using (64-bit) MATLAB R2017b with double precision floating point arithmetic.

Appendix A.1. Software Development and Usage

The MATLAB function given in this appendix are provided to give readers access to prototype source code that implements the algorithms discussed in this paper using m-code. In both cases, copyright is attributed to the authors and all rights are reserved. Redistribution and use in source and binary forms, with or without modification, are permitted provided that the following conditions are met:

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Appendix A.2. Function Lyapunov

```
function lambda=Lyapunov(data,N)
% Function to compute the Lyapunov Exponent - lambda -
%for a data stream of length N and time period defined by tau.
%
%Compute the log differences of the data.
for n=1:N-1
d(n)=log(data(n+1)/data(n)); end
d(N)=d(N-1);%Set end point value.
%Return the exponent.
lambda=sum(d);
```

Appendix A.3. Function Volatility

```
function sigma=Volatility(data,N)
%Function to compute the Volatility - sigma - for data of size N.
```

```
%
%Compute the log price differences.
for n=1:N-1
d(n)=log(data(n+1)/data(n)); end
d(N)=d(N-1);%Set end point value.
%Return the Volatility
sigma=sqrt(sum(abs(d).^2));
```

Appendix A.4. Function Movav

```
function Fdata=Movav(data,N,W)
%Function to compute the moving average of data of length N
%using a period (a look-back window) of size W
for n=W:N
%Window data.
for m=1:W
D(m)=data(n-W+m); end
%Compute the mean.
Fdata(n-W+1)=mean(D); end
```

Appendix A.5. Function Evaluator

```
function Evaluator(ZC,G,M,T)
%FUNCTION:
%Evaluates the accuracy of a short time trend analysis
%indicator in terms of the actual price differences that
%occurred. This~provides a measure of the accuracy in
%terms of the long and short positions identified that were
%successful in terms of the short time series dynamic
%relative to a net price difference.
%
%INPUTS:
%ZC - Array composed of zeros crossing point indicators;
% ZC = +1 flags the start of a positive trend,
% ZC = -1 flags the start of a negative trend.
%G - Array composed of the values of the time series at
% which the start of a trend is identified by a
% zero crossings.
%M - Data size.
%T - Period (moving window size used for data analysis).
%
%Read non-zero entries of input array ZC and G to vector arrays
%P and Q, respectively, thereby extracting all non-zero values.
n=1; N=1;%Initiate counters.
%Start process
for m=T+1:M-T-1
if G(m)>0.0
P(n)=G(m);n=n+1; Q(N)=ZC(m);N=N+1; end
end
%Count the number of times that an indication of a future upward
%trend led to a net positive price increase (up_good)and then the
%number of times that this failed to be the case (up_bad).
```

```
up_good=0; up_bad=0; %Initiate counters
%Start process.
for n=1:N-2
if Q(n)>0 & P(n+1)-P(n)>0
up_good=up_good+1; end
if Q(n)>0 & P(n+1)-P(n)<0
up_bad=up_bad+1; end
end
%Count the number of times that an indication of a future downward
%trend led to a net negative price decrease (down_good)and then the
%number of times that this failed to be the case (down_bad).
down_good=0; down_bad=0;%Initiate counters
%Start process
for n=1:N-2
if Q(n)<0 & P(n+1)-P(n)<0
down_good=down_good+1;
end
if Q(n)<0 & P(n+1)-P(n)>0
down_bad=down_bad+1;
end
end
%Provide outputs on the percentage accuracy of:
% - Successfully predicted upward trend - 'Entries_Accuracy'
% - Successfully predicted downward trend -'Exits_Accuracy'
% - The combine accuracy of both success rates - 'Combined_Accuracy'.
if (double(up_good)+double(up_bad))>0
Entries_Accuracy=100*double(up_good)/(double(up_good)+double(up_bad))
else
Entries_Accuracy=0.0
end
if (double(down_good)+double(down_bad))>0
Exists_Accuracy=100*double(down_good)/(double(down_good)+double(down_bad))
else
Exists_Accuracy=0
end
Combined_Accuracy=(Exists_Accuracy+Entries_Accuracy)/2
```

Appendix A.6. Function Backtester

```
function Backtester(W,T,L)
%FUNCTION: Back-testing procedure to compute accuracy of trend analysis
%INPUT PARAMETERS:
%(int) W > 0 - Size of window for pre-filtering using moving average of data
%(int) T > 1 - Size of window for computing financial indices.
%(int) L > 4 - Size of data stream to be processed (must be less than or
%equal length of data read from file).
%
%Read financial time series from Data.txt file into data array where ... denotes
%the path to the folder containing the file (for a windows operating system).
fid=fopen('...\Data.txt','r');%Open file
[series M]=fscanf(fid,'%g',[inf]);%Read time series data.
```

```
fclose(fid); series=flip(series); %Flip order of data (as required).
%Note: Some historical financial time series data which is available
%on the Internet is often given in terms in reverse time and for this
%reason the flip function is used.
%
%Extract L data components from the time series where L<=M.
for n=1:L
data(n)=series(n);
end
data=data./max(data);%Normalise the series for comparative
%display purpose involving the plotting of multiple data sets.
M=size(data,2);%Reset M to data size
%Filter the time series data using a moving average filter
Fdata=Movav(data,M,W);
%figure(100), plot(data); Figure~(200), plot(Fdata);
M=size(Fdata,2);%Reset M to size of filtered data
%Start moving window process.
for m=T:M-T
%Window the data.
for n=1:T
s(n)=Fdata(n-1+m);
end
%
L(m)=Lyapunov(s,T,1);%Compute the Lyapunov Exponent
V(m)=Volatility(s,T);%Compute the Volatility.
R(m)=L(m)/V(m);%Compute the Lyapunov to Volatility Ratio (LVR).
D(m)=Fdata(m-1+T);%Assign value of Fdata to D (for later use)
x(m)=m;%Set counter to x (for later use).
%
%Compute zero crossings
if m>T
k=1;
%Compute mean of LVRs - post filtering.
for n=m-T:m
Data(k)=R(n);
k=k+1;
end
F(m)=mean(Data);
%Evaluate zero crossings from negative to positive half-space
if F(m)>0 & F(m-1)<=0
ZC(m)=1;%Zero-crossing given positive flag.
G(m)=D(m);%Assignment for later evaluation
else
ZC(m)=0;%Set value to zero
G(m)=0;%set value to zero
end
%Evaluate zero crossings from positive to negative half-space
if F(m)<0 & F(m-1)>=0
ZC(m)=-1;%Zero-crossing given negative flag.
G(m)=D(m);%Assignment for later evaluation
```

```
end
%Plot filtered data D, filtered LVR and Zero-crossing flags
%using black dashdot, dotted and solid lines, respectively.
Figure~(1), plot(x,D,'k-.',x,F,'k:',x,ZC,'k-');
%For colour plots, plotting filtered data D, filtered LVR and
%Zero-crossing flags using red, green and blue lines, respectively,
%use Figure~(1), plot(x,D,'r-',x,F,'g-',x,ZC,'b-');
else
end
grid on%Display grid
pause(0.01);%Retain plot ffor 0.01 seconds
end%Repeat process and update plot
%Evaluate accuracy of strategy.
Evaluator(ZC,G,M,T);
clear;%Remove all variables from the workspace.
```

Appendix A.7. Function Levy

```
function gamma=Levy(data,N)
%Computation of the Levy Index using the least squares algorithm.
%Compute the Amplitude Spectrum
data=abs(fft(data));
%Compute the logarithm of the data for half-space data with DC
%component removed.
for n=2:round(N/2)
ydata(n)=log(data(n)); xdata(n)=log(n); end
%Compute each term of the least squares formula.
%associated with log scaling law gamma*log(data)
term1=sum(ydata).*sum(xdata); term2=sum(ydata.*xdata);
term3=sum(xdata)^2; term4=sum(xdata.^2);
%Compute alpha
gamma=(term1-(N*term2))/(term3-(N*term4));
%Compute Levy Index
gamma=-1/gamma;
```

Appendix A.8. Function Alpha

```
function alpha=Alpha(data,N)
%Computation of the Alpha Index using the least squares algorithm.
%Compute the logarithm of the input data
for n=1:N
ydata(n)=log(data(n)); xdata(n)=log(n); end
%Compute each term of the least squares formula.
%associated with log scaling law alpha*log(data)
term1=sum(ydata).*sum(xdata); term2=sum(ydata.*xdata);
term3=sum(xdata)^2; term4=sum(xdata.^2);
%Compute alpha
alpha=(term1-(N*term2))/(term3-(N*term4));
```

Appendix B. Relationship between the Lévy Index and the Fractal Dimension

Consider a simple Euclidean straight line ℓ of length $L(\ell)$ over which we 'walk' a shorter 'ruler' of length δ. The number of steps taken to cover the line $N[L(\ell),\delta]$ is then $L(\ell)/\delta$ which is not always an integer for arbitrary L and δ. Since

$$N[L(\ell),\delta] = \frac{L(\ell)}{\delta} = L(\ell)\delta^{-1}$$

$$\Rightarrow 1 = \frac{\ln L(\ell) - \ln N[L(\ell),\delta]}{\ln\delta} = -\left(\frac{\ln N[L(\ell),\delta] - \ln L(\ell)}{\ln\delta}\right)$$

which expresses the topological dimension $n = 1$ of the line. In this case, $L(\ell)$ is the Lebesgue measure of the line and if we normalize by setting $L(\ell) = 1$, the latter equation can then be written as

$$1 = -\lim_{\delta\to 0}\left[\frac{\ln N(\delta)}{\ln\delta}\right]$$

and, in the asymptotic limit

$$N(\delta) = \frac{1}{\delta},\ \delta \to 0 \tag{A1}$$

For extension to a fractal line f, the essential point is that the fractal dimension should satisfy an equation of the form

$$N[F(f),\delta] = F(f)\delta^{-D}$$

where $N[F(f),\delta]$ is 'read' as the number of rulers of size δ needed to cover a fractal set f whose measure is $F(f)$ which can be any valid suitable measure of the curve. Normalising, for $F(\ell) = 1$, we can then define the fractal dimension as

$$D = -\lim_{\delta\to 0}\left[\frac{\ln N(\delta)}{\ln\delta}\right]$$

and, in the asymptotic limit

$$N(\delta) = \frac{1}{\delta^D},\ \delta \to 0. \tag{A2}$$

Consider the scaling relationship between the amplitude $A(t)$ of a signal at a time $t \in [0,1]$ given by

$$A(t) = t^H,\ H \in [0,1]$$

where H is the Hurst dimension. If the time period is divided up into $N = 1/\Delta t$ equal intervals, then the amplitude increments ΔA are given by

$$\Delta A = \Delta t^H = N^{-H}$$

The number of boxes of size δ required to cover the area $\Delta A\Delta t$ is, using Equation (A1), given by $N^{-H}/\delta^2 = N^{2-H}$. Thus we can write

$$N(\delta) = \frac{1}{\delta^{2-H}},\ \delta \to 0$$

and, given Equation (A2), by inspection,

$$D = 2 - H.$$

Thus, for example, a signal where $H = 1/2$ has a fractal dimension of 1.5. For higher topological dimensions n, using a similar box counting measure, we have

$$D = n + 1 - H, \ \mathbf{r} \in \mathbb{R}^n \tag{A3}$$

Consider a random scaling fractal signal defined by a time dependent function $f(t)$. Let $f_T(t)$ denote a component of the function which is of finite support

$$f_T(t) = \begin{cases} f(t), & 0 < t < T; \\ 0, & \text{otherwise.} \end{cases}$$

where

$$F_T(\omega) \leftrightarrow f_T(x)$$

which has a power spectrum defined by

$$P_T(\omega) = \frac{1}{T} \mid F_T(\omega) \mid^2, \ P(\omega) = \lim_{T\to\infty} P_T(\omega).$$

Let the function $g(t)$ be the result of scaling the function $f(t)$ by $1/a^H$ for a real constant $a > 0$. Then we can write

$$g_T(t) = \begin{cases} g(t) = \frac{1}{a^H} f(at), & 0 < t < T; \\ 0, & \text{otherwise.} \end{cases}$$

where

$$G_T(\omega) \leftrightarrow g_T(x)$$

with power spectrum

$$Q_T(\omega) = \frac{1}{T} \mid G_T(\omega) \mid^2, \ Q(\omega) = \lim_{T\to\infty} Q_T(\omega).$$

We can therefore construct the equation

$$G_T(\omega) = \int_0^T g_T(t) \exp(-i\omega t) dt = \frac{1}{a^{H+1}} \int_0^T f(\tau) \exp\left(-\frac{i\omega\tau}{a}\right) d\tau, \ \tau = at$$

showing that

$$G_T(\omega) = \frac{1}{a^{H+1}} F_T\left(\frac{\omega}{a}\right).$$

The power spectrum of $g_T(t)$ is therefore given by

$$Q_T(\omega) = \frac{1}{a^{2H+1}} \frac{1}{aT} \left| F_T\left(\frac{\omega}{a}\right)\right|^2 \Rightarrow Q(\omega) = \frac{1}{a^{2H+1}} P\left(\frac{\omega}{a}\right), \ T \to \infty$$

and setting $\omega = 1$ and then replacing $1/a$ by ω we obtain

$$Q(\omega) \propto \frac{1}{\mid \omega \mid^\beta}, \ \beta = 2H + 1.$$

The corresponding amplitude spectrum $A(\omega)$ is therefore characterised by

$$A(\omega) \propto \frac{1}{\mid \omega \mid^\gamma}, \ \gamma = \beta/2.$$

The result $\beta = 2H + 1$ applies to case when $\mathbf{r} \in \mathbb{R}^1$ and for $\mathbf{r} \in \mathbb{R}^n$ generalises to $\beta = 2H + n$ so that from Equation (A3) we obtain Equation (33).

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The Mittag-Leffler Fitting of the Phillips Curve

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Abstract: In this paper, a mathematical model based on the one-parameter Mittag-Leffler function is proposed to be used for the first time to describe the relation between the unemployment rate and the inflation rate, also known as the Phillips curve. The Phillips curve is in the literature often represented by an exponential-like shape. On the other hand, Phillips in his fundamental paper used a power function in the model definition. Considering that the ordinary as well as generalised Mittag-Leffler function behave between a purely exponential function and a power function it is natural to implement it in the definition of the model used to describe the relation between the data representing the Phillips curve. For the modelling purposes the data of two different European economies, France and Switzerland, were used and an "out-of-sample" forecast was done to compare the performance of the Mittag-Leffler model to the performance of the power-type and exponential-type model. The results demonstrate that the ability of the Mittag-Leffler function to fit data that manifest signs of stretched exponentials, oscillations or even damped oscillations can be of use when describing economic relations and phenomenons, such as the Phillips curve.

Keywords: econometric modelling; identification; Phillips curve; Mittag-Leffler function

1. Introduction

It is because of, or thanks to, Paul Anthony Samuelson and Robert Merton Solow [1], that the economists all around the world call the negative correlation between the rate of wage change (or the price inflation rate) and the unemployment rate the Phillips curve (PC). It is lesser-known that the idea occurred in the work by Irving Fisher [2] more than 30 years before publishing the famous paper of Alban William Housego Phillips [3]. Fisher was not the only one who would deserve such an important discovery be named after him. Three years before Phillips paper, Arthur Joseph Brown [4] precisely described the inverse relation between the wage and price inflation and the rate of unemployment. Also Richard George Lipsey [5] played an important role by the birth, creation of the theoretical foundations and popularisation of the PC. In the empirical studies the inverse relationship between the rate of wage change and the unemployment rate was proven, e.g., for the United States of America [1,6,7] or United Kingdom [3–5]. The policy implications were for the first time mentioned by Samuelson and Solow [1]. The PC was in its beginnings widely used by the policy-makers to benefit from the trade-off to decrease the unemployment at a cost of minor increase of the inflation–the "sacrifice ratio". Since then the PC has been studied, extended and re-formulated by many authors. For example, the model representing the New Keynesian theory of the output-inflation trade-off allows the expectations to jump based on the current and anticipated changes in policy. The new Keynesian Phillips curve (NKPC) model uses the ideas coming from the models of staggered contracts [8,9] and the quadratic price adjustment cost model of Rotemberg and Woodford [10], all of which have a similar formulation as the expectations-augmented PC of Friedman and Phelps [11,12]. The work of Clarida et al. [13] illustrates the widely usage of this model in theoretical analysis of monetary policy. Shifting the focus from the unemployment rate to the output gap, the Phillips' relationship has become an aggregate supply curve. The NKPC stayed popular also in the late 1990s and at the beginning of the 21st century as a theory for understanding inflation dynamics (e.g., [14]).

When Magnus Gustaf Mittag-Leffler, in his works [15,16] proposed a new function $E_\alpha(x)$, he surely did not expect how important generalisation of the exponential function e^x he developed. The Mittag-Leffler (ML) function and its generalisations interpolate between a purely exponential law and a power-law-like behaviour, and they arise naturally in the solution of fractional-order integro-differential equations, random walks, Lévy flights, the study of complex systems, and in other fields. In numerous works the properties, generalisations and applications of the ML-type functions were studied e.g., [17–26], and computation procedures for evaluating the ML function were developed e.g., [27–29]. The ML function become of great use and importance not only for mathematicians, but thanks to its special properties and huge potential for solving applied problems it found its applicability also in the fields such as psychorheology [19], electrotechnics [30,31], modeling of processes (diffusion [32], combustion [33], universe expansion [34]), etc. The ML function is also widely used in the numerical methods for solving ordinary and partial fractional-order differential equations, and in the the field of "fractal calculus" [35]. The idea to use the fractional-order calculus and the ML function for modelling phenomenons from the fields of economics and econophysics was elaborated by several authors [36–42].

In this paper the one-parameter ML function is for the first time used to model the relation between the unemployment rate and the inflation rate - the Phillips curve, and its performance is compared to the power-type model and the exponential-type model. French and Swiss econometric data are taken for the period of time 1980–2017 from the portal *EconStats*^{TM} [43] to identify the PC of these economies. The dataset is split into two subsets, the "modelling" subset is used to identify the model parameters, and a shorter "out-of-sample" subset serves for evaluating the forecast-performance of the models. The performance of all three models is evaluated based on the fitting-criterion, i.e., the sum of squared errors (SSE). The results are presented in the form of figures and tables, where the SSE of the fitting curve to the "modelling" subset, SSE of the fitting curve to the "out-of-sample" subset, and SSE of the fitting curve to the complete dataset, as well as some other quality criterions for the goodness-of-fit are listed.

The paper is organised as follows. Section 2 gives an overview of Mittag-Leffler function and its generalisations. The original Phillips curve as well as the Mittag-Leffler model for fitting the Phillips curve is described in Section 3. The numerical results and the discussion on the experiments can be found in Section 4. Finally, concluding remarks are given in Section 5.

2. Preliminaries: Mittag-Leffler Function and Its Generalisations

In 1903 M. G. Mittag-Leffler [15,16] introduced a new function $E_\alpha(x)$, a generalisation of the classical exponential function e^x, which is till today known as the one-parameter ML function. Using Erdélyi's notation [44], where z is used instead of x, the function can be written as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \, \mathrm{Re}(\alpha) > 0, \, z \in \mathbb{C}, \tag{1}$$

where Γ denotes the (complete) Gamma function, having the property $\Gamma(n) = (n-1)!$. The one-parameter ML function and its properties were further investigated [45–48] followed by the generalisation to a two-parameter function of the ML-type, by some authors called the Wiman's function (some give the credit to Agarwal). Following the Erdélyi's handbook the formula has the form [44]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \, \mathrm{Re}(\alpha) > 0, \, \mathrm{Re}(\beta) > 0, \, z \in \mathbb{C}. \tag{2}$$

The main properties of the above mentioned functions, and other ML-type functions, can be found in the book by Erdélyi et al. [44], and a detailed overview in the book by Dzhrbashyan [17]. To demonstrate the concept of generality of the ML-type functions let us point out, that the ML function for one parameter (1), is a special case of the two-parameter ML function, i.e., if we substitute $\beta = 1$ in (2). Accordingly, the classical exponential function is a special case of the one-parameter ML function, where $\alpha = 1$:

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \equiv E_\alpha(z),$$

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Also, other authors introduced and investigated further generalisations of the ML function, but as these are not used in the following experiments, they are not discussed in details here.

3. Modelling the Phillips Curve

As in many fields of science and applications, so in economics, to describe a relation between two variables, regression analysis is often used. One can use different regression models from simple linear-type, throughout exponential- and power-type models, to polynomial ones, and many other more complex and sophisticated. The discussion on the linearity or nonlinearity, and on the convex or concave shape of the PC, if it is supposed to be nonlinear, is still ongoing. Some authors are in favour of convex shape [49–52], some of concave [53], and some of their combination [54]. The application of the ML-type function to describe the PC perfectly fits into this discussion.

3.1. The "Original" Phillips Curve

Phillips in [3] used British econometric data–the rate of change of money wage rates, provided by the Board of Trade and the Ministry of Labour (calculated by Phelps Brown and Sheila Hopkins [55]), and corresponding percentage employment data, quoted in [56]. But, for a simpler evaluation, the data were first preprocessed, i.e., the average values of the rate of change of money wage rates and of the percentage unemployment for six different levels of the unemployment (0–2, 2–3, 3–4, 4–5, 5–7, 7–11) were calculated. The crosses in the Figure 1 refer to these average values. Phillips then fitted a curve to the crosses using a model in the form:

$$\begin{aligned} y + a &= b\, x^c \Rightarrow \\ \log(y + a) &= \log b + c \log x, \end{aligned} \tag{3}$$

where y stands for the rate of change of wage rates and x for the percentage unemployment. The parameters b and c were estimated using the least squares to fit four crosses laying between 0–5% of unemployment, and the parameter a was chosen to fit the remaining two crosses laying in the interval 5–11% of unemployment. Based on this "fitting criterion" Phillips identified the parameters of the model (3) as follows:

$$\begin{aligned} y + 0.900 &= 9.638\, x^{-1.394} \Rightarrow \\ \log(y + 0.900) &= 0.984 - 1.394 \log\, x. \end{aligned}$$

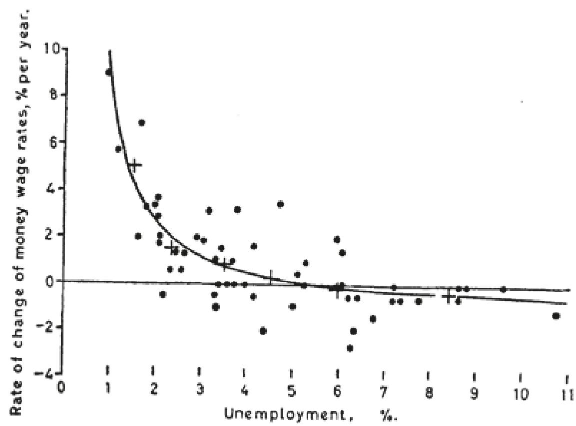


Figure 1. "Original" Phillips curve [3] (with permission from the John Wiley and Sons publisher).

3.2. The Mittag-Leffler Model for Fitting the Phillips Curve

The idea to use an ML-type function to describe the econometric data (representing the Phillips curve) results naturally from the observation of two facts:

- the simplicity of the model used by Phillips in his paper [3] given in (3), where a power-type regression is used to fit the data, and where the model can be defined in the form:

$$y(x) = b\,x^{c} - a, \quad a, b, c \in \mathbb{R}, \tag{4}$$

- the usual shape of the PC, used in the literature, which reminds on the exponential-type function:

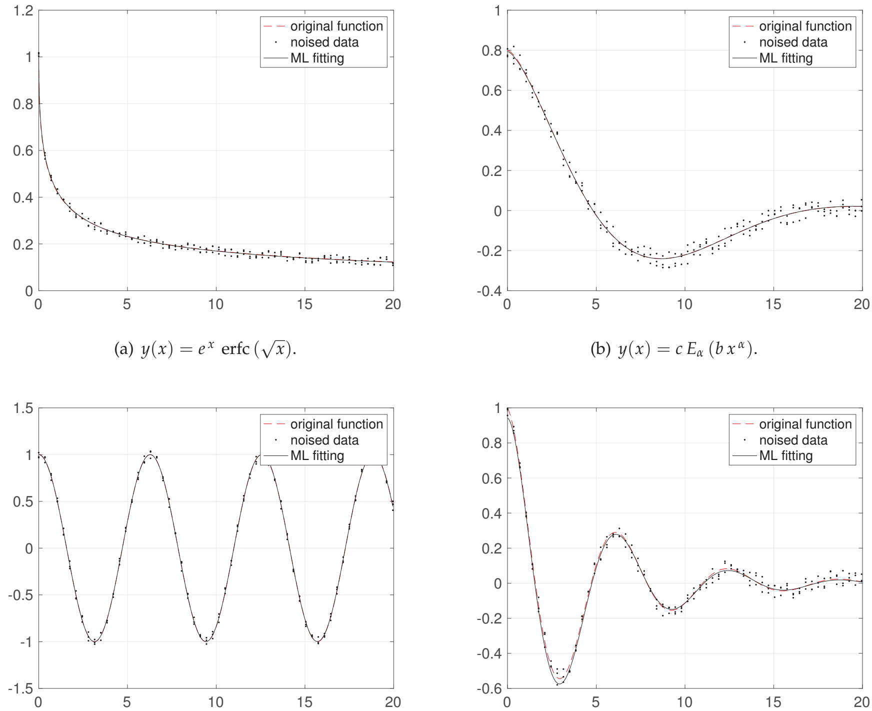
$$y(x) = b\,\mathrm{e}^{c\,x} - a, \quad a, b, c \in \mathbb{R}, \tag{5}$$

where for both cases, (4) and (5), x stands for the unemployment rate and y for the inflation rate.

Based on these facts, the one-parameter ML function appears to be a general model to fit the PC relation, as it behaves between a purely exponential function and a power function. The one-parameter ML function defined in (1), which includes the special case when $\alpha = 1$, i.e., the classical exponential function, is used to model the econometric data under study. Generally, the proposed fitting model can be written as follows:

$$y(x) = c\,E_{\alpha}\,(b\,x^{\alpha}), \quad \alpha \in \mathbb{C}, \quad \mathrm{Re}(\alpha) > 0, \quad b, c \in \mathbb{R}, \tag{6}$$

where the parameters α, b, c are subject to optimisation procedure minimising the squared sum of the vertical offsets between the data points and the fitting curve. Some of the possible manifestations of the ML model given in (6) are shown in Figure 2 (figures generated using the Matlab demo published by Igor Podlubny [57]), with the identified parameters listed in Table 1.



(a) $y(x) = e^x \operatorname{erfc}(\sqrt{x})$.

(b) $y(x) = c\, E_\alpha\,(b\, x^\alpha)$.

(c) $y(x) = \cos(x)$.

(d) $y(x) = e^{-dx} \cos(x)$.

Figure 2. Mittag-Leffler fitting using different functions $y(x)$ for generating data [57].

Table 1. Identified parameters of the Mittag-Leffler fitting model: $y(x) = c\, E_\alpha\,(b\, x^\alpha)$.

		$y(x) = e^x \operatorname{erfc}(\sqrt{x})$	$y(x) = c\, E_\alpha\,(b\, x^\alpha)$	$y(x) = \cos(x)$	$y(x) = e^{-dx} \cos(x)$
generating parameters	c	-	0.8	-	-
	α	-	1.5	-	-
	b	-	-0.2	-	-
	d	-	-	-	0.2
identified parameters	c	0.9982	0.7869	1.0045	0.9722
	α	0.5008	1.4999	2.0000	1.7538
	b	-0.9974	-0.1988	-0.9999	-1.0327

4. Numerical Results and Discussion

To evaluate the performance of the proposed ML model (6) in comparison to the power-type model (4), and the exponential-type model (5) the econometric data of two European countries (France and Switzerland) were used, that were obtained from the EconStats$^{\text{TM}}$ portal [43]. The unemployment rate and the inflation rate were taken for the period of time 1980–2017. The whole list of the processed data can be found in the Table A1.

4.1. Goodness-of-Fit Statistics and Data Preprocessing

The sum of squared errors (SSE) between the fitting models and the used data serves as the fitting-criterion, with values closer to 0 indicating a smaller random error component of the model.

Also some other quality measures were evaluated, i.e., the R-square from interval $[0, 1]$, that indicates the proportion of variance satisfactory explained by the fitting-model (e.g., R-square $= 0.7325$ means that the fit explains 73.25% of the total variation in the data about the average); the adjusted R-square statistic, with values smaller or equal to 1, where values closer to 1 indicate a better fit; the root mean squared error (RMSE), with values closer to 0 indicating a fit more useful for prediction [58].

The used dataset, where the unemployment rate corresponds to the x-coordinate and the inflation rate corresponds to the y-coordinate (each sample represents the state of these two indicators for each year from the period under study), is first split into two subsets, the "modelling" subset is used to identify the model parameters, the "out-of-sample" subset serves for evaluating the forecast-performance of the models. For both economies, French and Swiss, all three models were first fitted to the data from the "modelling" subset (composed of 31 samples), by minimising SSE, identifying the optimal parameters. The obtained parameters were then used to compute SSE of the identified models to the "out-of-sample" subset (composed of seven samples with the greatest values of unemployment rate) and SSE of the fitting model to the complete dataset.

4.2. Experiments

The first experiment was conducted using the French econometric data. The modelling subset of 31 samples, was used for the identification purposes. All three models, the power-type model (4), the exponential-type model (5), and the ML model (6), were fitted to these data minimising the SSE obtaining so the optimal parameters. The identified models were then used to compute the SSE to the complete dataset of 38 samples (including the "out-of-sample" subset). SSE results to the modelling subset as well as SSE to the "out-of-sample" subset and SSE to the complete dataset for the French Phillips curve are shown in Table 2, alongside the values of R-square, adjusted R-square, and RMSE. The ML model outperformed the compared models in all listed statistic indicators, with SSE to the "out-of-sample" subset double smaller than the exponential-type model, and almost three-times smaller than the power-type model (see Table 2, where bold stands for better result).

Table 2. The statistical results of the French Phillips curve fitting.

	Power-Type Model	Exponential-Type Model	ML Model
SSE to "modelling" subset	157.8422	155.8276	**149.6035**
SSE to "out-of-sample" subset	10.9024	8.0347	**3.9904**
SSE to complete dataset	168.7446	163.8623	**153.5939**
R-square	0.5634	0.5690	**0.5862**
adjusted R-square	0.5322	0.5382	**0.5567**
RMSE	2.3740	2.3590	**2.3110**
Model definition	$y(x) = b\,x^{c} - a$	$y(x) = b\,\mathrm{e}^{c\,x} - a$	$y(x) = c\,E_{\alpha}\,(b\,x^{\alpha})$
Identified parameters	$a = 1.552$ $b = 1.009e+04$ $c = -3.471$	$a = -0.0187$ $b = 563.6$ $c = -0.571$	$\alpha = 1.358$ $b = -0.6378$ $c = -149.9$

The result of the French Phillips curve fitting is also shown in Figure 3, where it is possible to observe a similar behaviour of all three models, i.e., with the increase of the unemployment rate the inflation rate exponentially decreases. However, for the last samples, the decrease of the ML model slows down in comparison to the power-type and exponential-type models. This behaviour of the ML model is obviously better describing the trend of the "out-of-sample" subset. This is also confirmed by the smallest SSE value of the ML fitting curve to the "out-of-sample" subset (see Table 2).

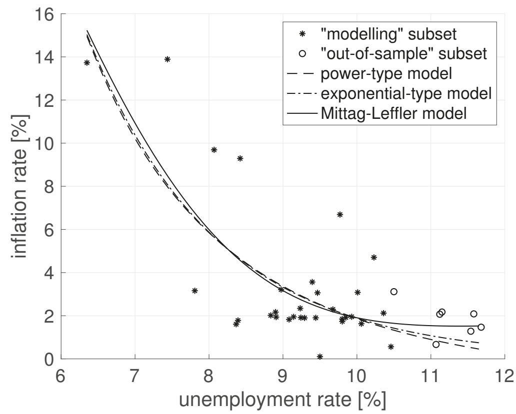


Figure 3. Fitting the French Phillips curve.

Identically as in the French case, the Swiss econometric data (unemployment rate and inflation rate) were first preprocessed. The complete dataset was split into the "modelling" subset composed of 31 samples, that was used to identify the optimal parameters of the power-type model (4), the exponential-type model (5), and the ML model (6). The SSE between the "modelling" subset and the fitting curves was again used as the fitting criterion. Using the identified model parameters the SSE of the evaluated models to the complete dataset of 38 samples (including the "out-of-sample" subset) was computed. In order to compare the forecast-performance of the models SSE to the "out-of-sample" subset, as well as the SSE values for the "modelling" subset fitting, and SSE to the complete dataset for the Swiss Phillips Curve are shown in Table 3, alongside the values of R-square, adjusted R-square, and RMSE.

Table 3. The statistical results of the Swiss Phillips curve fitting.

	Power-Type Model	**Exponential-Type Model**	**ML Model**
SSE to "modelling" subset	39.6506	40.0588	**39.2992**
SSE to "out-of-sample" subset	6.8961	**4.6826**	5.0041
SSE to complete dataset	46.5466	44.7414	**44.3033**
R-square	0.6389	0.6351	**0.6420**
adjusted R-square	0.6131	0.6091	**0.6165**
RMSE	1.1900	1.1960	**1.1850**
Model definition	$y(x) = b\,x^{c} - a$	$y(x) = b\,\mathrm{e}^{c\,x} - a$	$y(x) = c\,E_{\alpha}\,(b\,x^{\alpha})$
Identified parameters	$a = -551.3$ $b = -548.6$ $c = 30.85e - 04$	$a = -0.7809$ $b = 6.364$ $c = -1.376$	$\alpha = 0.7733$ $b = -1.468$ $c = 8.823$

Observing the result of the Swiss Phillips curve fitting shown in Figure 4, one can see an interesting case, where although all the compared models are exponentially decreasing, the curve representing the proposed ML model proceeds in-between the power-type and the exponential-type models, that form a kind of scissors. In respect to the "out-of-sample" subset it is possible to conclude that two points of that subset deviate, having higher inflation rate value then the others. This strongly influenced the fitting results. In this case the exponential-type model visually represents the "out-of-sample" subset slightly better then the ML model, that is also demonstrated by a smaller value of SSE of the exponential-type model to the "out-of-sample" subset (see Table 3). In spite of this, the ML model outperforms the compared models in all other used statistic indicators, including smaller SSE to the complete dataset, proving it's capability. Moreover, in case of filtering these two outliers from the "out-of-sample" subset, the ML model better fits the data-trend.

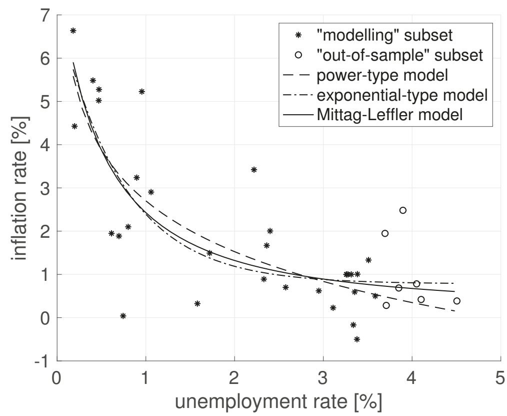


Figure 4. Fitting the Swiss Phillips curve.

5. Conclusions

The ability of the Mittag-Leffler function to behave between the power-type and the exponential-type function, and moreover to fit data that manifest signs of stretched exponentials, oscillations or damped oscillations is demonstrated in this paper, with application to fitting the econometric data (Phillips curve) of two European economies, where the proposed ML model outperforms the compared fitting-models in terms of the chosen performance criterions. Exploiting the full potential of the Mittag-Leffler function and it's generalisations, as well as associating the model parameters with the corresponding economic indicators will be the topic of further work.

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Conflicts of Interest: The author declares no conflict of interest.

Appendix A. The Econometric Dataset

Table A1. The complete dataset: Econometric data for years 1980–2017 [43].

Year	France		Switzerland	
	Unemployment Rate [%]	Inflation Rate [%]	Unemployment Rate [%]	Inflation Rate [%]
1980	6.3490	13.7300	0.1970	4.4260
1981	7.4380	13.8900	0.1810	6.6370
1982	8.0690	9.6910	0.4040	5.4850
1983	8.4210	9.2920	0.8010	2.1000
1984	9.7710	6.6900	1.0590	2.9040
1985	10.2300	4.7030	0.8970	3.2380
1986	10.3600	2.1210	0.7440	0.0400
1987	10.5000	3.1150	0.6970	1.8870
1988	10.0100	3.0810	0.6130	1.9490
1989	9.3960	3.5630	0.4690	5.0220
1990	8.9750	3.2120	0.4720	5.2760
1991	9.4670	3.0630	0.9550	5.2270

Table A1. *Cont.*

Year	France		Switzerland	
	Unemployment Rate [%]	Inflation Rate [%]	Unemployment Rate [%]	Inflation Rate [%]
1992	9.8500	1.9180	2.2190	3.4210
1993	11.1200	2.0700	3.8970	2.4820
1994	11.6800	1.4690	4.1020	0.4200
1995	11.1500	2.1720	3.6950	1.9480
1996	11.5800	2.0860	4.0510	0.7810
1997	11.5400	1.2820	4.5050	0.3860
1998	11.0700	0.6680	3.3380	−0.1680
1999	10.4600	0.5620	2.3620	1.6680
2000	9.0830	1.8270	1.7190	1.4930
2001	8.3920	1.7810	1.5810	0.3250
2002	8.9080	1.9380	2.3300	0.8910
2003	8.9000	2.1690	3.3530	0.5940
2004	9.2330	2.3420	3.5090	1.3320
2005	9.2920	1.9000	3.3840	1.0060
2006	9.2420	1.9120	2.9490	0.6210
2007	8.3670	1.6070	2.4000	2.0040
2008	7.8080	3.1590	2.5760	0.7010
2009	9.5000	0.1030	3.7090	0.2830
2010	9.8020	1.7360	3.8500	0.6860
2011	9.6750	2.2930	3.1100	0.2280
2012	9.9290	1.9520	3.3790	−0.5000
2013	10.0600	1.6300	3.5850	0.5000
2014	9.8010	1.8480	3.3150	1.0000
2015	9.4430	1.9040	3.2780	1.0000
2016	9.1440	1.9490	3.2590	1.0000
2017	8.8350	2.0150	3.2620	1.0000

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Stability and Bifurcation of a Delayed Time-Fractional Order Business Cycle Model with a General Liquidity Preference Function and Investment Function

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Abstract: In this paper, the business cycle (BC) is described by a delayed time-fractional-order model (DTFOM) with a general liquidity preference function and an investment function. Firstly, the existence and uniqueness of the DTFOM solution are proven. Then, some conditions are presented to guarantee that the positive equilibrium point of DTFOM is locally stable. In addition, Hopf bifurcation is obtained by a new method, where the time delay is regarded as the bifurcation parameter. Finally, a numerical example of DTFOM is given to verify the effectiveness of the proposed model and methods.

Keywords: business cycle model; stability; time delay; time-fractional-order; Hopf bifurcation

1. Introduction

Macroeconomics is an essential economic field that analyzes the general law of economics through macroeconomic indicators such as national income, market investment, and money supply [1]. As one of the most important issues of macroeconomics, business cycle (BC) theory has been studied by many economists because of its realistic meaning and practical value [2].

To obtain the factors involved in fluctuations in the business cycle (BC), many mathematical models are founded on nonlinear dynamics and relative theories that improve the development of BC theory. Using graph analysis, Kaldor [3] proved that a BC exists when the investment and saving function are time-varying nonlinear. Chang and Smyth [4] proved the results shown in [3] by using mathematical theory and gave the conditions required for the existence of limit cycles. J. R.Hicks and A. H. Hansen proposed the IS-LM model, which is an important tool for describing the macroeconomic analysis of the interlinked theoretical structure between the product market and the money market [5]. These early studies are of great significance to current and future BC.

Time delay is a phenomenon existing in practice that often appears in engineering applications [6–8]. Time delay is an important factor that also widely exists in economics. Since economics is not only affected by the present state, but also by the past state, the delayed mathematical model is more suitable for describing economic systems, and some significant results have been drawn in recent years [9–11]. Considering expectation and delay, Liu and Cai [12] studied a BC model and gave some conditions of stability and bifurcation. Hu and Cao [13,14] studied the Kaldor–Kalecki model of delayed BC. To summarize, time delay is an important reason for fluctuations in BC.

In recent years, the theory of fractional calculus (FC) has not only rapidly developed, but has also been widely applied in many fields [15–19]. In fact, most economic systems have long-term memories. Compared with integer derivatives, since fractional derivatives are related to the entire time domain of the economic process, the fractional-order systems are more suitable for describing economic systems. In the last few years, fractional calculus equations have been widely used to describe a class of economic processes with power law memory and spatial nonlocality. Some continuous-time mathematical models describing economic dynamics with long memory have been proposed [20,21], and some interesting results were obtained. Wang and Huang [22] studied a delayed fractional-order financial system and obtained some conditions of stability and chaos. Ma and Ren [23] studied a fractional-order macroeconomic system and obtained some conditions of stability and Hopf bifurcation. Considering negative parameters, Tacha and Munoz-Pacheco [24] studied a fractional-order finance system, and the cause of chaos was found. Motivated by the above considerations, a new delayed fractional-order model (DFOM) for BC with a general liquidity preference function and an investment function is considered in this paper.

The arrangements of the article are as follows: In Section 2, the model description and some definitions and lemmas are provided. Section 3 shows the main results. The existence and uniqueness of the solution and the local stability and bifurcation of the positive equilibrium point of DFOM for the BC are presented. Numerical simulations and conclusions are respectively presented in Sections 4 and 5.

2. Preliminaries and Model Descriptions

In this paper, the Caputo form of the fractional-order derivative is used.

Definition 1. *[25] For a continuous function $f(t)$ and a positive integer n, if $\alpha \in (n-1, n)$ is satisfied, the fractional-order Caputo's derivative is defined as:*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{1}$$

where $t_0 \in \mathbf{R}$ is the initial time and $\Gamma(\cdot)$ is the Gamma function.

Definition 2. *[26] For $0 < \alpha < 1$, under the initial condition $x(t_0) = x_{t_0}$, x^* is called an equilibrium point of system $D^{\alpha} x(t) = f(t, x)$ if and only if $f(t, x^*) = 0$.*

Lemma 1. *[26] For $\alpha \in (0, 1]$, under the initial condition $x(t_0) = x_{t_0}$, if $f(t, x)$ satisfies the local Lipschitz condition, then system $D^{\alpha} x(t) = f(t, x)$ has a unique solution for $t > t_0$.*

Lemma 2. *[27] Suppose that x^* is the equilibrium point of system $D^{\alpha} x(t) = f(x)$; if $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$ is satisfied, then x^* is asymptotically locally stable, where λ_i is any one of the eigenvalues for the Jacobian matrix $J = \partial f / \partial x$ evaluated at x^*.*

We consider the augmented IS-LM BC model given by Gabisch and Lorenz [28],

$$\begin{aligned} \dot{Y}(t) &= a[I(Y(t), K(t), R(t)) - S(Y(t), R(t))], \\ \dot{R}(t) &= b[L(Y(t), R(t)) - \bar{M}], \\ \dot{K}(t) &= I(Y(t), K(t), R(t)) - \delta K(t), \end{aligned} \tag{2}$$

where $Y(t)$, $R(t)$, and $K(t)$ represent the gross product, interest rate at time t, and capital stock, respectively. $I(Y(t), K(t), R(t))$ is the investment function, $L(Y(t), R(t))$ is the liquidity preference function, $S(Y(t), R(t))$ is the saving function, and $\bar{M}$ is a constant money supply. $a > 0$ and $b > 0$ denote the adjustment coefficient of goods and the monetary market, respectively. $0 < \delta < 1$ represents the depreciation rate of the capital stock.

We consider an investment delay $\tau > 0$, which is always encountered in capital stock, and suppose that $S(Y(t), R(t)) = s_1 Y(t) + s_2 R(t)$. The DFOM for the BC with a general liquidity preference function and an investment function under the initial conditions $Y(\theta) = \phi_1(\theta)$, $R(\theta) = \phi_2(\theta)$, $K(\theta) = \phi_3(\theta)$ is described as follows:

$$\begin{aligned} D^\alpha Y(t) &= a[I(Y(t), K(t), R(t)) - s_1 Y(t) - s_2 R(t)], \\ D^\alpha R(t) &= b[L(Y(t), R(t)) - \bar{M}], \\ D^\alpha K(t) &= I(Y(t-\tau), K(t-\tau), R(t-\tau)) - \delta K(t), \end{aligned} \tag{3}$$

where $-\tau \le \theta \le 0$, $0 < \alpha \le 1$ and s_1 and s_2 are positive constants.

Suppose that $I(Y(t), K(t), R(t))$, and $L(Y(t), K(t))$ are differentiable. $X(t) = K(t+\tau)$ is denoted as the expected capital stock. By substituting it into System (3), one obtains:

$$\begin{aligned} D^\alpha Y(t) &= a[I(Y(t), X(t-\tau), R(t)) - s_1 Y(t) - s_2 R(t)], \\ D^\alpha R(t) &= b[L(Y(t), R(t)) - \bar{M}], \\ D^\alpha X(t) &= I(Y(t), X(t-\tau), R(t)) - \delta X(t). \end{aligned} \tag{4}$$

3. Main Results

We firstly prove that the solution of DFOM for BC exists and is unique. In addition, we give some conditions to guarantee that the positive equilibrium point is stable and bifurcates.

3.1. Existence and Uniqueness of the Solution

Theorem 1. *If $C = C([-\tau, 0], \mathbf{R}^3)$ is the continuous function of the Banach space and $Z_0(t) \in C$ is an initial condition, then System (4) has a unique solution $Z(t)$, where $Z(t) = (Y(t), R(t), X(t))$.*

Proof. Consider a mapping $H(Z) = (H_1(Z), H_2(Z), H_3(Z))$, where:

$$\begin{aligned} H_1(Z(t)) &= -s_1 a Y(t) - s_2 a R(t) + aI(Y(t), X(t-\tau), R(t)), \\ H_2(Z(t)) &= -b\bar{M} + bL(Y(t), R(t)), \\ H_3(Z(t)) &= -\delta X(t) + I(Y(t), X(t-\tau), R(t)). \end{aligned} \tag{5}$$

The following conclusion can be made:

$$\begin{aligned} &\|H(Z(t_1)) - H(Z(t_2))\| \\ \le&|H_1(Z(t_1)) - H_1(Z(t_2))| + |H_2(Z(t_1)) - H_2(Z(t_2))| + |H_3(Z(t_1)) - H_3(Z(t_2))| \\ =&|a\{[I(Y(t_1), X(t_1-\tau), R(t_1)) - I(Y(t_2), X(t_2-\tau), R(t_2))] - s_1[Y(t_1) - Y(t_2)] - s_2[R(t_1) - R(t_2)]\}| \\ &+ |[I(Y(t_1), X(t_1-\tau), R(t_1)) - I(Y(t_2), X(t_2-\tau), R(t_2))] - \delta(X(t_1) - X(t_2))| \\ &+ |b[L(Y(t_1), R(t_1)) - L(Y(t_2), R(t_2))]| \\ \le&|Y(t_1) - Y(t_2)|[a|s_1| + (a+1)|\frac{\partial I(Y(T_1), X(t_1-\tau), R(t_1))}{\partial Y}| + b|\frac{\partial L(Y(T_2), R(t_1))}{\partial Y}|] \\ &+ \max\{|X(t_1-\tau) - X(t_2-\tau)|, |X(t_1) - X(t_2)|\}[|(a+1)\frac{\partial I(Y(t_2), X(T_3), R(t_1))}{\partial X}| + \delta] \\ &+ |R(t_1) - R(t_2)|[s_2 + |(a+1)\frac{\partial I(Y(t_2), X(t_2-\tau), R(T_4))}{\partial R}| + b|\frac{\partial L(Y(t_2), R(T_5))}{\partial R}|] \\ &\le L \parallel Z(t_1) - Z(t_2) \parallel . \end{aligned} \tag{6}$$

The constants $T_1, T_2, T_4, T_5 \in [t_1, t_2]$, $T_3 \in [t_1 - \tau, t_2 - \tau]$ exist, satisfying:

$$
\begin{aligned}
|\frac{\partial I(Y(T_1), X(t_1-\tau), R(t_1))}{\partial Y}| &= \max_{t\in[t_1,t_2]} |\frac{\partial I(Y(t), X(t_1-\tau), R(t_1))}{\partial Y}|,\\
|\frac{\partial L(Y(T_2), R(t_1))}{\partial Y}| &= \max_{t\in[t_1,t_2]} |\frac{\partial L(Y(t), R(t_1))}{\partial Y}|,\\
|\frac{\partial I(Y(t_2), X(T_3), R(t_1))}{\partial X}| &= \max_{t\in[t_1-\tau,t_2-\tau]} |\frac{\partial I(Y(t_2), X(t), R(t_1))}{\partial X}|,\\
|\frac{\partial I(Y(t_2), X(t_2-\tau), R(T_4))}{\partial R}| &= \max_{t\in[t_1,t_2]} |\frac{\partial I(Y(t_2), X(t_2-\tau), R(t))}{\partial R}|,\\
|\frac{\partial L(Y(t_2), R(T_5))}{\partial R}| &= \max_{t\in[t_1,t_2]} |\frac{\partial L(Y(t_2), R(t))}{\partial R}|,
\end{aligned}
\tag{7}
$$

where one chooses a positive constant,

$$
\begin{aligned}
L = \max\{&|s_2| + |(a+1)\frac{\partial I(Y(t_2), X(t_2-\tau), R(T_3))}{\partial R}| + b|\frac{\partial L(Y(t_2), R(T_5))}{\partial R}|,\\
&|(a+1)\frac{\partial I(Y(t_2), X(T_3), R(t_1))}{\partial X}| + \delta,\\
&a|s_1| + (\alpha+1)|\frac{\partial I(Y(T_1), X(t_1-\tau), R(t_1))}{\partial Y}| + b|\frac{\partial L(Y(T_2), R(t_2))}{\partial Y}|\}.
\end{aligned}
\tag{8}
$$

It is quite clear that $H(Z)$ satisfies the Lipschitz condition. According to lemma 1, System (4) with Z_{t_0} has a unique solution $Z(t)$. □

3.2. Stability and Bifurcation

Assume that System (4) contains positive equilibrium points, and let $E^* = (Y^*, R^*, X^*)$ be one of them. For convenience, we define:

$$
\begin{aligned}
&\frac{\partial I(Y^*, X^*, R^*)}{Y} = I_Y, \quad \frac{\partial L(Y^*, R^*)}{Y} = L_Y,\\
&\frac{\partial I(Y^*, X^*, R^*)}{R} = I_R, \quad \frac{\partial L(Y^*, R^*)}{R} = L_R,\\
&\frac{\partial I(Y^*, X^*, R^*)}{X} = I_X, \quad x(t) = X(t) - x^*,\\
&y(t) = Y(t) - Y^*, \qquad r(t) = R(t) - R^* \quad .
\end{aligned}
\tag{9}
$$

By substituting (9) into (4) and using linearization methods for (4), we get:

$$
\begin{aligned}
D^\alpha y(t) &= a[I_Y y(t) + I_R r(t) + I_X x(t-\tau) - s_1 y(t) - s_2 r(t)],\\
D^\alpha r(t) &= b[L_Y y(t) + L_R r(t) - \bar{M}],\\
D^\alpha x(t) &= I_Y y(t) + I_R r(t) + I_X x(t-\tau) - \delta x(t).
\end{aligned}
\tag{10}
$$

The Jacobian matrix of (10) is described as:

$$
J = \begin{bmatrix} a(I_Y - s_1) & a(I_R - s_2) & aI_X e^{-s\tau} \\ bL_Y & bL_R & 0 \\ I_Y & I_R & I_X e^{-s\tau} - \delta \end{bmatrix}.
$$

Then, the characteristic equation can be expressed as:

$$
s^{3\alpha} + a_1 s^{2\alpha} + a_2 s^\alpha + a_3 + (a_4 s^{2\alpha} + a_5 s^\alpha + a_6)e^{-s\tau} = 0, \tag{11}
$$

where:

$$\begin{aligned}
a_1 &= -a(I_Y - s_1) - bL_R + \delta, \\
a_2 &= a(I_Y - s_1)(bL_R - \delta) - bL_R\delta - abL_Y(I_R - s_2), \\
a_3 &= ab\delta[(I_Y - s_1)L_R - (I_R - s_2)L_Y], \\
a_4 &= I_X, \\
a_5 &= a(I_Y - s_1)I_X + bL_R I_X - aI_X I_Y, \\
a_6 &= -ab(I_Y - s_1)L_R I_X - abL_Y I_R I_X + abL_R I_Y I_X + ab(I_R - s_2)L_Y I_X.
\end{aligned} \tag{12}$$

Theorem 2. *For $\tau = 0$ and System (4), $E^* = (Y^*, R^*, X^*)$ is one of the positive equilibrium points. If the conditions:*

(1) $\Delta \le 0$, $a_1 + a_4 > 0$, $a_2 + a_5 > 0$, $a_3 + a_6 > 0$, *and* $(a_1 + a_4)(a_2 + a_5) - (a_3 + a_6) > 0$,
(2) $\Delta > 0$, $(A + B - \frac{a_1+a_4}{3}) < 0$, *and* $|\arg(A\omega + B\omega^2 - \frac{a_1+a_4}{3})| > \frac{\alpha\pi}{2}$,

exist, where $\Delta = \frac{q^2}{4} + \frac{p^3}{27}$, $p = \frac{3(a_2+a_5)-(a_1+a_4)^2}{3}$, $q = \frac{2(a_1+a_4)^3-9(a_1+a_4)(a_2+a_5)+27(a_3+a_6)}{27}$, $A = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$ *and* $B = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$, *then E^* is locally asymptotically stable.*

Proof. If $\tau = 0$, then (11) can be rewritten as:

$$s^{3\alpha} + (a_1 + a_4)s^{2\alpha} + (a_2 + a_5)s^{\alpha} + a_3 + a_6 = 0. \tag{13}$$

Let $s^{\alpha} = \lambda$, and substitute it into (13). This produces:

$$\lambda^3 + (a_1 + a_4)\lambda^2 + (a_2 + a_5)\lambda + a_3 + a_6 = 0. \tag{14}$$

Let $\lambda^* = \lambda + \frac{a_1+a_4}{3}$. Then, (14) can be transformed into:

$$(\lambda^*)^3 + p\lambda^* + q = 0, \tag{15}$$

where $p = \frac{3(a_2+a_5)-(a_1+a_4)^2}{3}$, $q = \frac{2(a_1+a_4)^3-9(a_1+a_4)(a_2+a_5)+27(a_3+a_6)}{27}$.

If $\Delta = \frac{q^2}{4} + \frac{p^3}{27} \le 0$, (15) has three real roots. Thus, (14) has three real roots. Then, according to the Routh–Hurwitz criterion and the theory of the equilibrium point [29,30], if:

$$\begin{aligned}
&a_2 + a_5 > 0, a_1 + a_4 > 0, a_3 + a_6 > 0, \\
&(a_1 + a_4)(a_2 + a_5) - (a_3 + a_6) > 0,
\end{aligned} \tag{16}$$

all three real roots are negative. Thus, $|\arg(\lambda_i)| = \pi > \frac{\alpha\pi}{2}$, and E^* is locally asymptotically stable.

If $\Delta = \frac{q^2}{4} + \frac{p^3}{27} > 0$, then $\lambda_1^* = A + B$, $\lambda_2^* = A\omega + B\omega^2$, and $\lambda_3^* = A\omega^2 + B\omega$, where $A = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$, $B = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$, and $\omega = \frac{-1+\sqrt{3}i}{2}$. Thus, $\lambda_1 = A + B - \frac{a_1+a_4}{3}$, $\lambda_2 = A\omega + B\omega^2 - \frac{a_1+a_4}{3}$, and $\lambda_3 = A\omega^2 + B\omega - \frac{a_1+a_4}{3}$. If $|\arg(A\omega + B\omega^2 - \frac{a_1+a_4}{3})| > \frac{\alpha\pi}{2}$ and $A + B - \frac{a_1+a_4}{3} < 0$, E^* is locally asymptotically stable. □

Next, we use the method in [31] to deal with the case where $\tau \neq 0$. Assume that (11) has a purely imaginary root $s = i\varphi$ $(\varphi > 0)$ and $a_4 s^{2\alpha} + a_5 s^{\alpha} + a_6 \neq 0$. By substituting $s = i\varphi$ into Equation (11), we get:

$$\begin{aligned}
&a_1\varphi^{2\alpha}(\cos\alpha\pi + \sin(\alpha\pi)i) + a_2\varphi^{\alpha}(\cos\frac{\alpha\pi}{2} + \sin\frac{\alpha\pi}{2}i) + a_3 + \varphi^{3\alpha}(\cos\frac{3\alpha\pi}{2} + \sin\frac{3\alpha\pi}{2}i) \\
&+(a_4\varphi^{2\alpha}(\cos\alpha\pi + i\sin\alpha\pi) + a_5\varphi^{\alpha}(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}) + a_6)(\cos\varphi\tau - i\sin\varphi\tau) = 0.
\end{aligned} \tag{17}$$

If we write the real and imaginary parts separately, then we have:

$$\begin{cases} \varphi^{3\alpha} \cos \dfrac{3\alpha\pi}{2} + a_1 \varphi^{2\alpha} \cos\alpha\pi + a_2 \varphi^{\alpha} \cos \dfrac{\alpha\pi}{2} + a_3 \\ = -[a_4 \varphi^{2\alpha} \cos(\alpha\pi - \varphi\tau) + a_5 \varphi^{\alpha} \cos(\dfrac{\alpha\pi}{2} - \varphi\tau) + a_6 \cos \varphi\tau], \\ \varphi^{3\alpha} \sin \dfrac{3\alpha\pi}{2} + a_1 \varphi^{2\alpha} \sin \alpha\pi + a_2 \varphi^{\alpha} \sin \dfrac{\alpha\pi}{2} \\ = -[a_4 \varphi^{2\alpha} \sin(\alpha\pi - \varphi\tau) + a_5 \varphi^{\alpha} \sin(\dfrac{\alpha\pi}{2} - \varphi\tau) - a_6 \sin \varphi\tau]. \end{cases} \tag{18}$$

By squaring the corresponding sides of (18) and adding them, we get:

$$\begin{aligned} &\varphi^{6\alpha} + 2a_1 \cos \frac{\alpha\pi}{2} \varphi^{5\alpha} + (a_1^2 + 2a_2 \cos \alpha\pi - a_4^2)\varphi^{4\alpha} + (2a_3 \cos \frac{3\alpha\pi}{2} + 2a_1 a_2 \cos \frac{\alpha\pi}{2} - 2a_4 a_5 \cos \frac{\alpha\pi}{2})\varphi^{3\alpha} \\ &+ (a_2^2 + 2a_1 a_3 \cos \alpha\pi - a_5^2 - 2a_4 a_6 \cos \alpha\pi)\varphi^{2\alpha} + (2a_4 a_3 \cos \frac{\alpha\pi}{2} - 2a_5 a_6 \cos \frac{\alpha\pi}{2})\varphi^{\alpha} + a_3^2 - a_6^2 = 0. \end{aligned} \tag{19}$$

If we let $h(\varphi) = \varphi^{6\alpha} + 2a_1 \cos \frac{\alpha\pi}{2} \varphi^{5\alpha} + (a_1^2 + 2a_2 \cos \alpha\pi - a_4^2)\varphi^{4\alpha} + (2a_3 \cos \frac{3\alpha\pi}{2} + 2a_1 a_2 \cos \frac{\alpha\pi}{2} - 2a_4 a_5 \cos \frac{\alpha\pi}{2})\varphi^{3\alpha} + (a_2^2 + 2a_1 a_3 \cos \alpha\pi - a_5^2 - 2a_4 a_6 \cos \alpha\pi)\varphi^{2\alpha} + (2a_4 a_3 \cos \frac{\alpha\pi}{2} - 2a_5 a_6 \cos \frac{\alpha\pi}{2})\varphi^{\alpha} + a_3^2 - a_6^2$, and suppose that $a_3^2 - a_6^2 < 0$, then φ_0 is one of the positive roots of $h(\varphi)$. Using (17), we get:

$$\tau_j = \frac{1}{\varphi_0}\{\arccos - \frac{CE + DF}{E^2 + F^2} + 2j\pi\}, \qquad j = 0, 1, 2, \ldots, n, \tag{20}$$

where:

$$\begin{aligned} C &= \varphi_0^{3\alpha} \cos \frac{3\alpha\pi}{2} + a_1 \varphi_0^{2\alpha} \cos \alpha\pi + a_2 \varphi_0^{\alpha} \cos \frac{\alpha\pi}{2} + a_3, \\ D &= \varphi_0^{3\alpha} \sin \frac{3\alpha\pi}{2} + a_1 \varphi_0^{2\alpha} \sin(\alpha\pi) + a_2 \varphi_0^{\alpha} \sin \frac{\alpha\pi}{2}, \\ E &= a_4 \varphi_0^{2\alpha} \cos \alpha\pi + a_5 \varphi_0^{\alpha} \cos \frac{\alpha\pi}{2} + a_6, \\ F &= a_4 \varphi_0^{2\alpha} \sin \alpha\pi + a_5 \varphi_0^{\alpha} \sin \frac{\alpha\pi}{2}. \end{aligned} \tag{21}$$

Then, τ appears to be a bifurcation parameter. If we assume that (11) has an eigenvalue $\lambda(\tau) = \omega(\tau) + i\varphi(\tau)$, then we get $\omega(\tau_0) = 0$ and $\varphi(\tau_0) = \varphi_0$, where $\tau_0 = \min\{\tau_j\}$.

Theorem 3. *If we assume that $a_3^2 - a_6^2 < 0$, if $h'(\varphi_0) \neq 0$ is satisfied, Hopf bifurcation occurs.*

Proof. By using the implicit function theorem and differentiating (11) with respect to τ, we get:

$$\frac{ds}{d\tau} = \frac{sP_2(s)e^{-s\tau}}{-\tau P_2(s)e^{-s\tau} + P_2'(s)e^{-s\tau} + P_1'(s)}, \tag{22}$$

where $P_1(s) = s^{3\alpha} + a_1 s^{2\alpha} + a_2 s^{\alpha} + a_3$, $P_2(s) = a_4 s^{2\alpha} + a_5 s^{\alpha} + a_6$. Therefore,

$$\begin{aligned} \frac{ds}{d\tau}^{-1} &= \frac{-\tau}{s} + \frac{P_2'(s)}{sP_2(s)} + \frac{P_1'(s)}{sP_2(s)e^{-s\tau}} \\ &= \frac{-\tau}{s} + \frac{P_2'(s)}{sP_2(s)} - \frac{P_1'(s)}{sP_1(s)}. \end{aligned} \tag{23}$$

Then,

$$\left(Re\frac{ds}{d\tau}\right)^{-1}\Big|_{s=i\varphi_0} = Re\left[\frac{P_2'(s)}{sP_2(s)} - \frac{P_1'(s)}{sP_1(s)}\right]\Big|_{s=i\varphi_0}$$
$$= Re\left[\frac{2\alpha a_4 s^{2\alpha} + \alpha a_5 s^{\alpha}}{s^2(a_4 s^{2\alpha} + a_5 s^{\alpha} + a_6)} - \frac{3\alpha s^{3\alpha} + 2\alpha a_1 s^{2\alpha} + \alpha a_2 s^{\alpha}}{s^2(s^{3\alpha} + a_1 s^{2\alpha} + a_2 s^{\alpha} + a_3)}\right]\Big|_{s=i\varphi_0} \tag{24}$$
$$= \frac{h'(\varphi_0)}{2\varphi_0(a_4^2\varphi_0^{4\alpha} + 2a_4a_5\cos\frac{\alpha\pi}{2}\varphi_0^{3\alpha} + (a_5^2 + 2a_4a_6)cos\alpha\pi\varphi_0^{2\alpha} + 2a_5a_6\cos\frac{\alpha\pi}{2}\varphi_0^{\alpha} + a_6^2)}.$$

By assuming $a_4 s^{2\alpha} + a_5 s^{\alpha} + a_6 \neq 0$, we get $2\varphi_0(a_4^2\varphi_0^{4\alpha} + 2a_4a_5\cos\frac{\alpha\pi}{2}\varphi_0^{3\alpha} + (a_5^2 + 2a_4a_6)\cos\alpha\pi\varphi_0^{2\alpha} + 2a_5a_6\cos\frac{\alpha\pi}{2}\varphi_0^{\alpha} + a_6^2) \neq 0$. Thus, if $h'(\varphi_0) \neq 0$, $Re\frac{ds}{d\tau} \neq 0$, then the transversality condition holds. Therefore, Hopf bifurcation occurs at $\tau = \tau_0$. □

In conclusion, we have the following theorem.

Theorem 4. *If we assume that $a_3^2 - a_6^2 < 0$, $h'(\varphi_0) \neq 0$ and the conditions in Theorem 2 are satisfied, we get the following results:*

(1) *If $\tau < \tau_0$, E^* is stable;*
(2) *If $\tau > \tau_0$, E^* is unstable;*
(3) *A Hopf bifurcation exists at $\tau = \tau_0$.*

Remark 1. *According to (20), it is easy to see that τ_0 relates to the order α. Therefore, if τ is selected, the order α may be the cause of bifurcation.*

4. Numerical Simulation

There are many numerical simulation methods like the Monte Carlo [19] and the predict-evaluate and correct-evaluate (PECE) method [32,33], and the PECE method is used for numerical simulation presented in this section. According to [12], the following Kaldor investment function is used:

$$I(Y(t), R(t), K(t)) = \frac{e^{Y(t)}}{1 + e^{Y(t)}} - cR(t) - dK(t), \tag{25}$$

where $c, d > 0$. According to the literature [12,34], the following liquidity preference function is chosen:

$$L(Y(t), R(t)) = mY(t) + \frac{n}{R(t) - \hat{R}}, \tag{26}$$

where $m, n, \hat{R} > 0$. The following parameter values are chosen:

$$\begin{array}{llllll} a = 2, & b = 1.5, & s_1 = 0.2, & s_2 = 0.1, & c = 0.1, & d = 0.2, \\ m = 0.05, & n = 0.0005, & \hat{R} = 0.001, & \delta = 0.1, & \bar{M} = 0.05, & \end{array} \tag{27}$$

one can get the following system:

$$\begin{aligned} D^{\alpha}Y(t) &= 2\left[\frac{e^{Y(t)}}{1 + e^{Y(t)}} - 0.1R(t) - 0.2X(t-\tau) - 0.2Y(t) - 0.1R(t)\right], \\ D^{\alpha}R(t) &= 1.5\left[0.05Y(t) + \frac{0.0005}{R(t) - 0.001} - 0.05\right], \\ D^{\alpha}X(t) &= \frac{e^{Y(t)}}{1 + e^{Y(t)}} - 0.1R(t) - 0.2X(t-\tau) - 0.1X(t). \end{aligned} \tag{28}$$

It is easy to see that $E^* = (0.97, 0.36, 2.30)$ is a positive equilibrium point of (28). In this case, $a_3^2 - a_6^2 < 0$, $h'(\varphi_0) \neq 0$, and the conditions in Theorem 2 are satisfied. By calculating, the critical value of System (28) is determined to be $\tau_0 = 3.10$ when $\alpha = 0.98$. The initial values are chosen to be $Y_0 = 1$, $R_0 = 0.35$, and $X(\theta) = 2.3$ where $\theta \in (-\tau, 0)$. This is shown in Figures 1–3.

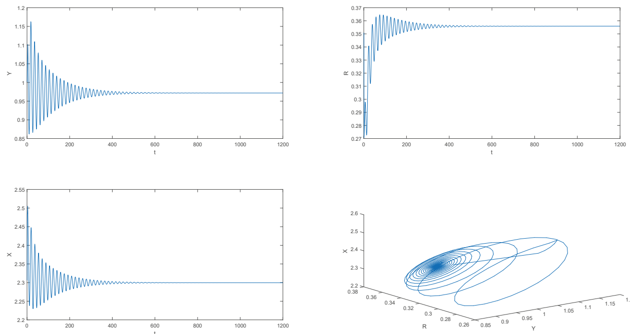

Figure 1. E^* is asymptotically stable, when $\alpha = 0.98$, $\tau = 3$.

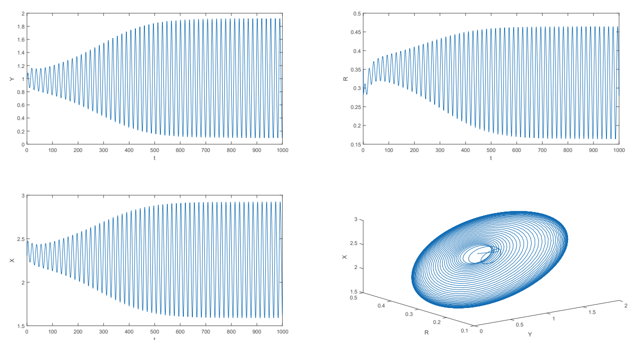

Figure 2. Stable periodic orbit of System (28), when $\alpha = 0.98$, $\tau = 3.2$.

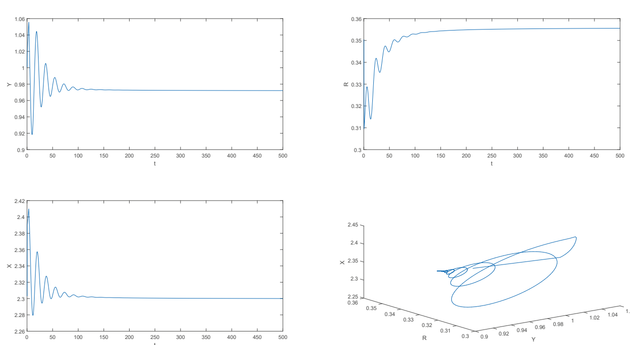

Figure 3. E^* is asymptotically stable, when $\alpha = 0.9$, $\tau = 3.2$.

In contrast with the results shown in Figures 1 and 2, if $\tau < \tau_0$, E^* is stable, and if $\tau > \tau_0$, E^* is unstable, which conforms to Theorem 4. When comparing Figures 2 and 3, it can be seen that the order α is an important factor in the stability of E^* when τ is selected, which conforms to Remark 1.

By virtue of this important discovery, it can be seen that it is necessary to establish a fractional financial model with an appropriate fractional-order. Moreover, through analyzing the current capital stock, predicting the future capital stock is beneficial to weaken the level of fluctuations.

5. Conclusions

In this paper, a DFOM for BC was established to describe the interaction of markets using the interest rate. The existence and uniqueness of the proposed model were obtained. By mathematical analysis, we found that the investment delay is an important factor in the stability and bifurcation of the economic equilibrium in dynamic macroeconomics. Moreover, we also found that the fractional-order affects the economic equilibrium's stability and bifurcation.

In order to reduce the factors of macroeconomic instability and to promote stable development, the government can adjust its investment activities from the following aspects, according to the conclusions of this paper. On the one hand, through a series of measures, such as improving the production equipment and working efficiency to reduce investment delay, the economic fluctuations can be weakened. On the other hand, considering the output efficiency of various industries under the current macroeconomic environment, by calculating the average investment delay, the shortage of short-term capital stock in the future can be reasonably predicted, so investment policy and the expected results can be combined to restrain the economic fluctuations caused by investment delay.

Author Contributions: Y.X. studied the stability and the Hopf bifurcation analysis. Z.W. was in charge of the fractional calculus theory and the simulation. B.M. provided guidance and recommendations for research; and, lastly, Y.X. contributed to the contents and writing of the manuscript. All authors have read and approved the final manuscript.

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Article

Fractional Dynamics and Pseudo-Phase Space of Country Economic Processes

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Abstract: In this paper, the fractional calculus (FC) and pseudo-phase space (PPS) techniques are combined for modeling the dynamics of world economies, leading to a new approach for forecasting a country's gross domestic product. In most market economies, the decline of the post-war prosperity brought challenging rivalries to the Western world. Considerable social, political, and military unrest is today spreading in major capital cities of the world. As global troubles including mass migrations and more abound, countries' performance as told by PPS approaches can help to assess national ambitions, commercial aggression, or hegemony in the current global environment. The 1973 oil shock was the turning point for a long-run crisis. A PPS approach to the last five decades (1970–2018) demonstrates that convergence has been the rule. In a sample of 15 countries, Turkey, Russia, Mexico, Brazil, Korea, and South Africa are catching-up to the US, Canada, Japan, Australia, Germany, UK, and France, showing similarity in many respects with these most developed countries. A substitution of the US role as great power in favor of China may still be avoided in the next decades, while India remains in the tail. The embedding of the two mathematical techniques allows a deeper understanding of the fractional dynamics exhibited by the world economies. Additionally, as a byproduct we obtain a foreseeing technique for estimating the future evolution based on the memory of the time series.

Keywords: fractional calculus; pseudo-phase space; economy; system modeling

1. Introduction

The last crisis (2007–2008) was severe, and it came without adequate warning to markets and policymakers. High unemployment rates could not be avoided, and the austerity programs had enormous impacts on standards of living, and caused extensive suffering. While sound democracies have been implanted in the Western world, sacrifice associated with austerity throughout the 2008 financial crisis could not be alleviated, and great discredit has befallen economics as a social science.

The use of gross domestic product (GDP), and GDP per capita, to assess economic performance and prosperity has long been discussed. The indicator does not include any out-of-market production, as self-consumption escapes accounting efforts and methods altogether. At the same time, negative externalities of economic growth, such as inequality, resource depletion, pollution, environmental degradation, and effects on climate, have been forgotten, although they always affect future economic growth [1].

The claim for better prosperity indicators and welfare measurement includes political fears about the capacity of democratic political regimes to implement them [2]. While there are no better metrics to reflect people's lives and aspirations that will be able to inspire better policies for better lives after the 2008–2009 crisis, the US and the European economies have been growing slightly more slowly than in earlier years, according to GDP metric.

As slower economic growth has been coupled with urban riots, strikes, warring, and mass migrations around the world, analysts have been led to doubt the possibility of overcoming social problems. Pessimistic views from Fogel 2007 even discuss the US capacity to go on performing as the world great power [3]. They predict an acceleration of convergence, and suggest that by 2040 the world may experience a global geopolitical turnover related with the relative economic decline of the US. Beyond alternative techniques to be invented to measure prosperity, the relative share of the US GDP in the global GDP will fall from 22% to 14%, according to Fogel [3]. For Europe, the relative share of the European GDP in the global GDP will fall from 21% to 5%, it is said.

The contrast of post-war prosperity with post 2007–2008 crisis is evident. Social, political, and military conflicts, including urban riots, terrorism, guerrilla actions, and outright warfare, are afflicting all continents and regions of the world. Independently of the construction of other indicators to assess economic performance and social progress, as suggested by Stiglitz et al. [1], the successful catching-up of Asian partners to core countries, coupled with their expansionary demography, will bring about fear for the survival of democracy, as this will depend on the political options of those Asian countries, according to Fogel.

Experts have recalled other aspects related with past crisis indicators, and concluded that the ongoing crisis has a new scale in comparison with any others in the past: Banking systems in Europe have faced episodes of instability several times in the nineteenth and twentieth centuries, but those crises were much less severe. The new millennium has brought monetary policies based on low interest rates that do not compensate for individuals' saving efforts, which is consistent with the ongoing banking crisis and the financial instability, which have increased in this period [4–7].

From the perspective of market economies, crises are normal episodes. Modern markets in capitalism have developed frequent episodes of booms and busts [8]. Surely, booms mean prosperity (sometimes with bubbles) and give origin to busts (or even crises, recessions, and depressions), that historically were followed by new prosperity [9]. Marxian views on the end of capitalism never materialized [10]. The failure of the less-adapted firms during crises can explain Capitalism's resilience and survival (until now). Business cycles alternating between prosperity and crisis are intrinsic features of market economies.

Independently of the invention of new measures for human welfare and happiness, economic convergence of national economies deserves generalized approval, because, in improving the lowest standards of living around the world, convergence contributes to global welfare.

In the 1960s Alexander Gerschenkron, economic historian at Harvard, devoted his attention to economic growth in a historical perspective, and identified the meaning of industrialization for prosperity to recall the higher rates of economic growth in the late-comers. According to Gerschenkron's hypothesis, the adoption of new technologies commands industrialization, helping imitators to grow faster [11]. This observation from the post-WWII perspective is remarkable, even if governments assumed strong roles in national economies. In many countries, such as France or the UK, the government's policies did not stimulate capital markets in the 1950s and 1960s because of nationalization of important sectors of economic activity [12]. However, the American role in joining European countries to the Marshall Plan offer in the OEEC, in 1948, guided them to considerable co-operation, putting Western partners into an openness attitude towards global capital markets, again.

The 1948–1973 period experienced the most successful economic growth of the entire history of humankind. It allowed the defeated Japan and late-industrialized countries to catch up to the developed European countries. Genuine economic modernization and urbanization occurred [13]. Individual aggressive competitive behavior was extended to more traditional societies. Co-operation with governments in strategies of growth, which included the formation of conglomerates and cross-participation of multinational firms, provided safety and more robust opportunities in Asian markets. In adopting new technologies for productive investment, Asian countries were successful in catching-up with the most-developed nations. A convergence process took place in each country in industrialization.

Fractional calculus (FC) is a mathematical formalism that models efficiently phenomena with non-locality and long term memory. On the other hand, the pseudo-phase space (PPS) is a tool for studying dynamics while avoiding the calculation of derivatives. Recently, various authors applied the tools of classical dynamical systems to the economy. Petráš and Podlubny [14] used the state space for describing national economies, while adopting GDP, inflation, and unemployment rates as state variables. Machado and Mata [15,16] analyzed the Economic and Monetary Union countries and their similarities during the integration process, and investigated the Portuguese short-run business cycles over the last 150 years, using the multidimensional scaling method for visualizing the results. Škovránek et al. [17] proposed an approach to macroeconomic modeling based on the state space, fractional-order differential equations, and orthogonal distance fitting. The GDP, inflation, and unemployment rates were adopted as state variables. Machado and Mata [18] presented a bond-graph approach to model economy. The generalization of the principle of conservation of power and the assignment of causalities were circumvented by means of a variable fractional-order element. Machado et al. [19] investigated the economic growth using the multidimensional scaling method and state space portrait analysis. The GDP per capita was adopted as the main indicator for economic growth and prosperity, and the long-run perspective from 1870 to 2010 for identifying the main similarities among countries' economic growth. Machado and Mata [20] proposed the PPS and FC for modeling the Western global economic downturn. Tarasova and Tarasov [21] introduced a generalization of the economic model of logistic growth by considering the effects of memory and crises. The memory effects are modeled with fractional order derivatives. Using the equivalence of fractional differential equations and the Volterra integral equations they obtained discrete maps with memory that were exact discrete analogs of fractional differential equations of economic processes. Tarasov and Tarasova [22] designed a model of economic growth with fading memory and continuous distribution of delay time. Their approach can be considered as a generalization of the standard Keynesian macroeconomic model. Tejado et al. [23] presented models of economic growth for the countries of the Group of Seven (G7) for the period 1973–2016. Such models consisted of differential equations of both integer and fractional order, where the GDP was a function of the country's land area, arable land, population, school attendance, gross capital formation, exports of goods and services, general government final consumption expenditure, and broad money. Ming et al. [24] applied the Caputo fractional derivative to simulate China's GDP growth, while comparing the effectiveness of both fractional and integer order models. Škovránek [25] proposed a mathematical model based on the one-parameter Mittag–Leffler function to describe the relation between the unemployment and the inflation rates, known as the Phillips curve. For a comprehensive literature review see [26] and the references therein.

Hereafter, the aforementioned questions are analyzed using the FC and PPS in the scope of the complex dynamics of world economies and the perspective of 15 important countries. Furthermore, a new approach for forecasting the GDP is proposed. The common adoption of the two approaches leads to (i) a clear visualization and a straightforward interpretation of the dynamic effects, and (ii) foreseeing the future dynamics of the time series (TS). The 15 selected countries are Australia, Brazil, Canada, China, France, Germany, India, Japan, Korea, Mexico, Russia, South Africa, Turkey, the United Kingdom, and the United States.

The paper has the following organization. Section 2 introduces the fundamentals of the FC and the PPS representation. Section 3 develops the methodology and the analytical formulation for studying the dynamics of the TS. Section 4 discusses the proposed estimators and evaluates their efficiency. Finally, Section 5 summarizes the conclusions.

2. Fundamental Concepts

2.1. The Fractional Calculus

The FC generalizes the derivative of a function f, $D^{\alpha} f(x)$, to orders $\alpha \in \mathbb{R}$ [27,28]. During the last decades FC has become a popular tool [29–31], and new areas of application have emerged. The modeling finance [32] and economy [18,21,22,26,33–35] phenomena became of particular relevance.

The Riemann–Liouville and Caputo are classical definitions of a fractional derivative (FD), being given by [36]:

$$ {}_{a}^{RL}\mathcal{D}_{t}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{x(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau,\ n-1<\alpha<n, \tag{1} $$

$$ {}_{a}^{C}\mathcal{D}_{t}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau,\ n-1<\alpha<n,\ \alpha>0, \tag{2} $$

where $\Gamma(\cdot)$ denotes the gamma function and $\{t, a\} \in \mathbb{R}$ $(t > a)$ are the upper and lower limits of the calculation interval, respectively.

Another classical expression is the Grünwald–Letnikov (GL) definition. The GL formulation has the advantage of leading to a straightforward digital implementation and is given by [36]:

$$ {}_{a}^{GL}\mathcal{D}_{t}^{\alpha} x(t) = \lim_{h \to 0^+} \frac{1}{h^{\alpha}} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} \gamma(\alpha, k)\, x(t-kh),\ t>a,\ \alpha>0 \tag{3a} $$

$$ \gamma(\alpha, k) = (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}, \tag{3b} $$

so that $\left[\frac{t-a}{h}\right]$ is the integer part of $\frac{t-a}{h}$ and h stands for the time increment.

The calculation of Equation (3) can be approximated by the truncated series

$$ \mathcal{D}^{\alpha}[x(t)] \approx \frac{1}{T^{\alpha}} \sum_{k=0}^{r} \gamma(\alpha, k)\, x(t-kT), \tag{4} $$

where r represents the truncation order and T is the sampling period. Therefore, in the $\mathcal{Z}$-domain we have:

$$ \mathcal{Z}\{D^{\alpha}[x(t)]\} \approx \left(\frac{1}{T^{\alpha}} \sum_{k=0}^{r} \gamma(\alpha, k) z^{-k} \right) \mathcal{Z}\{x(t)\}, \tag{5} $$

with $\mathcal{Z}\{\cdot\}$ and z denoting the $\mathcal{Z}$ transform and variable, respectively.

Equation (5) results from adopting the Euler (or first backward difference) in the discrete approximation. However, other approximations are possible, such as the Tustin (or bilinear) and the Simpson rules, as well as their combinations [37]. To obtain rational expressions the approximants need to be expanded into Taylor series and the final algorithm corresponds either to a truncated series, or to a rational Padé fraction.

In the follow-up we shall adopt Equations (4) and (5) to approximate the FD, due to their direct applicability in TS analysis.

2.2. The Pseudo-Phase Space

The PPS is useful for studying complex and non-linear dynamics. The PPS is more robust against noise than the classical phase space (PS) method and allows representations using a small number of measurements. The PPS reconstruction follows the Takens' embedding theorem [38]. If a TS represented by $x(t)$ is an attractor component represented by a smooth d-dim manifold, then the topological properties of the TS are equivalent to those of the embedding n-dim vector:

$$v(t) = [x(t) \;\; x(t+\tau) \;\; x(t+2\tau) \;\; \cdots \;\; x[t+(n-1)\tau]], \tag{6}$$

where t stands for time, $d, n \in \mathbb{N}$, $n > 2d+1$, and $\tau > 0$. Moreover, the symbols n and τ represent the embedding dimension and time delay, respectively. For $n = 2$ or $n = 3$, the vector $v(t)$ can be represented in an n-dim plot, so that the vectors $[x(t) \;\; x(t+\tau)]$ and $[x(t) \;\; x(t+\tau) \;\; x(t+2\tau)]$ in the PPS reflect the classical dynamic described by $[x(t) \;\; \dot{x}(t)]$ and $[x(t) \;\; \dot{x}(t) \;\; \ddot{x}(t)]$ in the PS.

An aspect of utmost importance in the PPS method is the time delay τ. Let us assume, for example, that the signal $x(t)$ has a limited superimposed noise. For small values of τ, the variables $x(t)$, $x(t+\tau)$ and $x(t+2\tau)$ have close values for the same sample and we obtain a straight line in the PPS. On the other hand, for large values of τ the TS are almost independent, but their intersection almost vanishes. Indeed, for a TS of length $L \in \mathbb{N}$, the intersection becomes $L-(n-1)\tau$, when considering an n-dim representation. Therefore, some kind of compromise needs to be established to obtain the time delay, τ_m, that minimizes a given index, that is, for calculating $\min_{\tau} \{J[v(t)]\}$.

One possible way for selecting J is to adopt the autocorrelation function between $x(t)$ and $x(t+\tau)$, for $\tau = 0, 1, 2, \ldots$. The value of τ_m is given by the first minimum. The fractal dimension and the mutual information [19,39] were also adopted, but it is not clear which index is superior and we often find several difficulties, such as low precision or small sensitivity, the influence of noise, or problems due to the limited length L.

3. The Description of the Time Series Dynamics

We consider the dynamics of the TS characterizing a set of 15 countries during the years 1970–2018. The set is made up of Australia, Brazil, Canada, China, France, Germany, India, Japan, Korea, Mexico, Russia, South Africa, Turkey, the United Kingdom and the United States, herein denoted by {AUS, BRA, CAN, CHN, FRA, DEU, IND, JPN, KOR, MEX, RUS, ZAF, TUR, GBR, USA}. The performance of the economy is captured through the GDP per capita. The data were obtained at the World Bank website on 15 November, 2019. In the following, each TS is represented by $x_i(t)$, where $i = 1, \ldots, 15$.

A classical method for analyzing the dynamics consists of the PS. For example, Figures 1 and 2 depict the GDP of China versus time, $x_4(t)$, and the corresponding 2- and 3-dim PS during years 1970–2018.

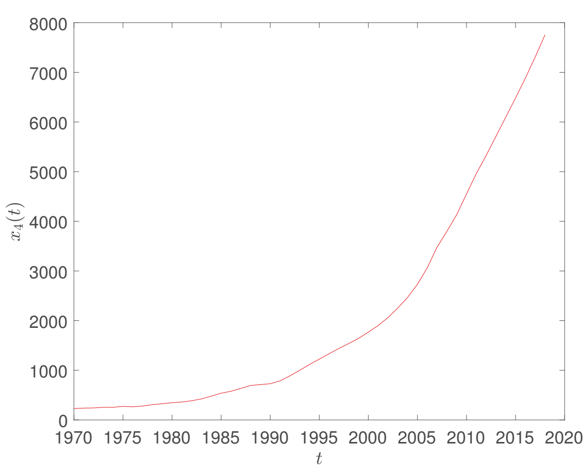

Figure 1. The gross domestic product (GDP) of China, $x_4(t)$, during 1970–2018.

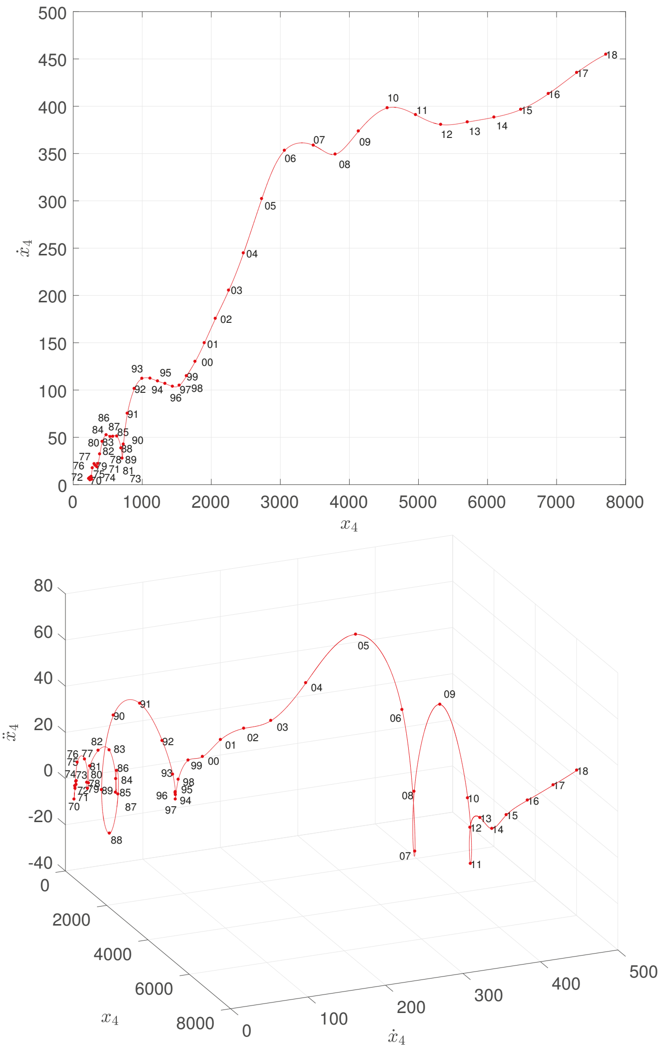

Figure 2. The 2- and 3-dim PS of China during 1970–2018.

For the numerical calculation of $\dot{x}_4(t)$ and $\ddot{x}_4(t)$ we adopt the algorithm proposed in [40], since it mitigates the effect of noise:

$$\dot{x}_4(t) = \frac{1}{8T}\left\{2\left[x_4(t+T) - x_4(t-T)\right] + x_4(t+2T) - x_4(t-2T)\right\}, \tag{7a}$$

$$\ddot{x}_4(t) = \frac{1}{4T^2}\left\{\left[x_4(t+2T) + x_4(t-2T)\right] - 2x_4(t)\right\}, \tag{7b}$$

where T stands for the sampling period.

We note in the PS considerable variations in the derivatives $\dot{x}_4(t)$ and $\ddot{x}_4(t)$. We can adopt other filtering techniques, but they introduce undesirable delay and reveal limited performance. Consequently, the PPS technique discussed in Section 2 emerges as a relevant strategy to solve such issues.

Two important limitations imposed by the TS are the large sampling period of one year, and the limited length of the TS of 49 years. To have a smaller sampling period, but avoiding artificial numerical artifacts, we implement a half-year piecewise cubic interpolation. Such an approach is consistent with the TS evolution, without revealing large fluctuations among consecutive samples, and allows passing from a TS with $L = 49$ to another one with $L = 97$ samples.

China's economic growth after 1970 was very successful, as Figure 1 illustrates [24]. While extensive literature calls into question the veracity and accuracy of Chinese GDP data based on officially published information, often suggesting that the declared economic growth rate of Chinese statistics greatly overstates the Chinese real rate of economic growth, it is true that China could adopt technological innovation and move to a position of strong participation in international markets. Thanks to institutional adjustments and entrepreneurial initiative, industrial growth based largely on bank loans, and increasing consumption of electricity and freight, Chinese economic growth has been remarkable, and clearly confirms resilience to adverse economic shocks [41]. Moreover, China has experienced structural transformations that encourage productivity growth in producing new commodities and services (especially those for electronic delivery), which explain an annual average accumulated growth rate above 10%. While quick ongoing economic growth may be an important reason to believe that China will not be able to continue indefinitely at such a pace, it is a fact that Chinese expansion consistently out-performed most analysts' expectations over the past 50 years [42].

To determine the delay τ, the cosine correlation [43] is adopted:

$$r(\tau) = \frac{\sum_{t=1}^{L-\tau} x(t)\, x(t+\tau)}{\sqrt{\sum_{t=1}^{L-\tau} x^2(t) \sum_{t=1}^{L-\tau} x^2(t+\tau)}}. \tag{8}$$

The first minimum of r versus τ provides the value τ_m. Several numerical experiments confirmed its good performance when compared with the mutual information and the Pearson or the Kendall tau rank correlations.

Both the FD and the PPS capture the memory of past dynamics. Indeed, the FD implicitly includes the past in Equation (4) by means of the series of signal samples at the time instants kT, $k = 1, \cdots, r$. On the other hand, the PPS captures memory through the delay τ_m in Equation (4). Therefore, we can compare $\mathcal{D}^{\alpha}[x(t)]$ with τ_m by some kind of average, γ_{av}, representing the TS. Nonetheless, the arithmetic mean reveals difficulties due to the slow convergence of the GL coefficients $\gamma(\alpha, k)$. On the other hand, the geometric mean was tested numerically, revealing good convergence for obtaining γ_{av}. We "average" the signal samples in Equation (4) by the geometric average of their terms, yielding [20]:

$$\gamma_{av}(\alpha, r)\, z^{-\tau_m} = \left(\prod_{k=1}^{r} \gamma(\alpha, k) \right)^{\frac{1}{r}} z^{-\frac{r(r+1)}{2}}, \tag{9}$$

in the $\mathcal{Z}$ domain.

Equation (9) leads directly to $\tau_m = \frac{r(r+1)}{2}$. Figure 3 depicts the evolution of γ_{av} versus (α, r), where the line connects the points with maximum value $\gamma_{av} = \max(\gamma_{av})$.

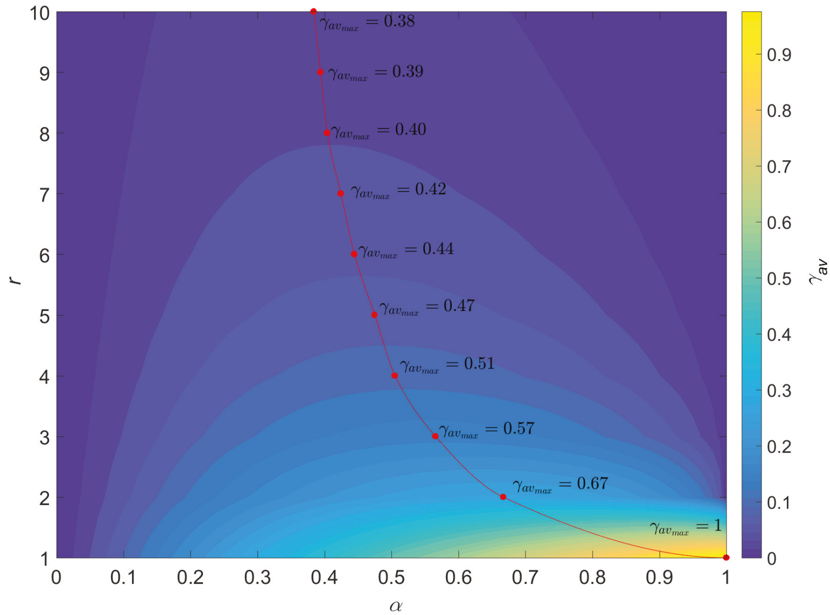

Figure 3. Locus of γ_{av} versus (α, r) and the points with maximum value $\gamma_{av} = \max(\gamma_{av})$.

For obtaining the geometric average of the GL series Equation (9) we interpolate $\gamma_{av} = \max(\gamma_{av})$ at τ_m and determine the corresponding α. The values of τ_m versus α are listed in Table 1 for the set of 15 countries.

Table 1. List of τ_m versus α for the set of 15 countries.

	AUS	BRA	CAN	CHN	FRA	DEU	IND	JPN	KOR	MEX	RUS	ZAF	TUR	GBR	USA
τ_m (years)	13	9.5	10	8	17.5	20	24.5	20.5	22	10	15	19	6	12	13
α	0.430	0.454	0.450	0.470	0.406	0.400	0.390	0.398	0.395	0.449	0.419	0.402	0.492	0.435	0.430

After calculating the values of τ_m for each TS we can plot the corresponding PPS. Figure 4 represents the PPS $[x(t) \;\; x(t+\tau_m)]$ of the GDP for the 15 countries during the period 1970–2018. We observe the emergence of three main clusters, namely $\mathcal{S}_1 = \{\text{CHN, IND}\}$, $\mathcal{S}_2 = \{\text{BRA, KOR, MEX, RUS, TUR, ZAF}\}$ and $\mathcal{S}_3 = \{\text{AUS, CAN, DEU, FRA, GBR, JPN, USA}\}$. We note a clear trend towards the 45 degree line, since the TS values do not vary quickly and, in general, recent values are higher than previous ones, due to the global economic progress. Another detail is the clusters' location, with $\mathcal{S}_1$ representing countries with a fast and steady growth, $\mathcal{S}_2$ standing for economies both with progress and recession, and $\mathcal{S}_3$ for countries with a small but sustained improvement.

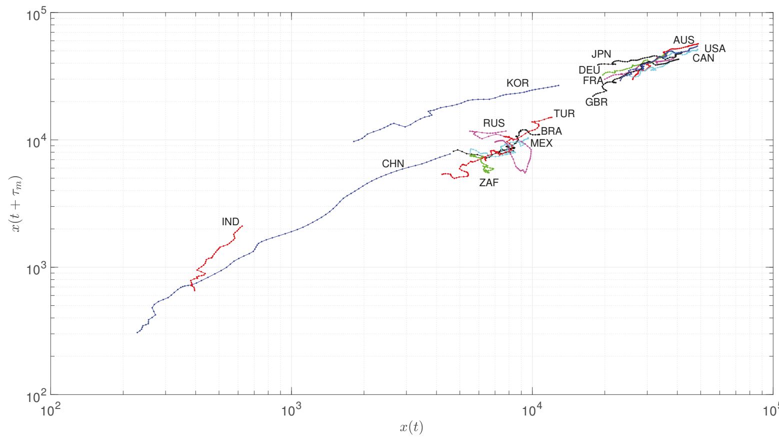

Figure 4. The pseudo-phase space (PPS) $[x\,(t)\;\; x\,(t+\tau_m)]$ of GDP per capita for the set of 15 countries during 1970–2018.

In a different perspective, we approximate the PPS $[x\,(t)\;\; x\,(t+\tau_m)]$ by means of power law functions, $x\,(t+\tau_m) \approx c \cdot [x\,(t)]^b$, using a non-linear least-squares fit algorithm. We verify that the parameters b are correlated with α, as shown in Figure 5, reflecting the fractional behavior of the economy dynamics. With the exception of IND, RUS and ZAF, we verify an almost linear correlation given by $b = -1.9737 + 6.2083\,\alpha$ for the rest of the countries.

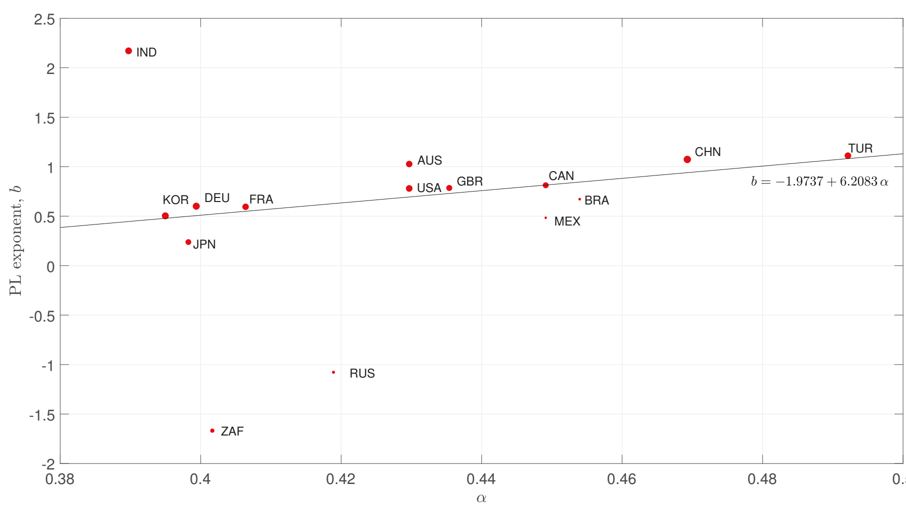

Figure 5. The locus of parameters b versus α. The markers' size is proportional to the coefficient of determination R^2.

The usual PPS portrait requires the calculation of τ_m, but lacks giving assertive information about the memory embedded in the TS. Since a heuristic relationship between τ_m and α was formulated previously, we expand the PPS by including the fractional dynamics information. Figure 6 shows the 3-dim locus $[x\,(t)\;\; x\,(t+\tau_m)\;\; \alpha]$ for the 15 countries during 1970–2018. This 3-dim locus extends the usual 2-dim PPS, by placing $x(t)$ and $x(t+\tau_m)$ in the x- and y-axes, respectively, and α in the z-axis. We find countries exhibiting slow dynamics (i.e., low α) in the bottom (e.g., IND, JPN, and KOR), and countries with fast dynamics (i.e., high α) in the top (e.g., BRA, CHN, and TUR).

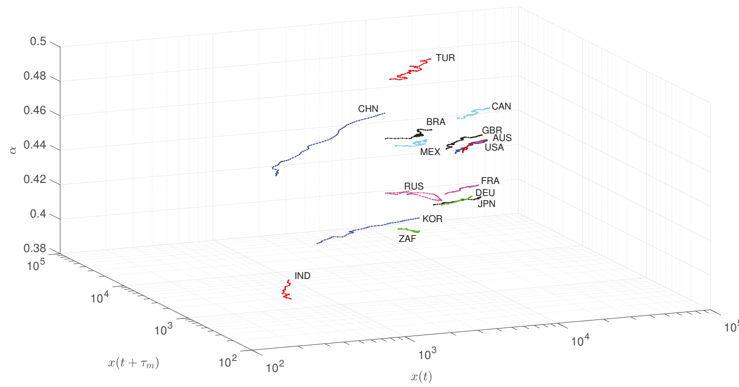

Figure 6. The locus $[x\,(t) \;\; x\,(t+\tau_m) \;\; \alpha]$ of GDP per capita for the set of 15 countries during 1970–2018.

The sample is made up of the G5 partners (UK, France, Germany, US, and Japan) and the three new successful Asian partners (India, China, and Korea), and also includes the heirs of old empires (Turkey and Russia), and four old European offshores (Australia, Canada, Mexico, Brazil, and South Africa). Figure 4 proves that in their long-term economic growth, gaps have been blurred. This means that globalization has brought acculturation and a contagious spread of economic growth. All over the world, a process of diffusion of technology has brought an extension of capitalism:

- The European partners and Japan could catch-up to the US and two other old European offshores (Australia and Canada). Before the late 1980s and the fall of the Berlin Wall, fighting communism may be considered to have been crucial to the national political strategies in most Western European countries and in East Asia. The anti-communist strategies clearly stimulated national policies in drawing them toward stock markets after the late 1980s. Many decision makers working at the World Bank and other international development agencies have even criticized codification of capital markets as meaning overregulation for the purpose of extending capitalism to communist-socialist areas.
- The heirs of old empires (Turkey and Russia), Korea, and three of the European offshores (Mexico, Brazil, and South Africa) also converged. In spite of cultural differences, genetic specificity, and climatic influences, they experimented consumption uniformization, with barriers that result from inequality in the distribution of revenue. Russia's political change has made its transition to convergence with the most-developed economies difficult, and the country exhibits more erratic economic growth behavior, comprising periods of strong rates of growth separated by frequent and severe crises.
- The two Asian historical civilizations, China and India, have proceeded at fast and regular economic growth rates; China and India's comparatively less-modern sectors have been catching up disproportionately faster to the world productivity frontier [44].

4. Estimation

In the last years the interest on models and algorithms for economic data forecasting has been growing [45,46]. The available methods include regression analysis [47], moving average [48], artificial neural networks [49], evolutionary computing [50], and empirical analysis [51], among others [52,53].

The proposed strategy leads implicitly to one estimation method based on the PPS. While our main objective was to establish a relationship between the time delay and the fractional order, it is relevant to explore this method and to compare results with other standard schemes.

After obtaining the value of τ_m for each economy and the corresponding 2-dim PPS, we have distinct alignments between the two TS, $x\,(t)$ and $x\,(t+\tau_m)$. This effect is negative, because it reduces the size of the intersection between the two vectors. Nonetheless, we can use the time shift between $x(t)$ and $x(t+\tau_m)$ for foreseeing purposes, so that the future values $[\hat{x}\,(L+1)\cdots\hat{x}\,(L+\tau_m)]$ are estimated on the basis of the old values $[x\,(L+1-\tau_m)\cdots x\,(L)]$. In this perspective, the new values are those that maximize the index of Equation (8) for the pair $[\hat{x}\,(L+1)\cdots\hat{x}\,(L+\tau_m)]$ and $[x\,(L+1-\tau_m)\cdots x\,(L)]$, as shown in the diagram of Figure 7.

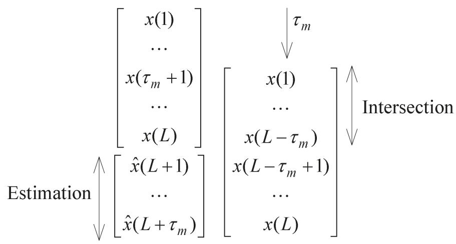

Figure 7. Estimation of future GDP values.

The values of the delay depend on the dataset and the estimation period varies accordingly. Consequently, the estimation horizon is different for each country.

4.1. Assessing the Estimation Method

For assessing the proposed scheme we divide the available data into two subsets: 1970–2013 for prediction and 2013.5–2018 for testing. First, we calculate the delays τ_m^* for the prediction TS, since they are slightly different from the ones resulting for the complete dataset, and we predict the future values. Second, we compare the real and predicted values for the period 2013.5–2018 by means of the metrics $\mathcal{E}_1$ and $\mathcal{E}_2$, given by:

$$\mathcal{E}_1 = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left[x(t)-\hat{x}(t)\right]^2}, \tag{10}$$

$$\mathcal{E}_2 = \frac{1}{T}\sum_{t=1}^{T}\frac{|x(t)-\hat{x}(t)|}{x(t)+\hat{x}(t)}, \tag{11}$$

where $T=10$ corresponds to the number of estimated points in the interval 2013.5–2018. These expressions closely follow the so-called Euclidean and Canberra distances [54] often adopted to assess the differences between numerical data.

Table 2 includes the complete list of values $\{\tau_m^*, \alpha^*\}$ of the estimation TS and the errors $\{\mathcal{E}_1, \mathcal{E}_2\}$ for all countries. Since the prediction algorithm is supported by the past, we do not obtain unrealistic values often proposed in the literature.

Table 2. The values of τ_m^*, α^*, $\mathcal{E}_1$, and $\mathcal{E}_2$ for the set of 15 countries: Real data for 1970–2013 and comparison between real and estimated values for 2013.5–2018.

	AUS	BRA	CAN	CHN	FRA	DEU	IND	JPN	KOR	MEX	RUS	ZAF	TUR	GBR	USA
τ_m^* (years)	13	9.5	8.5	8.5	9	19	20.5	20	21	10	13.5	17	5	8	12.5
α^*	0.430	0.454	0.464	0.464	0.459	0.402	0.398	0.399	0.397	0.449	0.427	0.409	0.534	0.469	0.433
$\mathcal{E}_1$	3759.3	848.4	3170.7	315.3	2879.7	3278.0	325.0	6031.7	3995.9	374.3	1950.5	947.6	3764.0	2216.6	4089.5
$\mathcal{E}_2$	0.028	0.031	0.026	0.019	0.029	0.032	0.083	0.050	0.060	0.017	0.086	0.061	0.095	0.022	0.031

The errors $\mathcal{E}_1$ and $\mathcal{E}_2$ are compared with those obtained with common regression analysis. Usually, regression methods perform adequately with economic data, since the TS evolves smoothly in time [55]. We adopt the nonlinear least-squares to fit 62 distinct models to the prediction TS, while discarding those that have a large number of parameters or that depict clear divergent behavior outside the fitting interval. The best fit is obtained with the rational and the third degree polynomial functions:

$$\hat{x}_R(t) = \frac{a + bt}{1 + ct + dt^2}, \tag{12}$$

$$\hat{x}_P(t) = a + bt + ct^2 + dt^3, \tag{13}$$

where $a, b, c, d \in \mathbb{R}$ are parameters to be estimated for each time series.

Equations (12) and (13) are then used for predicting the countries' GDP per capita for the period 2013.5–2018 and for calculating $\mathcal{E}_1$ and $\mathcal{E}_2$.

Figure 8 depicts the locus of the errors obtained with the PPS method and Equations (12) and (13). We verify that the PPS method does not lead always to the best prediction. This result was expected since Equations (12) and (13) were chosen as the best between a large number of possible functions, while the PPS-based estimation is just a byproduct of the relationship between the time delay and the fractional order. Nonetheless, we must note that the PPS method yields values of $\mathcal{E}_1$ and $\mathcal{E}_2$ in the narrower interval, being more robust to the variations between the distinct country GDP per capita.

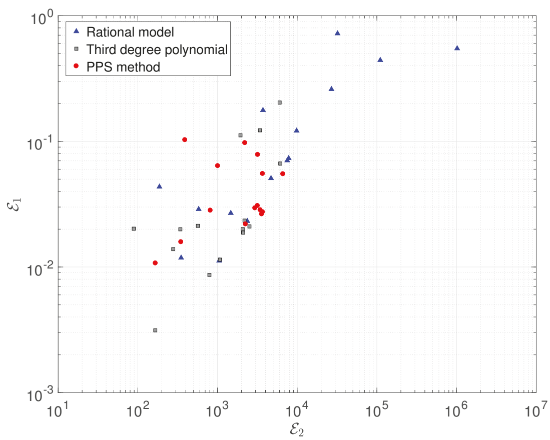

Figure 8. The locus of $\mathcal{E}_1$ and $\mathcal{E}_2$ obtained with the PPS method and Equations (12) and (13) for the set of 15 countries.

Figure 9 depicts $\mathcal{E}_2$ versus α^*, revealing three clusters that comprise {IND, RUS, TUR}, {JPN, KOR, ZAF}, and {AUS, BRA, CAN, CHN, FRA, DEU, MEX, GBR, USA}, respectively. We verify the small values of $\mathcal{E}_2$ particularly for the third (and larger) set of countries.

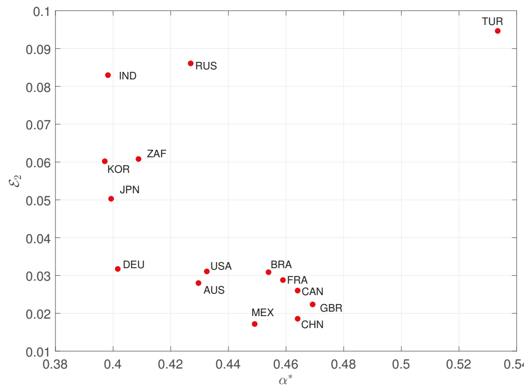

Figure 9. The locus of $\mathcal{E}_2$ versus α^* for the set of 15 countries.

4.2. Complete Estimation of GDP per Capita

We estimate GDP per capita of the set of 15 countries up to the maximum accomplished by the PPS method. This means that we have distinct prediction horizons for each country, going from 2024 for TUR up to 2042.5 for IND, as listed in Table 3. Figure 10 depicts the GDP per capita of China versus time. For this country we have $\tau_m = 8$ years and, therefore, we have real and estimation data for the periods 1970–2018 and 2018.5–2026, respectively. Figure 11 illustrates the real and estimated PPS for all countries. Again, we verify that the PPS representation produces good evolution without abrupt changes or uncommon behavior.

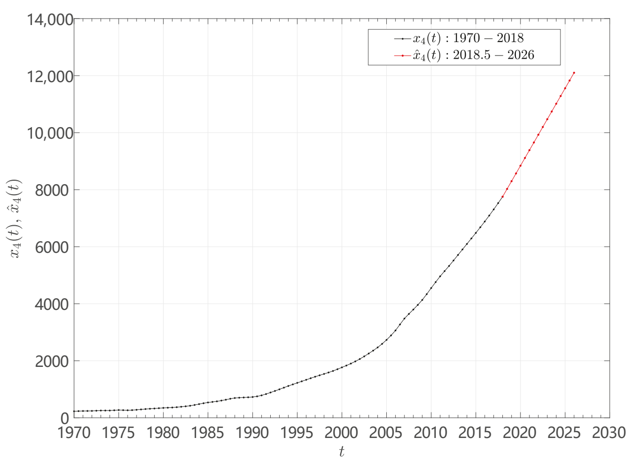

Figure 10. The GDP per capita of China versus time: Real data for 1970–2018 and estimated values for 2018.5–2026.

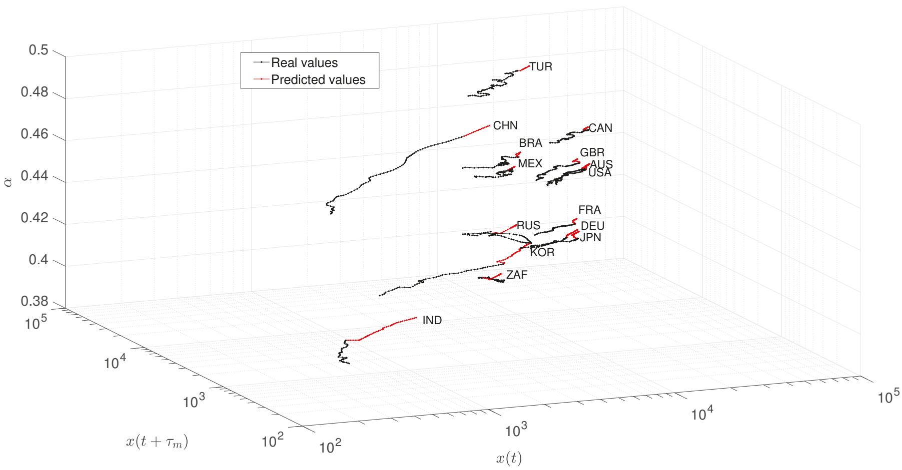

Figure 11. The real and predicted PPS for the set of 15 countries.

Table 3. Estimation of GDP per capita for the set of 15 countries.

α	AUS 0.430	BRA 0.454	CAN 0.450	CHN 0.469	FRA 0.406	DEU 0.400	IND 0.390	JPN 0.398	KOR 0.395	MEX 0.449	RUS 0.419	ZAF 0.402	TUR 0.492	GBR 0.435	USA 0.430
2018.5	57,908.8	11,509.6	52,625.9	8026.6	44,457.5	49,163.0	2084.8	50,188.3	27,743.1	10,273.7	10,915.1	7221.4	14,975.7	43,929.1	55,651.7
2019	58,898.2	11,992.9	53,387.6	8298.1	45,251.4	50,824.2	2065.3	51,457.0	28,724.3	9928.3	10,385.6	7002.8	15,469.5	44,872.2	56,761.7
2019.5	59,887.6	12,476.2	53,769.9	8569.7	46,045.3	52,449.0	2045.9	52,725.7	29,705.5	10,071.7	10,728.3	6784.3	15,741.7	45,815.3	57,871.6
2020	60,877.1	12,959.6	54,424.7	8841.3	46,839.2	53,163.3	2026.5	53,994.4	30,686.7	10,286.1	11,092.5	6565.7	15,993.1	46,758.4	58,981.6
2020.5	61,866.5	13,362.1	55,079.4	9112.9	47,633.1	53,701.7	2007.1	55,263.2	31,667.9	10,406.9	11,545.3	6629.8	16,368.9	47,701.5	60,091.6
2021	62,855.9	13,431.3	55,593.1	9384.5	48,427.0	53,973.0	1992.4	56,531.9	32,649.1	10,515.7	12,041.0	6710.8	16,686.6	47,397.5	61,000
2021.5	63,845.3	13,497.0	55,785.7	9656.1	49,220.9	53,947.0	2012.9	57,800.6	33,630.3	10,662.1	12,566.2	6765.2	16,809.6	47,562.3	59,875.8
2022	64,834.7	13,662.4	55,963.7	9927.7	50,014.8	53,884.5	2049.3	58,994.7	34,611.5	10,752.8	13,086.0	6824.4	16,938.8	47,831.8	58,935
2022.5	65,058.1	13,781.4	56,268.1	10,199.3	50,808.7	53,658.1	2098.5	58,961.4	35,592.7	10,754.4	13,533.4	6924.8	17,436.8	48,037.5	59,360
2023	65,271.7	13,768.1	56,665.0	10,470.9	51,602.6	53,474.2	2172.2	59,226.3	36,573.9	10,756.0	13,774.1	7049.4	17,912.0	48,238.8	59,946.1
2023.5	65,553.7	13,732.4	57,290.6	10,742.5	52,169.3	53,727.3	2243.0	59,732.2	37,555.1	10,813.4	13,233.9	7183.7	18,045.4	48,408.1	60,196.4
2024	65,956.3	13,493.9	57,706.9	11,014.1	52,749.4	54,107.9	2267.5	60,358.5	38,536.3	10,916.1	12,696.3	7330.1	18,100.0	48,595.7	60,437.7
2024.5	66,662.2	13,135.1	57,690.3	11,285.7	53,105.6	54,294.2	2288.3	61,030.6	39,517.5	11,024.9	12,907.5	7485.4		48,883.4	60,871.6
2025	67,345.2	12,795.3	57,673.7	11,557.2	53,060.8	54,523.1	2322.7	61,559.4	40,498.7	11,136.6	13,261.0	7643.9		49,262.1	61,345.8
2025.5	67,656.5	12,596.3	57,661.9	11,828.8	52,942.5	55,401.3	2357.7	62,036.6	41,479.9	11,239.3	13,544.7	7810.5		49,786.5	61,685.6
2026	67,925.5	12,608.7	57,655.0	12,100.4	52,069.1	56,601.2	2380.1	62,462.1	42,461.1	11,323.9	13,820.3	7949.7		50,333.6	62,043.4
2026.5	68,282.5	12,628.1	58,110.3		51,195.6	57,679.8	2406.4	62,881.9	43,442.3	11,378.4	14,090.5	8046.8		50,753.3	62,526.5
2027	68,639.5	12,647.4	58,672.6		51,428.2	58,527.7	2469.8	63,473.8	44,423.5	11,425.2	14,307.1	8094.7		51,104.7	63,099.1
2027.5	68,931.3	12,670.1	58,855.4		51,901.3	59,035.7	2543.5	63,847.7	45,404.7	11,473.6	14,459.3	7977.9		51,387.8	63,819.6
2028	69,239.4		58,936.1		52,441.3	59,270.6	2617.1	63,630.8	46,385.9	11,522.0	14,533.6	7860.3		51,652.7	64,441.5
2028.5	69,687.7				52,780.7	57,903.2	2690.8	63,115.6	47,367.1		14,481.8	7905.0		51,949.5	64,718.7
2029	70,106.7				52,736.0	56,535.8	2764.5	61,522.1	48,348.3		14,376.4	7982.0		52,223.6	64,977.9
2029.5	70,330.0				52,691.2	56,940.7	2838.2	59,928.6	49,329.5		14,168.9	8058.1		52,438.4	65,425.2
2030	70,550.1				52,699.9	58,474.4	2911.8	60,950.7	50,310.7		14,015.1	8119.8		52,612.4	65,993.4
2030.5	70,923.4				52,721.9	60,135.6	2985.5	62,207.9	51,291.9		14,019.7	8145.9			66,664.6
2031	71,417.3				52,817.5	61,742.2	3059.2	62,231.9	52,273.1		14,035.0	8168.8			67,461.1
2031.5					52,973.2	61,861.7	3132.8	62,250.2	53,254.3		14,114.7	8207.0			
2032					53,145.6	61,932.8	3206.5	62,629.9	54,235.5		14,248.1	8238.7			
2032.5					53,338.1	61,991.7	3279.4	63,284.8	55,216.7		14,394.9	8252.7			
2033					53,500.4	62,068.8	3353.1	64,043.1	56,197.9		14,571.2	8258.6			
2033.5					53,709.6	62,546.4	3426.8	64,640.7	57,179.1			8250.0			
2034					54,259.6	63,151.6	3500.5	64,813.8	58,160.3			8230.9			
2034.5					54,916.5	63,429.0	3574.1	64,969.8	59,049.7			8185.2			
2035					55,383.0	63,700.3	3647.8	65,383.8	59,799.7			8149.9			
2035.5					55,762.5	64,111.2	3721.5	65,832.3	60,609.0			8150.2			
2036						64,601.0	3795.1	66,062.8	61,409.5			8149.6			
2036.5						65,194.0	3868.8	66,310.5	62,106.2			8135.4			
2037						65,750.6	3942.5	66,982.7	62,791.0			8103.3			
2037.5						66,155.4	4016.2	67,700.8	63,547.0						
2038						66,469.2	4089.8	68,114.8	64,341.5						
2038.5							4163.5	68,374.0	65,183.4						
2039							4237.2		66,028.3						
2039.5							4310.8		66,813.9						
2040							4384.5		67,566.9						
2040.5							4458.2								
2041							4531.9								
2041.5							4605.5								
2042							4679.2								
2042.5							4752.9								

5. Discussion and Conclusions

A novel approach for analyzing the dynamics of economies was developed. The method embeds the concepts of FC and PPS for processing information recorded in the TS.

From the economic history perspective it is possible to say that by 2030 the world ranking of countries' GDPs per capita will not present significant differences from the ranking that prevails today. Australia and the US will be at the top of the ladder, followed by Canada and Japan. However, the three European countries (Germany, France, and UK), will be closer to them. Clearly, in a recovery period, optimism has prevailed in Western economies [56], but commercial warring means that some competition worries are in the air. Catching-up also expresses Western lifestyles convergence, confirming Gerschenkron [11].

Among the newly successful partners, their relative positions will change quite clearly: Today the Brazilian GDP per capita is about the same as Russia's, much above China's, while India remains on the tail. In the same way, distances among Brazil, Russia, and China will disappear, to make a pool of competitors. Global development of stock markets has allowed even the less-developed Russian regions to improve and ameliorate poverty [57], while the economic growth expectations in the Brazilian case will continue slowly, and growth from China relies on intensive diplomatic commercial dealings with the US [56].

Looking again to the presented forecasts, 2030 will begin a global turnover. Germany will grow and catch up to Japan. These two countries, both of which were defeated in WWII, will move ahead of France and the UK, both of which were victors in that conflict. The results from the PPS methodology indicate that India will not catch up, although it will benefit from the stimulus of strong external demand from developing economies such as China, and fast dynamics. By 2030 and even 2040 India will remain at the back, on the tail. The reason, as Battisti et al. [58] say, is that, although a country converges to its long-run growth path, such a path may be not enough and can even diverge from those countries at the global economic growth frontier. This means that for long-run performance, low levels of wealth at departure do really matter.

In a modeling perspective, exploring the combination of FC and PPS led to a new strategy for describing the dynamics of the economies while highlighting their fractional order. As a byproduct of the PPS approach, a robust algorithm for estimating the future of the TS was also obtained.

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Fractional Derivatives for Economic Growth Modelling of the Group of Twenty: Application to Prediction

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Abstract: This paper studies the economic growth of the countries in the Group of Twenty (G20) in the period 1970–2018. It presents dynamic models for the world's most important national economies, including for the first time several economies which are not highly developed. Additional care has been devoted to the number of years needed for an accurate short-term prediction of future outputs. Integer order and fractional order differential equation models were obtained from the data. Their output is the gross domestic product (GDP) of a G20 country. Models are multi-input; GDP is found from all or some of the following variables: country's land area, arable land, population, school attendance, gross capital formation (GCF), exports of goods and services, general government final consumption expenditure (GGFCE), and broad money (M3). Results confirm the better performance of fractional models. This has been established employing several summary statistics. Fractional models do not require increasing the number of parameters, neither do they sacrifice the ability to predict GDP evolution in the short-term. It was found that data over 15 years allows building a model with a satisfactory prediction of the evolution of the GDP.

Keywords: fractional calculus; modelling; economic growth; prediction; Group of Twenty

1. Introduction

In this paper, models of economic growth are developed. The economies considered are those of the countries members of the Group of Twenty (G20). The period under consideration consists of the years from 1970 until 2018. The gross domestic product (GDP) is obtained as the output of a dynamic system with eight input variables. The best models found employ derivatives of fractional order. These models are compared with alternative versions with integer order derivatives only. The comparison employs several statistical tools, commonly used to assess the quality of a model to predict future outputs. In this manner, the ability of predicting the evolution of the GDP in the short-term is demonstrated. This paper comes in the sequence of similar models obtained for Portugal, Spain [1], France, Italy [2], all the EU member-states [3], the Group of Seven [4], and China [5]. The success of fractional order models for this purpose has been justified as consistent with the mechanisms of economic growth, and is supported by the results.

Of the countries mentioned above, only China is not a highly developed country. Hence, this paper has, for the first time, long term fractional order economic growth models for several countries which are not highly developed, contributing to show that such models are suitable also for this particular case. It also includes a study of the best number of years included in each model to optimise results.

The remainder of this paper is organised in the following manner. Section 2 describes the G20 and presents a short state-of-the-art. Section 3 explains the methodology followed for economic growth modelling of the G20. Section 4 gives and discusses the results obtained. The main conclusions of this paper are drawn in Section 5.

2. Preliminaries

2.1. The G20

The G20 is a group of 19 countries and one international institution (the European Union), that together account for over $\frac{9}{10}$ of the world's GDP. Membership comprises all the members of the G7, which are the seven wealthiest advanced countries in the world (the European Union being a permanent invitee, though not an eighth member). Other economies, which are not developed economies (according to the criteria either of the United Nations or of the World Bank), play an important role in the world economic and political scenes because of their size, or at least of their regional importance. For this reason, the G7 conceived an extended international forum comprising such countries, which became the G20. Membership was established by invitation upon foundation of the G20 in 1999. Several summits have taken place every year since then. The most visible get together the heads of state or of the heads of government of the members. Others take place at ministerial level, or further business and trade partnerships.

The G20 members are, in alphabetical order: Argentina (ARG), Australia (AUS), Brazil (BRA), Canada (CAN), China (CHI), the European Union (EUU), France (FRA), Germany (DEU), India (IND), Indonesia (IDN), Italy (ITA), Japan (JPN), Mexico (MEX), Russian Federation (RUS), Saudi Arabia (SAU), South Africa (ZAF), South Korea (KOR), Turkey (TUR), the United Kingdom (GBR), and the United States of America (USA). Notice that France, Germany, Italy and the United Kingdom are also member states of the European Union, and are thus represented both directly and indirectly at meetings. (As of writing, the United Kingdom is about to leave European Union membership, and thus to become only directly represented).

The final year for which data is available is 2018. While the G20 exists since 1999 only, it was decided to extend the analysis further back to 1970. There are two reasons for this. First, data for less than 20 years might suffice to establish some models, but not to validate them, verify their performance over some extended period of time, or test their prediction abilities. Second, the period since 1970 is one for which data is easily found for nearly all countries in the G20. Still, for four members, viz. China, Russia, Saudi Arabia and Turkey, it was impossible to find data for this entire period. (This is not surprising for Russia given that it was part of the Union of Soviet Socialist Republics until 1991. In the other cases poor statistics may be due precisely to the lack of economic development). Restricting the period so that there should be data for all countries would result in too short a period, as mentioned above. Thus, only the remaining sixteen members of the G20 will be considered below. Despite this fact, we will refer to those as the G20.

2.2. Fractional Calculus in Economic Modelling

Several financial and economic models of fractional order have been developed. A current review can be found in [6]. An economic interpretation of fractional derivatives is given in [7]. A review of methods for financial models is given in [4]. The particular case of economic growth was addressed using fractional state-spaces [8–11], variable order derivatives [12], and pseudo-phase plane and state space analysis [13,14]; the effect of memory obtained with fractional derivatives was studied in [15,16]; fractional diffusion models were used for economic crises [17] and financial markets [18,19].

3. Methodology

This section describes the methodology followed in this paper for economic growth modelling of the G20.

3.1. GDP Models

The models presented below rely on the following assumption: the evolution of the GDP is a result of variables of two types. Variables of the first type reflect available resources; variables of the second type reflect impacts on the economy. Consequently, the first model structure conceived for the GDP is a linear integer order differential equation, which is, for each country of the G20, given by:

$$
\begin{aligned}
y(t) = \quad & C_1 x_1(t) + C_2 x_2(t) + C_3 x_3(t) + C_4 x_4(t) + C_5 \int_{t_0}^{t} x_5(t)\mathrm{d}t + C_6 x_6(t) + C_7 x_7(t) \\
& + C_8 \frac{\mathrm{d}x_8(t)}{\mathrm{d}t} + C_9 \frac{\mathrm{d}x_9(t)}{\mathrm{d}t},
\end{aligned}
\tag{1}
$$

Variables are as follows:

- $y(t)$: GDP, in 2010 US$;
- C_k: weights, constant in time, for each of the input variables x_k;
- t_0: first year considered—1970 in this case;
- x_k: inputs of the model, viz.:
- x_1: land area, in km^2—measures the natural resources available;
- x_2: arable land, in km^2—measures of the quality of the natural resources;
- x_3: population—measures the human resources available;
- x_4: school attendance, in years—measures the quality of human resources;
- x_5: gross capital formation (GCF), in 2010 US$—measures manufactured resources (the model considers the accumulated manufactured resources);
- x_6: exports of goods and services, in 2010 US$—measures external impacts on the economy;
- x_7: general government final consumption expenditure (GGFCE), in 2010 US$—measures budgetary impacts on the economy;
- x_8: broad money (M3), in 2010 US$—measures monetary impacts on the economy (the model considers the variation of monetary impacts);
- x_9: variation of x_5, in 2010 US$—the variation of GCF measures the impact of investment on the economy.

Keynesian models for the dynamics of economies usually consider as inputs variables that have short-term impacts in the economy. Growth accounting usually favors a more long-term approach. (See examples in [20–22], and the discussion in [23] about the factors economic growth depends upon). The variables above combine both. Notice that, to make the role of GCF clearer, since it appears twice in the model with different roles, two different variables (x_5 and x_9) are used to denote it.

As explained below in Section 4.1, not all variables in (1) have the same importance for the accuracy of the model. Their relative importance was found for each country for the whole time period. In this manner simpler models could be obtained. In particular, a second integer order model, with five variables only, was taken as an alternative:

$$
y(t) = \sum_{k=1,3,6,7} C_k x_k(t) + C_5 \int_{t_0}^{t} x_5(t)\mathrm{d}t. \tag{2}
$$

Impacts on the economy have effects that are felt for an extended period of time. Of course, this effect wanes away. Such a behavior can be modelled with fractional derivatives, since fractional derivatives are operators with memory [15,24]. In other words, the fractional derivative of a function is not a local operator, but its value depends on past values of the function. Depending on the particular order employed, this memory of past values can correspond to weights of the said past values that vanish for older time instants. This is the reason fractional derivatives are used to model phenomena such as distributions corresponding to power laws, long tails in general, or chaotic systems [25].

Hence, a fractional generalization of model (2) was considered. Rather than using more variables, and thus recovering model (1), the variables of model (2) representing impacts, and only those, are affected with a fractional derivative. Such variables are x_5, x_6 and x_7, so the considered model was:

$$y(t) = \sum_{k=1,3} C_k x_k(t) + \sum_{k=5,6,7} C_k \mathrm{D}^{\alpha_k} x_k(t). \tag{3}$$

The sign of the differentiation orders α_5, α_6 and α_7 can be positive or negative. This type of generalization has already been successfully used in our works [1–4].

Notice that the resulting fractional model has eight parameters. This is one parameter less than the number of variables of the original integer model (1). As the number of variables is similar, the comparison between the performance of the two is fair. The extra parameter of the integer model gives it a slight advantage; consequently, should the performance turn out to be the same, the fractional model will be considered better, since it achieves the same results with one parameter less.

The fractional differentiation operator D^{α_k} was numerically implemented following the Caputo definition [24], as ${}_0\mathrm{D}_t^{\alpha_k} x_k(t)$. Years are counted from 1970, which thus corresponds to the lower terminal 0. Terms for initial conditions were not included. Consequently, the effects of inputs are considered only from 1970 on. This approximation reduces statistical data needed to develop models and was used in previously published works, where it has provided acceptable results [4].

3.2. Optimizing and Assessing Performance

A fitting procedure implemented in MATLAB was used to find models (1)–(3) for each of the G20 countries. This procedure relies on Nelder-Mead's simplex search method. MATLAB's implementation from function `fminsearch` was used. The objective was the minimization of the mean square error (MSE):

$$\mathrm{MSE} = \frac{\sum_{j=1}^{N} (y_j - \hat{y}_j)^2}{N}, \tag{4}$$

where N is the number of years—$N = 49$ in this case—and y_j and $\hat{y}_j$ are the GDP and the model's GDP estimate, respectively. Several performance indexes other than the MSE were used from function `regstats` to further evaluate the quality of the resulting models, viz.:

1. The mean absolute deviation (MAD):

$$\mathrm{MAD} = \frac{\sum_{j=1}^{N} |y_j - \hat{y}_j|}{N}. \tag{5}$$

2. The coefficient of determination ($\mathrm{R}^2 \in [0, 1]$):

$$\mathrm{R}^2 = 1 - \frac{\sum_{j=1}^{N} (y_j - \hat{y}_j)^2}{\sum_{j=1}^{N} (y_j - \bar{y})^2}, \tag{6}$$

   where $\bar{y}$ is the mean of the GDP.

3. The t-values and p-values for each variable.

In Section 4 it will be shown that not all nine variables x_1, x_2, ..., and x_9 were necessary for every single model given by (1). This was already the case in models for other countries [1–4]. This result was established in three ways. First, from the t- and p-values for each variable. Second, from performance indexes MAD and R^2, that should not be significantly worst when one or more variables are removed from the model, if they are indeed necessary. Third, from the Akaike information criterion (AIC):

$$\text{AIC} = N \log \frac{\sum_{j=1}^{N} \left(y_j - \hat{y}_j\right)^2}{N} + 2K + \frac{2K(K+1)}{N-K-1}, \tag{7}$$

where K is the number of model parameters. The value of the AIC itself does not give information about the quality of a model. But comparing the AIC values of different models does. With such a comparison, it is possible to find out which models have a higher probability be good models for the data. In fact, a lower value of the AIC denotes a higher probability of a model being the best. Assuming that there are M models, this probability of model i being the best can be normalized as the Akaike weight w_i, $i = 1, \ldots, M$, by:

$$w_i = \frac{\exp\left(-\frac{\text{AIC}_i - \min\limits_M \text{AIC}}{2}\right)}{\sum_{j=1}^{M} \exp\left(-\frac{\text{AIC}_j - \min\limits_M \text{AIC}}{2}\right)}. \tag{8}$$

In this way, models given by (2) and (3) were developed.

3.3. Models Found from Data for Different Numbers of Years

For each of the expressions (1)–(3), four models were obtained. The first uses the data for the entire 1970–2018 period, so as to obtain a long term fit. This method, however, may lead to overfitting. This can be caused by an excessive influence in parameters of data of too many years into the past. Furthermore, it is impossible to assess the capability of this model of economic growth to predict the future evolution of the economy, because there are no additional years of data for testing the prediction ability.

To improve on this, three models for shorter time ranges were obtained. To find out which numbers of years could be reasonable, trend lines were found for the GDP of each country. Both linear and exponential trend lines were obtained; the former provided a better fit for some countries, and the latter for others. Finally, a fast Fourier transform (FFT) was used to obtain, for the different tendencies, the spectral content of the oscillations $y(t) - \tilde{y}(t)$. This was done in [4] to obtain the best time ranges of models. In the present case, Figure 1 shows the spectral content of these oscillations for all countries, normalized so that every curve peaks at 1. It can be seen that, in the G20, economies do not have similar periods of oscillations around the corresponding tendencies. Within the frequencies where most peaks take place, three reasonable values of time ranges were chosen: periods of 5, 10, and 15 years. In this way, for each country, using (4) as cost function, 34 models were found for $N = 15$, for the periods 1970–1985, 1971–1986, 1972–1987, and so on, such as a moving average; and similarly for $N = 10$ (39 models) and $N = 5$ (44 models). And this was done separately for models given by (1)–(3). Each of these models can be tested, using for this purpose the data of years in the future. In this manner, it is possible to check how good the model is predicting GDP values which were not used to adjust its parameters. The quality of the prediction was measured with performance indicators MSE, R^2, MAD, AIC, and w for each country.

The GDP of different countries has different orders of magnitude. To make model performance comparison easier, the figures below present the R^2 performance index, to show the quality of predictions obtained with each N–year model. The R^2 is always in a normalized range, irrespective of the magnitude of the variable under study, which makes it particularly suited for this visual purpose. In this way, all the important characteristics of the different models in relation to the others can be studied.

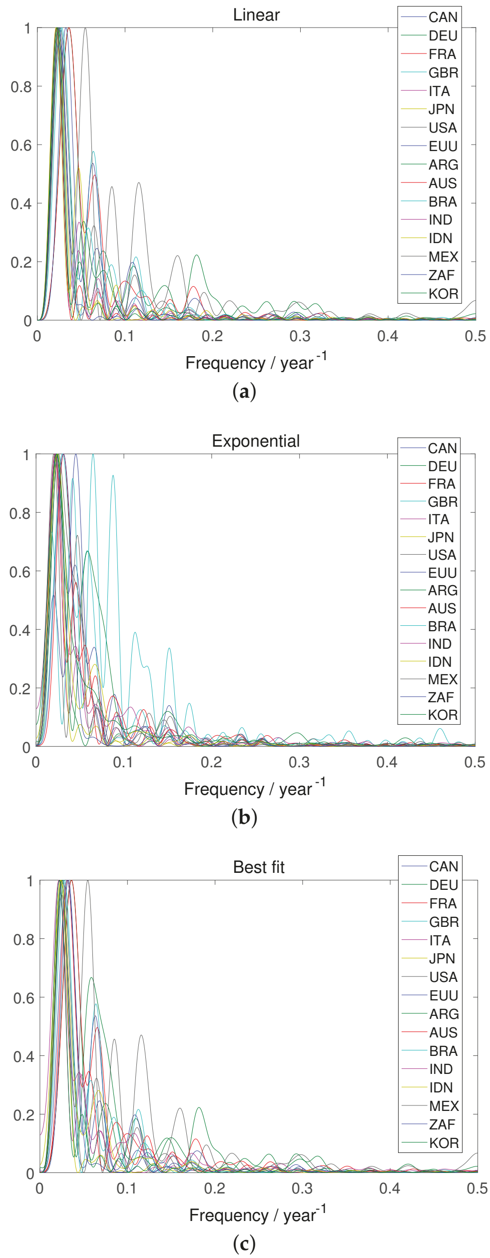


Figure 1. Spectral content of the oscillations $y(t) - \tilde{y}(t)$ for all countries, normalized so that every curve has a peak with amplitude 1, where $y(t)$ is the GDP, and $\tilde{y}(t)$ is a trend line: (**a**) Linear tendency (**b**) Exponential tendency (**c**) The best of the previous two.

4. Results

This section presents the models obtained, as well as their performance predicting the GDP of G20 countries. Due to its extension, a full tabulation of results is not included in the paper, but is available in [26]. Data sources are described in Appendix A.

4.1. Models for the 1970–2018 Period

Figures 2–4 show the results of the models obtained for each country from data for the entire 1970–2018 period. The performance indices are tabulated in Tables 1–3. In those tables, the t-values given in bold are those corresponding to variables which, assuming a 5% significance level, are necessary for the model. This information is also given in Table 4. It turns out that variables important for modelling six or more countries are x_1, x_3, x_5, x_6, and x_7. That is why model (1) could be simplified into model (2), which is in its turn generalized to fractional orders by model (3), only considering x_5, x_6, and x_7 to have fractional influence.

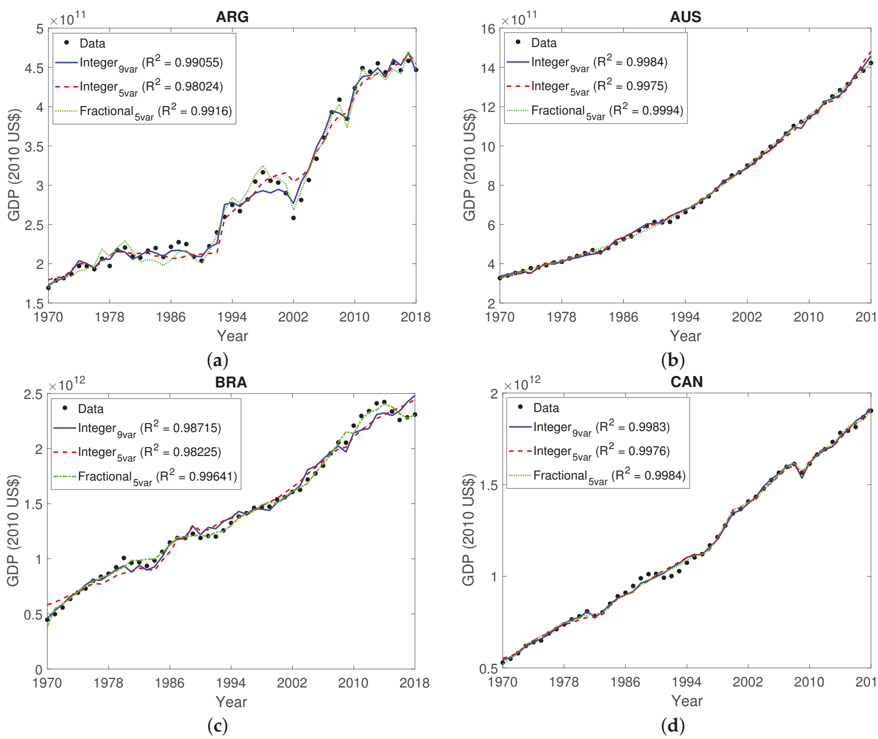


Figure 2. *Cont.*

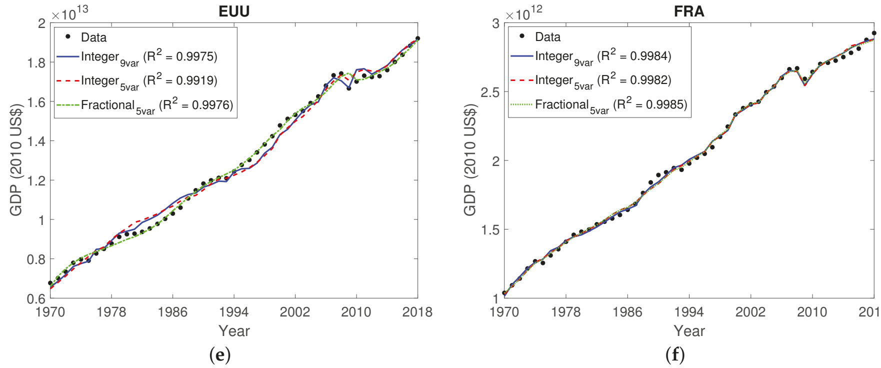


Figure 2. Results obtained with integer and fractional models for the following members of the G20: (**a**) Argentina (**b**) Australia (**c**) Brazil (**d**) Canada (**e**) European Union (**f**) France. GDP estimations were obtained with integer models (1) and (2) and with fractional model (3). R^2 values are given to show the quality of the results of each model. As GDPs have different orders of magnitude, different scales were used in the *y*-axis for different countries.

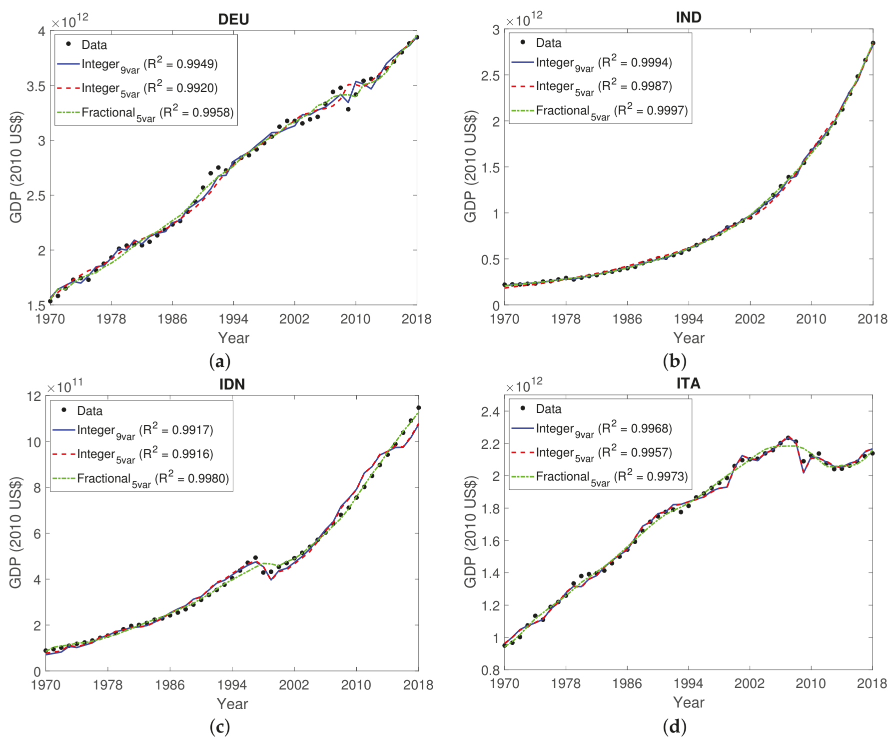
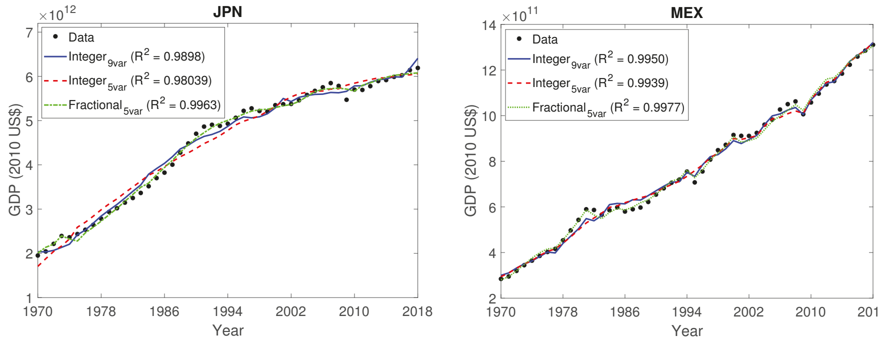
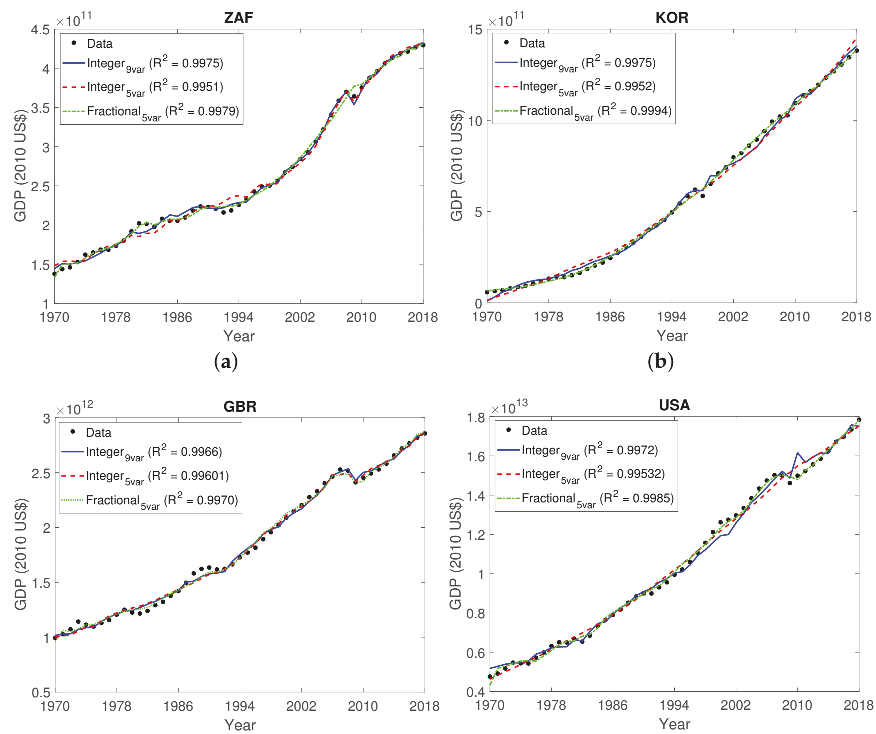


Figure 3. *Cont.*



(**e**) (**f**)

Figure 3. Results obtained with integer and fractional models for the following members of the G20: (**a**) Germany (**b**) India (**c**) Indonesia (**d**) Italy (**e**) Japan (**f**) Mexico. GDP estimations were obtained with integer models (1) and (2) and with fractional model (3). R^2 values are given to show the quality of the results of each model. As GDPs have different orders of magnitude, different scales were used in the y-axis for different countries.



(**c**) (**d**)

Figure 4. Results obtained with integer and fractional models for the following members of the G20: (**a**) South Africa (**b**) Korea (**c**) United Kingdom (**d**) United States of America. GDP estimations were obtained with integer models (1) and (2) and with fractional model (3). R^2 values are given to show the quality of the results of each model. As GDPs have different orders of magnitude, different scales were used in the y-axis for different countries.

Table 1. Performance indices of the different models obtained for the G20 members in Figure 2; for an explanation of performance assessment see Section 3.2.

		Argentina			Australia		
Index/Statistic	**Variable**	**Integer (1)**	**Integer (2)**	**Fractional (3)**	**Integer (1)**	**Integer (2)**	**Fractional (3)**
MSE ($\times 10^{19}$)		8.355	17.475	7.423	17.608	27.232	6.235
R^2		0.9906	0.9802	0.9916	0.9984	0.9975	0.9994
MAD ($\times 10^{9}$)		7.761	10.101	7.614	10.208	11.986	5.467
	x_1	−0.385	**3.393**	**−3.005**	**3.506**	**3.161**	**3.511**
	x_2	2.154	–	–	1.124	–	–
	x_3	−1.000	−1.424	**8.360**	−1.319	−2.160	**6.392**
	x_4	2.065	–	–	−1.490	–	–
t-values	x_5	0.910	**3.128**	**9.086**	2.193	1.316	**−13.267**
	x_6	1.694	**3.794**	2.214	2.194	**3.222**	**4.974**
	x_7	**6.784**	**5.138**	**3.232**	**5.404**	**6.257**	**7.013**
	x_8	1.139	–	–	**3.916**	–	–
	x_9	2.013	–	–	0.646	–	–
AIC ($\times 10^{3}$)		2.270	2.295	2.253	2.307	2.317	2.245
w (%)		0.02	0	**99.98**	0	0	**100**

		Brazil			Canada		
Index/Statistic	**Variable**	**Integer (1)**	**Integer (2)**	**Fractional (3)**	**Integer (1)**	**Integer (2)**	**Fractional (3)**
MSE ($\times 10^{20}$)		41.684	57.550	11.633	2.783	4.046	2.745
R^2		0.9871	0.9823	0.9964	0.9983	0.9976	0.9984
MAD ($\times 10^{10}$)		5.006	6.357	2.551	1.167	1.542	1.296
	x_1	−0.516	2.291	**−8.401**	−1.299	**3.645**	**5.391**
	x_2	1.583	–	–	**2.808**	–	–
	x_3	1.039	−0.063	**10.666**	**−2.823**	**−3.194**	**−3.830**
	x_4	−2.038	–	–	2.150	–	–
t-values	x_5	1.702	**2.829**	**10.562**	**5.079**	**6.318**	**8.927**
	x_6	1.067	1.816	**8.516**	**8.195**	**11.611**	**14.040**
	x_7	**3.272**	**3.338**	**−6.579**	**5.487**	**8.069**	**6.094**
	x_8	0.130	–	–	2.200	–	–
	x_9	**2.509**	–	–	**2.371**	–	–
AIC ($\times 10^{3}$)		2.462	2.466	2.388	2.329	2.336	2.317
w (%)		0	0	**100**	0.50	0	**99.50**

		European Union			France		
Index/Statistic	**Variable**	**Integer (1)**	**Integer (2)**	**Fractional (3)**	**Integer (1)**	**Integer (2)**	**Fractional (3)**
MSE ($\times 10^{20}$)		926.375	1122.995	334.785	4.942	5.812	4.852
R^2		0.9975	0.9919	0.9986	0.9984	0.9982	0.9985
MAD ($\times 10^{10}$)		25.326	29.613	12.679	1.816	1.897	1.731
	x_1	−1.385	0.697	−0.679	**−2.704**	**−5.648**	**−4.586**
	x_2	−1.384	–	–	**−2.384**	–	–
	x_3	0.901	−0.805	2.200	**5.069**	**5.968**	**5.294**
	x_4	**8.103**	–	–	1.606	–	–
t-values	x_5	**−3.213**	−1.869	**−11.728**	**−8.077**	**−9.564**	**−10.868**
	x_6	**3.636**	**2.915**	**7.922**	**9.667**	**14.155**	**14.625**
	x_7	2.242	**5.126**	**13.059**	**5.649**	**7.512**	**3.418**
	x_8	0.994	–	–	0.770	–	–
	x_9	2.195	–	–	0.592	–	–
AIC ($\times 10^{3}$)		2.614	2.612	2.553	2.357	2.354	2.345
w (%)		0	0	**100**	0.22	1.18	**98.60**

Table 2. Performance indices of the different models obtained for the G20 members in Figure 3; for an explanation of performance assessment see Section 3.2.

		Germany			India		
Index/Statistic	**Variable**	**Integer (1)**	**Integer (2)**	**Fractional (3)**	**Integer (1)**	**Integer (2)**	**Fractional (3)**
MSE ($\times 10^{20}$)		25.579	39.739	21.073	2.911	6.925	1.395
R^2		0.9949	0.9920	0.9958	0.9994	0.9987	0.9997
MAD ($\times 10^{10}$)		4.086	4.265	3.675	1.231	2.051	0.958
t-values	x_1	−1.204	−1.441	**−4.573**	1.527	−0.234	**−3.781**
	x_2	−1.993	—	—	−1.173	—	—
	x_3	**3.265**	**3.289**	**7.579**	−0.764	**4.915**	**7.666**
	x_4	0.710	—	—	1.670	—	—
	x_5	1.970	**6.050**	**−10.136**	**16.351**	**10.091**	**−5.143**
	x_6	**−3.322**	**−3.310**	**2.629**	**−2.325**	−0.570	**5.745**
	x_7	2.127	1.097	**14.676**	1.147	**2.883**	**6.815**
	x_8	**2.385**	—	—	1.889	—	—
	x_9	**3.933**	—	—	**3.630**	—	—
AIC ($\times 10^3$)		2.438	2.448	2.417	2.331	2.363	2.284
w (%)		0	0	**100**	0	0	**100**
		Indonesia			**Italy**		
Index/Statistic	**Variable**	**Integer (1)**	**Integer (2)**	**Fractional (3)**	**Integer (1)**	**Integer (2)**	**Fractional (3)**
MSE ($\times 10^{20}$)		7.362	7.166	1.741	6.567	6.716	5.670
R^2		0.9917	0.9916	0.9980	0.9968	0.9957	0.9973
MAD ($\times 10^{10}$)		2.057	2.036	0.896	1.918	1.999	1.887
t-values	x_1	0.532	1.177	**7.401**	1.426	**3.620**	**−4.646**
	x_2	0.094	—	—	−0.143	—	—
	x_3	−0.010	−1.134	**−5.814**	**−3.038**	**−3.776**	**7.412**
	x_4	−0.511	—	—	**3.588**	—	—
	x_5	1.632	**3.955**	**3.968**	**−5.028**	**−3.627**	**−15.949**
	x_6	**3.408**	**5.490**	1.280	**6.788**	**10.848**	**13.414**
	x_7	**6.959**	**7.320**	**40.381**	**10.987**	**32.791**	**18.617**
	x_8	−0.031	—	—	0.249	—	—
	x_9	0.184	—	—	0.829	—	—
AIC ($\times 10^3$)		2.377	2.364	2.295	2.371	2.361	2.353
w (%)		0	0	**100**	0	1.56	**98.44**
		Japan			**Mexico**		
Index/Statistic	**Variable**	**Integer (1)**	**Integer (2)**	**Fractional (3)**	**Integer (1)**	**Integer (2)**	**Fractional (3)**
MSE ($\times 10^{20}$)		177.805	332.447	60.170	4.194	5.077	1.935
R^2		0.9898	0.9804	0.9996	0.9950	0.9939	0.9977
MAD ($\times 10^{10}$)		11.103	15.812	6.289	1.649	1.678	1.119
t-values	x_1	**−5.556**	**−3.962**	**10.164**	1.582	−0.951	**−3.354**
	x_2	**3.615**	—	—	0.388	—	—
	x_3	0.456	**3.324**	**−9.361**	−1.214	**3.162**	**5.822**
	x_4	1.159	—	—	2.077	—	—
	x_5	0.004	−0.027	**−15.740**	1.189	−0.446	**8.484**
	x_6	0.074	−0.138	0.892	**3.199**	**3.515**	**7.318**
	x_7	0.874	1.250	**16.207**	**3.055**	**3.919**	**5.921**
	x_8	−0.930	—	—	0.794	—	—
	x_9	0.878	—	—	1.997	—	—
AIC ($\times 10^3$)		2.533	2.552	2.469	2.349	2.347	2.303
w (%)		0	0	**100**	0	0	**100**

Table 3. Performance indices of the different models obtained for the G20 members in Figure 4; for an explanation of performance assessment see Section 3.2.

		South Africa			Korea		
Index/Statistic	**Variable**	**Integer (1)**	**Integer (2)**	**Fractional (3)**	**Integer (1)**	**Integer (2)**	**Fractional (3)**
MSE ($\times 10^{19}$)		1.972	3.852	1.679	44.263	85.248	10.634
R^2		0.9975	0.9951	0.9979	0.9975	0.9952	0.9994
MAD ($\times 10^{9}$)		3.201	4.548	3.163	16.928	24.418	7.162
t-values	x_1	**3.369**	**7.434**	**19.420**	0.994	**−4.833**	**−3.940**
	x_2	−1.880	—	—	**−4.284**	—	—
	x_3	0.874	**−5.803**	**−12.655**	0.715	**4.116**	**6.034**
	x_4	**−4.015**	—	—	−0.484	—	—
	x_5	1.017	1.711	**4.929**	0.353	**2.794**	**−10.182**
	x_6	**9.658**	**9.529**	**17.246**	1.659	−0.553	**−18.575**
	x_7	**4.470**	**8.247**	**−14.366**	1.252	1.230	**13.575**
	x_8	−0.639	—	—	1.222	—	—
	x_9	1.377	—	—	**2.531**	—	—
AIC ($\times 10^{3}$)		2.199	2.221	2.180	2.352	2.373	2.271
w (%)		0	0	**100**	0	0.03	**99.97**
		United Kingdom			**United States of America**		
Index/Statistic	**Variable**	**Integer (1)**	**Integer (2)**	**Fractional (3)**	**Integer (1)**	**Integer (2)**	**Fractional (3)**
MSE ($\times 10^{21}$)		1.155	1.355	1.022	100.813	76.825	24.287
R^2		0.9966	0.9960	0.9970	0.9972	0.9953	0.9985
MAD ($\times 10^{9}$)		2.828	2.917	2.578	235.854	221.266	127.878
t-values	x_1	**9.140**	**12.407**	**12.899**	**−4.909**	**−5.387**	**−14.723**
	x_2	−0.976	—	—	1.013	—	—
	x_3	**−6.393**	**−10.798**	**−11.453**	**4.969**	**5.474**	**19.974**
	x_4	2.225	—	—	**−2.622**	—	—
	x_5	**8.255**	**9.848**	**4.313**	0.696	1.720	**9.501**
	x_6	**3.738**	**5.963**	**7.597**	−0.237	0.766	**−5.742**
	x_7	−1.567	−0.369	**10.167**	−0.017	1.458	**6.522**
	x_8	2.093	—	—	**3.485**	—	—
	x_9	0.692	—	—	2.003	—	—
AIC ($\times 10^{3}$)		2.399	2.396	2.382	2.618	2.593	2.537
w (%)		0.02	0.10	**99.88**	0	0	**100**

As can be observed, the MSE, R^2 and MAD allow reaching the same conclusion: the performance of models given by (3) is clearly better than the performance of integer models, in what concerns the quality of the fit during the period used to build each model. This happens for all sixteen countries. The Akaike weight, summarized in the last row of every country, also supports that models (3) are the best of the three for this purpose.

Table 4. Relevance of the independent variables of model (1), from which the GDP depends, for each country; for an explanation of how variable relevance was determined, see Section 4.1.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
Argentina							✓		
Australia	✓						✓	✓	
Brazil							✓		✓
Canada		✓	✓		✓	✓	✓		✓
European Union				✓	✓	✓			
France	✓	✓	✓		✓	✓	✓		
Germany			✓			✓		✓	✓
India					✓	✓			✓
Indonesia						✓	✓		
Italy			✓	✓	✓	✓	✓		
Japan	✓	✓							
Mexico						✓	✓		
South Africa	✓			✓		✓	✓		
Korea		✓							✓
United Kingdom	✓		✓		✓	✓			
United States of America	✓		✓	✓				✓	

4.2. Models for N–Year Period

Figures 5–9 show the performance of integer and fractional models of a group of selected countries (one per continent), namely Australia, the European Union, India, South Africa, and the United States of America, for $N = 5$, 10 and 15 years, predicting the future evolution of the GDP. Showing results for all countries would take too much space; then, results obtained with all models and performance indices can be found in [26] for all countries.

Notice that models obtained with data from periods beginning in the 1970s can be used to predict the GDP for many years until 2018. On the other hand, models developed with data from periods ending in the 21st century can be used to predict the GDP for a few years only. Furthermore, predictions for many years into the future have, as can be expected, a lower performance than those for years close to the end of the data from which the model was got. In fact, the performances in Figures 5–9 deteriorate over time, but are quite good at prediction for a short period, and here again fractional models show their better performance, as R^2 values do not decrease so significantly.

As far as the number of years for prediction is concerned, it was observed that the smaller the value of N, the better fitting—MSE obtained for every N–year period was really close to zero—but the lower the ability to predict GDP in future: the values of R^2 were the smallest of the three cases. This was especially clear for integer model (1). Conversely, the largest the value of N, the lower the value of MSE, but the better the prediction. Notice that the values of R^2 for $N = 15$ were close to 1, especially for predictions with fractional model (3).

Hence, in order to predict the economic growth of a country of G20 with certainty, it is necessary to consider a relatively large period of years.

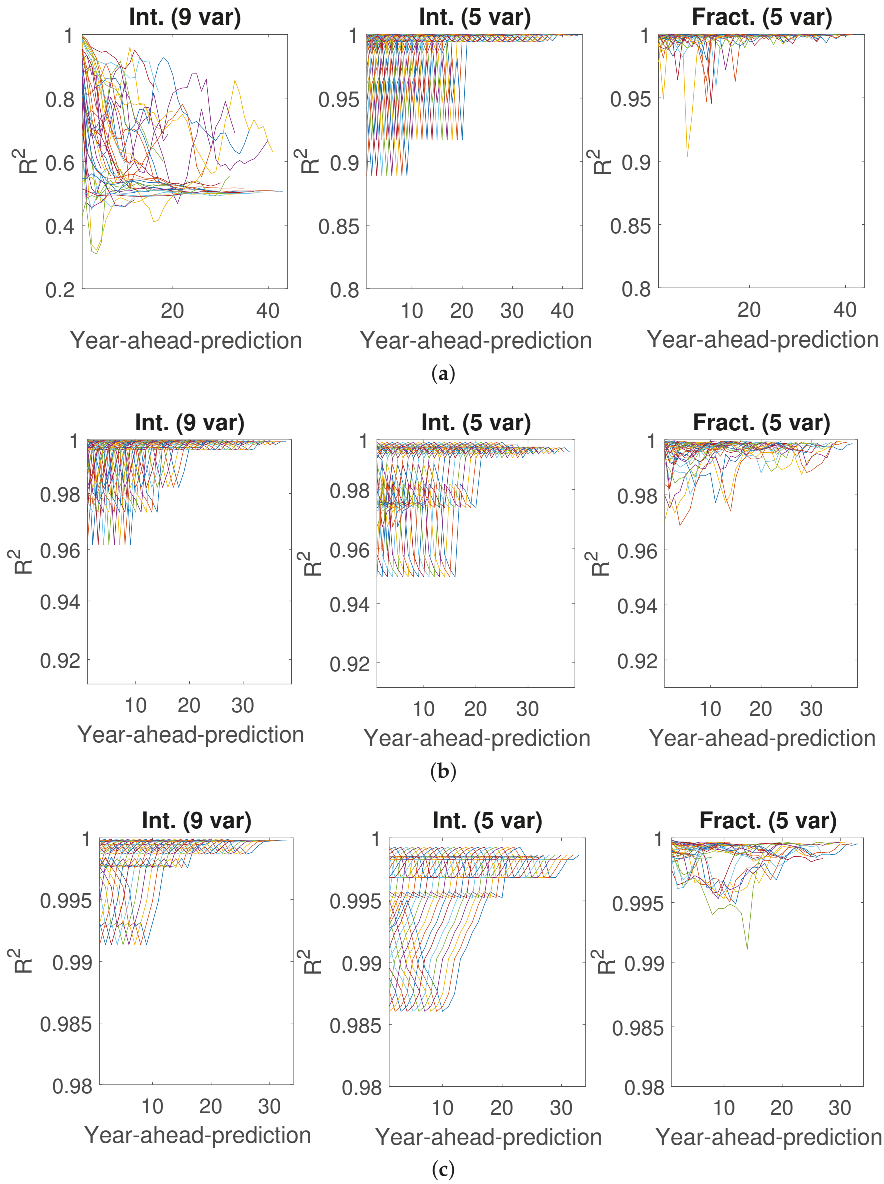


Figure 5. R^2 values of GDP estimates for Australia. Estimates were obtained with models (1) (left), (2) (middle), and (3) (right). The models were obtained with data for different numbers of years: (**a**) $N = 5$ (top) (**b**) $N = 10$ (center) (**c**) $N = 15$ (bottom). The models were used to estimate the GDP for as many years as possible after the period for which they were built. The scale of the y-axis is not the same for all models and all values of N.

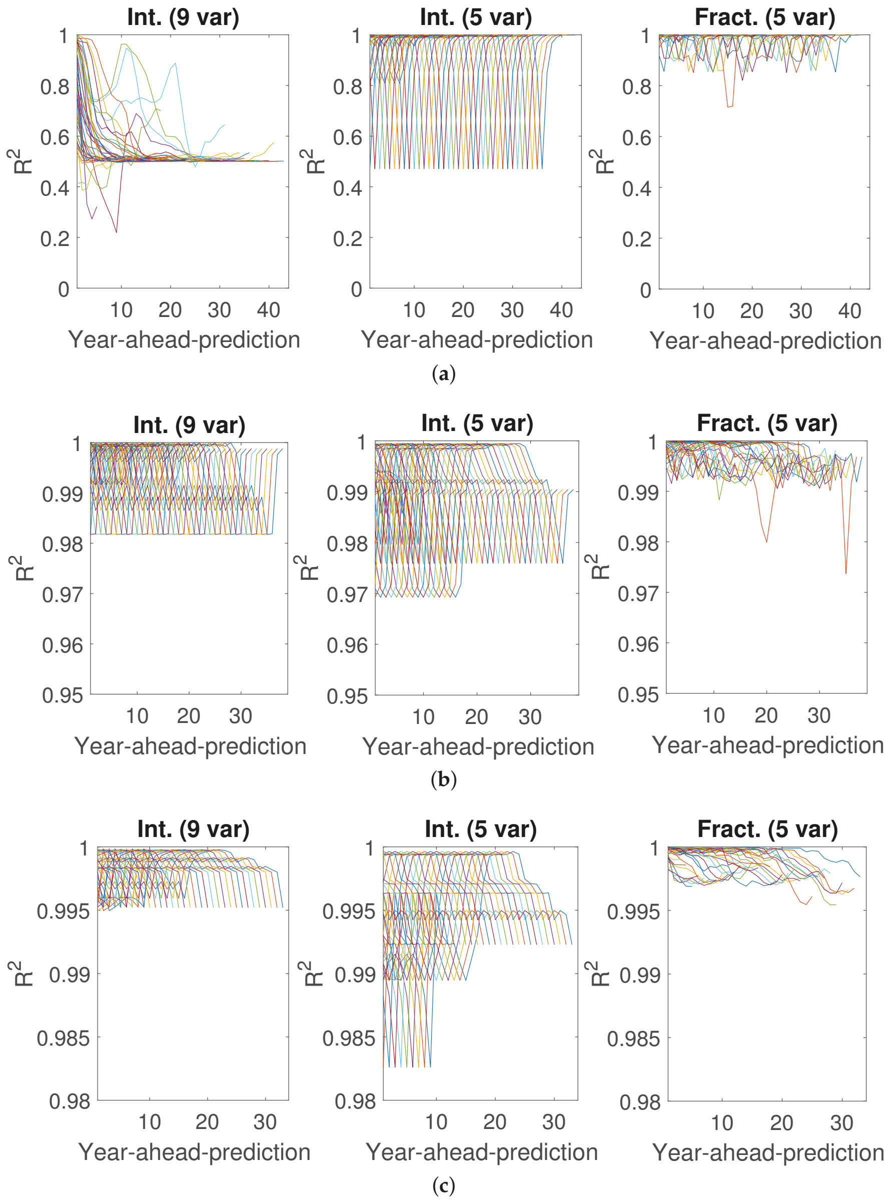
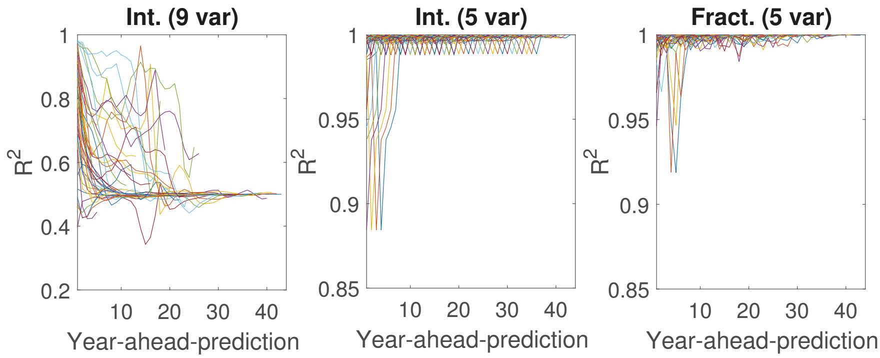
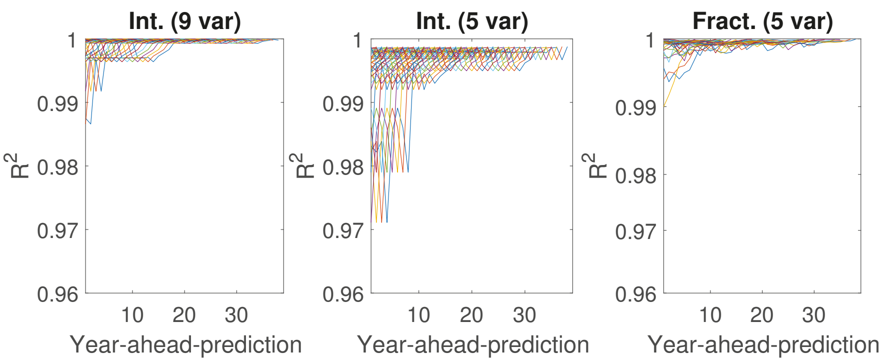


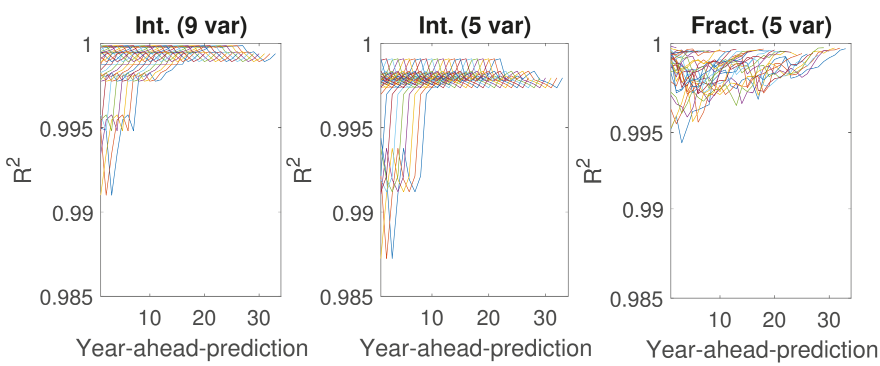
Figure 6. R^2 values of GDP estimates for the European Union. Estimates were obtained with models (1) (left), (2) (middle), and (3) (right). The models were obtained with data for different numbers of years: (**a**) $N = 5$ (top) (**b**) $N = 10$ (center) (**c**) $N = 15$ (bottom). The models were used to estimate the GDP for as many years as possible after the period for which they were built. The scale of the y-axis is not the same for all models and all values of N.



(**a**)



(**b**)



(**c**)

Figure 7. R^2 values of GDP estimates for India. Estimates were obtained with models (1) (left), (2) (middle), and (3) (right). The models were obtained with data for different numbers of years: (**a**) $N = 5$ (top) (**b**) $N = 10$ (center) (**c**) $N = 15$ (bottom). The models were used to estimate the GDP for as many years as possible after the period for which they were built. The scale of the y-axis is not the same for all models and all values of N.

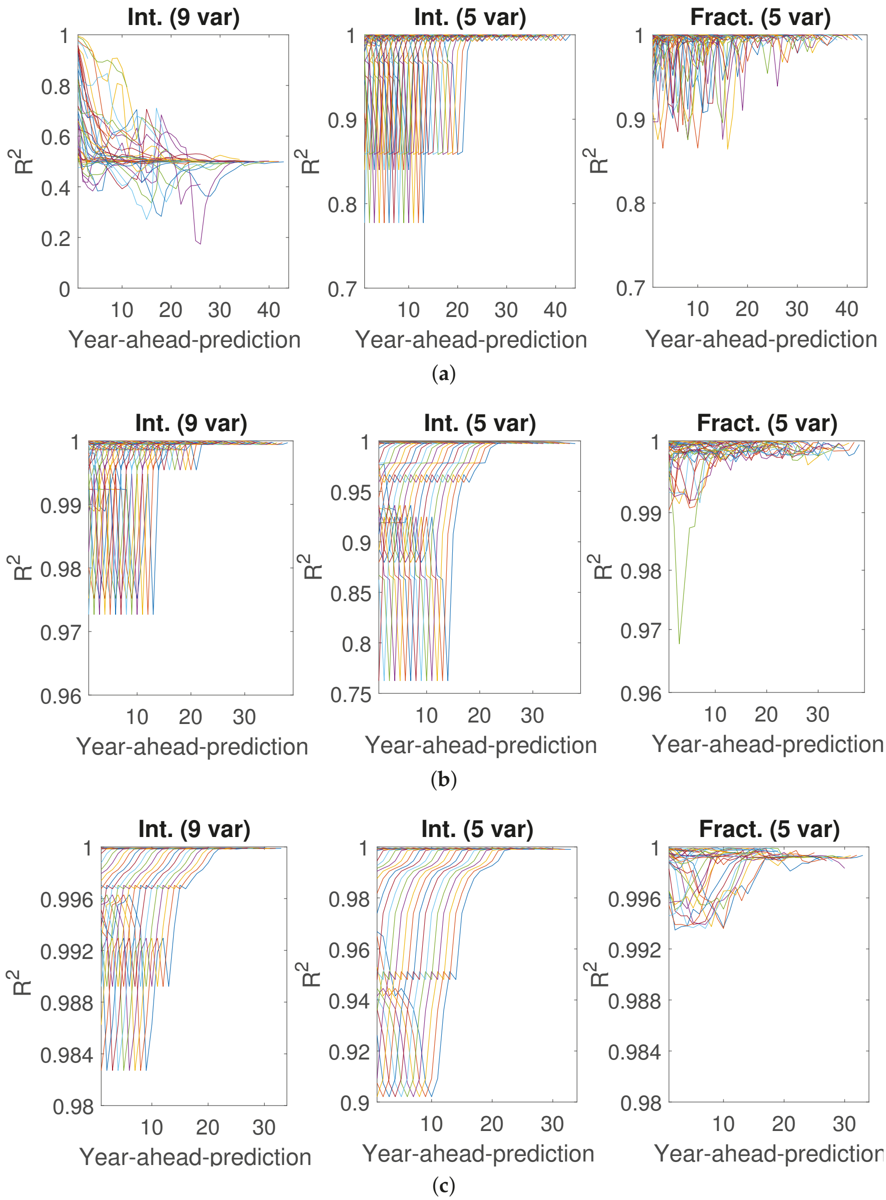


Figure 8. R^2 values of GDP estimates for South Africa. Estimates were obtained with models (1) (left), (2) (middle), and (3) (right). The models were obtained with data for different numbers of years: (**a**) $N = 5$ (top) (**b**) $N = 10$ (center) (**c**) $N = 15$ (bottom). The models were used to estimate the GDP for as many years as possible after the period for which they were built. The scale of the y-axis is not the same for all models and all values of N.

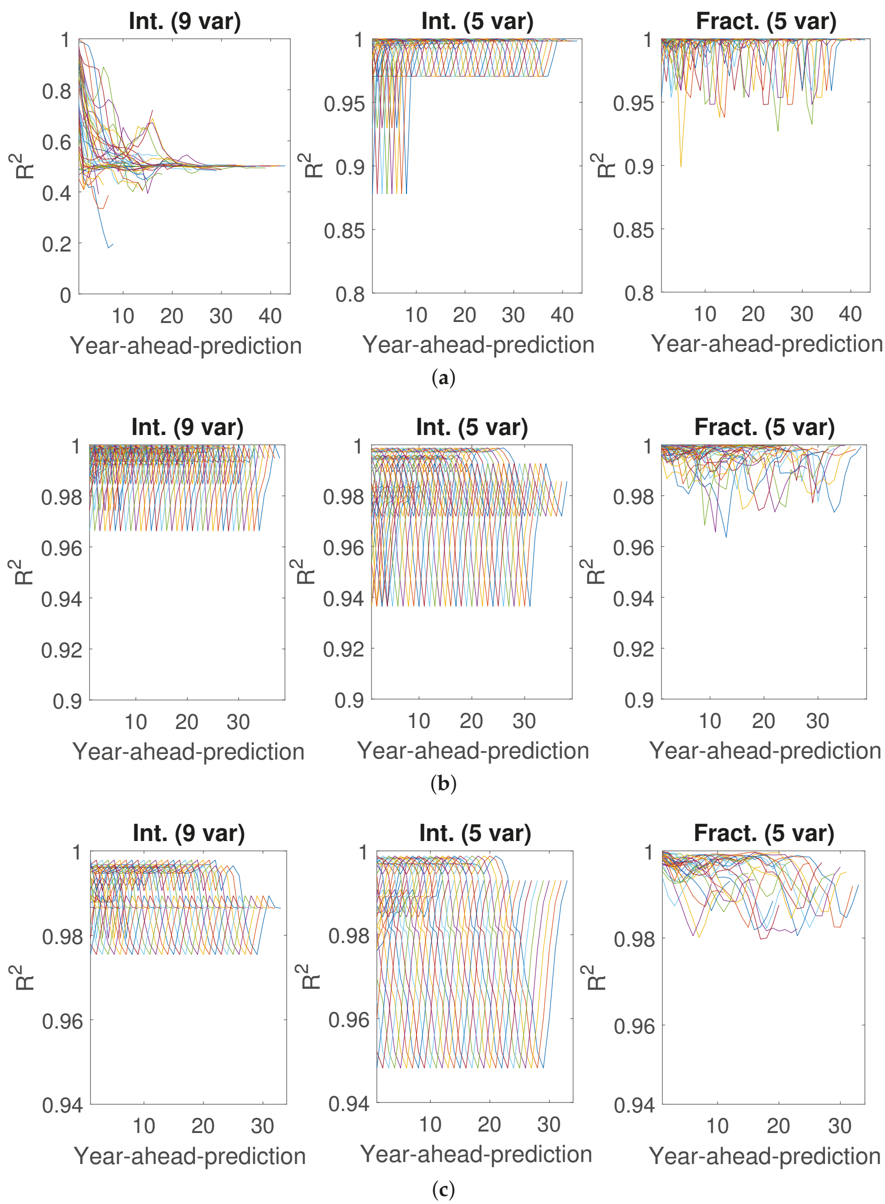


Figure 9. R^2 values of GDP estimates for the United States of America. Estimates were obtained with models (1) (left), (2) (middle), and (3) (right). The models were obtained with data for different numbers of years: (**a**) $N = 5$ (top) (**b**) $N = 10$ (center) (**c**) $N = 15$ (bottom). The models were used to estimate the GDP for as many years as possible after the period for which they were built. The scale of the y-axis is not the same for all models and all values of N.

5. Conclusions

The models of economic growth, of both integer and fractional order, presented in this paper for countries of the Group of Twenty (G20), from 1970–2018, are satisfactory. The variables chosen to predict variations of gross domestic product (GDP) prove to be suitable to the desired purpose.

It is clear from the results obtained that the performance of fractional models is superior. This statement is qualitatively backed by several indexes. Fractional models do not require an additional number of parameters, neither do they sacrifice the ability to predict the evolution of the GDP in the short-term. As to the number of years needed to build acceptable models, results show that $N = 15$ years lead to the best results.

The methodology followed in this paper can be further applied to more countries, and eventually generalised to more variables. Database [27], for instance, includes many time series, usually of good coherence, for all countries, that could be tested in the systematic manner described. The main difficulty is the disparity in the number of years for which time series are available; while for the G20 we could complete the missing values for eight variables, sixteen countries, and forty-nine years, this would likely be very difficult or even impossible if the number of variables, countries or years should be increased. So it would be necessary to improve this methodology in a manner that would cope with missing data and still be able to find, validate and compare models.

Author Contributions: This work involved all coauthors. I.T. wrote the original draft of this manuscript and contributed to the obtaining and the analysis of data and models. E.P. contributed to the obtaining of data and the coding in MATLAB. D.V. supervised all the work and contributed to frequency analysis of the data and the manuscript editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Data Sources

This appendix lists the sources of data used in this paper, which is not tabulated in this paper because of its size. It is available in [28].

- As in [4], variables for the EUU were the sum of the figures for its member states in each year. The only exception was x_4, addressed below.
- The source for the GDP, x_1, x_2, x_3, x_5, x_6, and x_7 was [27].
- Variable x_2 was available until 2016 only. It was assumed that $x_2(2017 : 2018) = x_2(2016)$. For Belgium and Luxemburg, which are member-states of the EUU, there is no x_2 data until 2000. Thus, x_2 was assumed constant until that year. This approximation corresponds, in the worst case, to an error in x_2 of 1.9% for the EUU during those years.
- The source for x_1 and x_3 for DEU until 1990 was [29]. In the same period, figures for x_2 were reduced in the same proportion.
- The source for x_4 was [30] until 2010. Figures are available with a 5-year period only, and were interpolated with a third-order spline. The figure for 2010 was extended into the future, using the increase rate of the figures in [31], also interpolated with a third-order spline. However, Figures for the following member-states of the EUU are not found in [30]: Croatia, Estonia, Latvia, Lithuania, Slovenia, Slovak Republic. The source for x_4 for these states was [27]. The EUU figure for x_4 is a weighted average of the figures for the member states in each year. The weight is the share of each state in x_3.

- Figures for x_5, x_6, and x_7 for JPN and USA for 2018 are those of 2017, updated with the yearly growth rate of the index in [32].
- The source for x_7 for ARG until 1992 was [27]. In the 1993–2018 period, the figure for 1992 was updated with the yearly growth rate of the index in [32].
- The source for x_8 for ARG, AUS, BRA, IDN, IDN, JPN, MEX, ZAF, GBR, and USA was [27].
- The source for x_8 for CAN until 2008 was [27]. In the 2009–2018 period, the figure for 2008 was updated with the yearly growth rate of the index in [32].
- The source for x_8 for DEU, FRA, ITA and other states of the EUU until 2015 was [33]. Figures were converted to 2010 US$ using the price index in [27]. In the 2016–2018 period, the figure for 2015 was updated with the growth rate in [34–36] for DEU, FRA, and ITA, respectively. However, figures for x_8 for Luxembourg and Romania in [33] are only available until 2011 and 2013, respectively. The figure for the last year was updated with the growth rate of [27].

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The Application of Fractional Calculus in Chinese Economic Growth Models

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Abstract: In this paper, we apply Caputo-type fractional order calculus to simulate China's gross domestic product (GDP) growth based on R software, which is a free software environment for statistical computing and graphics. Moreover, we compare the results for the fractional model with the integer order model. In addition, we show the importance of variables according to the BIC criterion. The study shows that Caputo fractional order calculus can produce a better model and perform more accurately in predicting the GDP values from 2012–2016.

Keywords: Caputo fractional derivative; economic growth model; least squares method

MSC: 26A33

1. Introduction

As one of the most important macroeconomic statistics indicators, GDP is an effective tool for people to understand and grasp the macroeconomic operation of a country; it is also an important basis for formulating economic policies. However, the calculation of GDP is very complicated, so a good economic growth model (EGM) can effectively form the economic progress problem, and it can reduce the loss of human and material resources.

Derivatives and integrals are often used to describe the process of economic development. However, there are still some shortcomings in using classical calculus to model real data. In recent years, the existence of solutions to fractional order differential equations have been studied in [1–3]. In addition, fractional calculus is widely used to construct economic models; it incorporates the effects of memory in evolutionary processes; experimental results show that the fractional order model is superior to integer order model, such as [4–13].

Recently, Luo et al. [14] improved the fractional EGM model in [5] and adopted different computational methods to simulate GDP via MATLAB, SPSS, and R software. The simulation results showed that the newly-established fractional hybrid model had better performance than the classical model.

In this paper, we adopt the idea in [14] to apply Caputo fractional order EGM and integer order to study China's GDP growth, as well as the minimum mean-squared-error (MSE) to estimate the parameters in the model. In order to compare the fitting effect between the integer order and the fractional order model, we establish the minimum absolute error coefficient, determination, and the BIC index. Finally, we use the prediction effect of the absolute relative error evaluation model.

Summarizing, based on fractional calculus, this paper conducts modeling of China's economic growth. Through a case study, it shows that fractional calculus has a better effect than integral calculus in modeling. It would be possible to use Monte Carlo simulation to generate sample data (see [15,16]) and then conduct modeling comparison. In this paper, real data are used for modeling and then showing the advantage of fractional calculus. The purpose of the two methods is the same, but the case analysis is often more complex and difficult than simulation, so a simulation is not used in this paper.

2. Models Description

We select six explanatory variables in this paper, and they are land area (LA) (km^2), cultivated area (CL) (km^2), total population (TP) (million), total capital formation (TCF) (billion), exports of goods and services (EGS) (billion), and general government final consumer spending (GGFCS) (billion), and the explained variable is GDP (billion). The data used in this paper were all Chinese data from the world bank from 1961–2016.

In order to simplify the expression, we define the following symbols:

x_1	x_2	x_3	x_4	x_5	x_6	y	n	k	t
LA	CL	TP	TCF	EGS	GGFCS	GDP	NVM	NPM	year

The general expression of the EGM is $y = f(x_1, x_2, \cdots)$, where f is the given function. Thus, the integer order model (IOM) and Caputo fractional order model (CFOM) are considered as:

- IOM:
$$y(t) = \sum_{j=1,2,3,5,6} c_j x_j(t) + c_4 \int_{t_0}^{t} x_4(t)dt + c_7 \frac{dx_7(t)}{dt},$$
- CFOM:
$$y(t) = \sum_{j=1}^{7} c_k (D_{t_0,t}^{\alpha_k} x_k)(t),$$

where t_0 and α_k represent the starting year and order respectively; in addition, the Caputo derivative $D_{t_0,t}^{\alpha_k} x_k$ for a given function x_k is defined as (see [1]):

$$D_{t_0,t}^{\alpha_k} x_k(t) = \frac{1}{\Gamma(1-\alpha_k)} \int_{t_0}^{t} \frac{\frac{dx_k(s)}{ds}}{(t-s)^{\alpha_k}} ds, \; t > t_0, \; 0 < \alpha_k \leq 1.$$

In order to facilitate the comparison of GDP between different years, the GDP, TCF, EGS, and GGFCE used here were converted into unchangeable local currency. The data from 1961–2011 were selected as the training sample, and data from 2012–2016 were used as the test sample. Moreover, we used the average absolute deviation (MAD) and the coefficient of determination (R^2) to evaluate the model, and the absolute relative error criterion was used to compare the prediction effect of the model. Recall the following definitions:

$$MAD = \frac{\sum_{i=1}^{n} |y_i - \hat{y}_i|}{n},$$

and:

$$ARE_i = \left| \frac{y_i - \hat{y}_i}{y_i} \right|, \quad i = 1, 2, \cdots, n,$$

and:

$$\mathrm{R}^2 = 1 - \frac{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}{\sum_{i=1}^{n} (y_i - \bar{y})^2}.$$

We often used the Akaike information criterion (AIC) and Bayesian information criterion (BIC) for the selection of variables in the model. Compared with the BIC criterion, the AIC criterion has the phenomenon of over-fitting. Therefore, we adopted the following BIC criterion:

$$BIC = \log\left(\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2\right) + \frac{p\log n}{n},$$

and:

$$\omega_j = \frac{\exp\left(-\frac{(BIC_j - BIC_{min})}{2}\right)}{\sum_{j=1}^{p}\exp\left(-\frac{(BIC_j - BIC_{min})}{2}\right)}.$$

3. Main Results

3.1. Economic Data for China

By using the Chinese economic data from 1961–2016 in unchangeable local currency, we apply R software to get the following figure (see Figure 1).

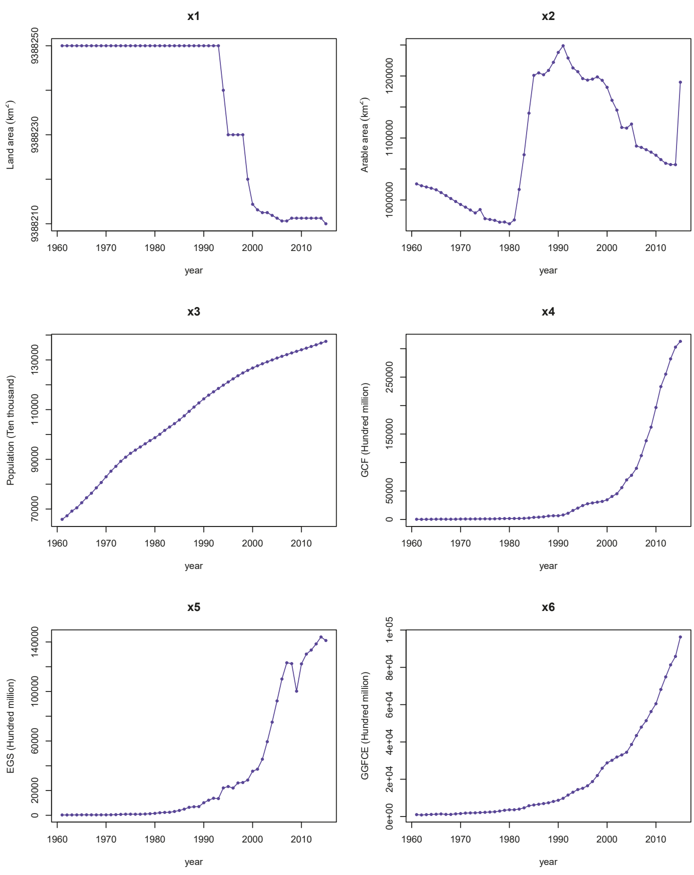

Figure 1. Data for China from 1961–2016. EGS, exports of goods and services.

3.2. Parameter Estimation

In this paper, we used R software and the least squares method to obtain the coefficient estimation in the integer order and Caputo fractional order models. Moreover, according to the MSE criteria, we gave the order of the Caputo fractional order model, and the following data were obtained (see Table 1).Table 2 shows the significance test results of the IOM and CFOM coefficients.

Table 1. The coefficients and orders of the integer order model (IOM) and the Caputo fractional order model (CFOM).

	IOM	CFOM		IOM	CFOM
α_1	0	−0.5389	c_1	−0.0051	0.0027
α_2	0	−1.3704	c_2	0.0286	0.0029
α_3	0	−0.6873	c_3	0.3220	−0.3619
α_4	−1	0.0960	c_4	0.1147	0.6058
α_5	0	−0.7777	c_5	0.4229	0.2329
α_6	0	0.0331	c_6	3.5943	1.3208
α_7	1	3.5251	c_7	0.7978	0.1087

Table 2. Significance level of the Caputo model.

	IOM		CFOM	
Variable	t-Value	p-Value	t-Value	p-Value
x_1	−5.066	7.76×10^{-6}	10.750	6.86×10^{-14}
x_2	2.853	6.58×10^{-3}	12.048	1.58×10^{-15}
x_3	4.127	1.61×10^{-4}	−10.697	8.04×10^{-14}
x_4	6.601	4.41×10^{-8}	16.408	2.00×10^{-16}
x_5	5.498	1.83×10^{-6}	19.996	2.00×10^{-16}
x_6	10.692	8.14×10^{-14}	6.645	3.80×10^{-8}
x_7	2.128	3.90×10^{-2}	2.304	2.60×10^{-2}

The results in Table 2 show that when the significance level was 0.05, the coefficients of IOM and CFOM passed the significance test.

3.3. Model Evaluation

In order to compare the performance of limited samples between IOM and CFOM, we present the values of MAD, R^2, and BIC index in the training sample set (see Table 3).

Table 3. Sample performance of IOM and CFOM.

Index	MSE	MAD	R^2	BIC
IOM	15,497,849	2,430.793	0.9991	17.0959
CFOM	1,906,429	1070.643	0.9999	15.0004

We adopted the BIC criterion to select variables in the model, and the importance of each variable was obtained, represented by ω (see Table 4).

Table 4. The importance of variables based on BIC.

	Variable	IOM	CFOM
BIC without one variable	x_1	17.47825	15.81924
	x_2	17.18849	16.06651
	x_3	17.34602	15.91584
	x_4	17.70702	15.02607
	x_5	17.54180	15.73556
	x_6	18.29923	16.06451
	x_7	17.11674	15.02607
ω found from the BIC without one variable	x_1	14.40%	12.92%
	x_2	16.64%	11.42%
	x_3	15.38%	12.31%
	x_4	12.84%	19.22%
	x_5	13.95%	13.48%
	x_6	9.55%	11.43%
	x_7	17.25%	19.22%

3.4. Fitting Results

Now, we give the fitting results of IOM and CFOM based on R software (see Figure 2).

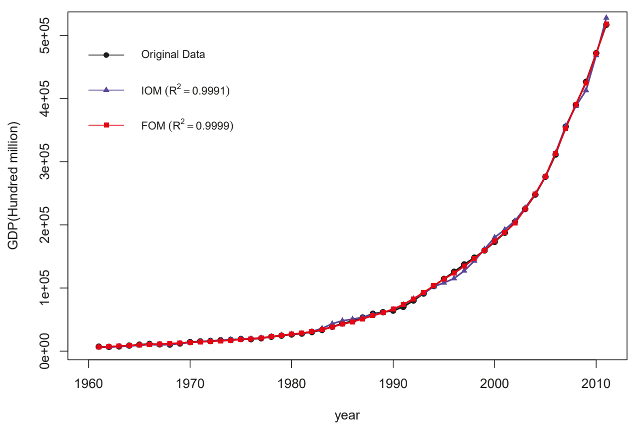

Figure 2. Data fitting.

3.5. Predicted Results

Finally, we present the forecast results of the IOM and CFOM models for China's GDP data from 2012–2016, and we calculate ARE index values, as shown in Table 5.

Table 5. Our results.

Year	Real Value	IOM		CFOM	
		Predict Value	ARE_i	Predict Value	ARE_i
2012	557,487.6	554,010.4	0.6237%	558,937.6	0.2601%
2013	600,735.4	611,769.9	1.8368%	601,514.2	0.1296%
2014	644,575.1	665,424.7	3.2346%	638,231.2	0.9842%
2015	689,052.1	741,602.5	7.6265%	675,637.3	1.9468%
2016	735,218.6	810,156.9	10.1927%	714,153.2	2.8652%

4. Conclusions

We selected six economic indicators in this paper and used IOM and CFOM to model China's GDP growth from 1961–2011. The fitting results showed that CFOM was significantly better than

IOM. To further illustrate the forecasting effect of the CFOM model, we presented the GDP forecast for China from 2012–2016 and compared it with the real value. It was found that the CFOM model not only had an advantage in fitting China's GDP growth, but also predicted it better. Finally, since all data were discrete, we intend to extend our study by applying the Caputo differences to create a fractional discrete time EGM.

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mathematics

Article

Deep Assessment Methodology Using Fractional Calculus on Mathematical Modeling and Prediction of Gross Domestic Product per Capita of Countries

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Abstract: In this study, a new approach for time series modeling and prediction, "deep assessment methodology," is proposed and the performance is reported on modeling and prediction for upcoming years of Gross Domestic Product (GDP) per capita. The proposed methodology expresses a function with the finite summation of its previous values and derivatives combining fractional calculus and the Least Square Method to find unknown coefficients. The dataset of GDP per capita used in this study includes nine countries (Brazil, China, India, Italy, Japan, the UK, the USA, Spain and Turkey) and the European Union. The modeling performance of the proposed model is compared with the Polynomial model and the Fractional model and prediction performance is compared to a special type of neural network, Long Short-Term Memory (LSTM), that used for time series. Results show that using Deep Assessment Methodology yields promising modeling and prediction results for GDP per capita. The proposed method is outperforming Polynomial model and Fractional model by 1.538% and by 1.899% average error rates, respectively. We also show that Deep Assessment Method (DAM) is superior to plain LSTM on prediction for upcoming GDP per capita values by 1.21% average error.

Keywords: deep assessment; fractional calculus; least squares; modeling; GDP per capita; prediction; LSTM

1. Introduction

In the last quarter of the century, the data exchange with not only person to person but also, machine to machine has increased tremendously. Developments in technology and informatics in parallel with the development of data science lead the companies, institutions, universities and especially, the countries to give priority to evaluating produced data and predicting what can be forthcoming. The modeling of all technical, economic, social events and data has been the interest of scientists for many years [1–4]. Many authors have been investigating the modeling and predicting events, options, choices and data. Especially, there is a huge research interest in finding any relation between telecommunication, economic growth and financial development [5–12]. One of the approaches to model a physical phenomenon or a mathematical study is to model the dependent variable satisfying differential equation with respect to the independent variable. However, the differential equations with an integer-order proposed for mathematical economics or data modeling cannot describe processes with memory and non-locality because the integer-order derivatives have the property of the locality. On the other hand, the fractional-order differential equation is a branch of mathematics that focuses on fractional-order differential and integral operators and can be used to address the limitations of integer order differential models. Using the fractional calculus or converting the integer-order differential equation into the non-integer order differential equations lead to a very essential advantage which is memory property of the fractional-order derivative. This is very crucial for models related to economics which in general, deal with the past and the effect of the past and now on future [12,13]. The memory capability of the fractional differential approach is the foundation of our motivation.

Fractional calculus (FC) as a question to Wilhelm Leibniz (1646–1716) first arose in 1695 from French Mathematician Marquis de L'Hopital (1661–1704) [11]. The main question of interest was what if the order of derivative were a real number instead of an integer. After that, the FC idea has been developed by many mathematicians and researchers throughout the eighteenth and nineteenth centuries. Now, there exist several definitions of the fractional-order derivative, including Grünwald-Letnikov, Riemann-Liouville, Weyl, Riesz and the Caputo representation. The fractional approach is used in many studies because the fractional derivative represents the intermediate states between two known states. For example, zero order-derivative of the function means the function itself while the first-order derivative represents the first derivative of the function. Between these known states, there are infinite intermediate states [11]. The use of semi-derivatives and integrals in the mass and heat transfer become an important instant in the field of fractional calculus due to employing the mathematical definitions into physical phenomena [12,13]. In the last decade, using fractional operators which explain the events, situations or modes between two different stages or the phenomena with memory provide more accurate models in many branches of science and engineering including chemistry, biology, biomedical devices, nanotechnology, diffusion, diffraction and economics [12–31]. In References [25–31], the modeling and comparison of the countries and trends in the sense of economics and its parameters are implemented. In References [25,26], economic processes with memory are discussed and modeling is obtained by using the fractional calculus. The studies with similar purposes as we aim such as modeling or prediction exist. In these studies, the fractional calculus is employed to model the given dataset and to predict for the forthcoming. In Reference [28], the orthogonal distance fitting method is used. The study is trying to minimize the sum of the orthogonal distance of data points in order to obtain an optimized continuous curve representing the data points. In Reference [32], the one-parameter fractional linear prediction is studied using the memory of two, three or four samples, without increasing the number of predictor coefficients defined in the study. In Reference [33], the generalized formulation of the optimal low-order linear prediction by using the fractional calculus approach is developed with restricted memory. All these studies focus on modeling or prediction for a phenomenon with fractional calculus. Also, in our previous studies, methods based on FC that works for modeling were introduced. In these studies, the children's physical growth, subscriber's numbers of operators, GDP per capita were modeled and compared with other modeling approaches such as Fractional Model-1 and Polynomial Models [34–36]. According to the results, proposed fractional models had better results compared to the results obtained from Linear and Polynomial Models [34–36]. Our previous works do not take into account the previous values of the dataset for any time instant. Their purpose is to model the dataset with minimum error and faster way compared with classical methods such as Polynomial and Linear Regression.

In this study, we extend our prior works by predicting the next incoming values as well as modeling the data itself. We introduce a new mathematical model, namely "Deep Assessment," based on the fractional differential equation for modeling and prediction by using the properties of fractional calculus. Different to the literature and our early studies mentioned above, this model can be used for prediction as well as modeling. The proposed approach is built on the fractional-order differential equation and corresponding Laplace transform properties are utilized. Here, the modeling is implemented with mathematical tools similar to those developed in the previous study [4] with a different approach in which the finite numbers of previous values and the derivatives are taken into account. Then, the prediction is obtained by assuming a value in a specific time can be expressed as the summation of the previous values weighted by unknown coefficients and the function to be modeled is continuous and differentiable. In this way, the proposed method takes previous values and variation rates between different time samples (derivative) of the dataset into account while modeling the data itself and predicting upcoming values. Combining the previous values with the variations weighted by the unknown coefficients lead to calling the method "deep assessment."

In this study, we assessed the proposed method by the modeling, testing and predicting GDP per capita of the following countries and the European Union: Brazil, China, European Union, India, Italy, Japan, the UK, the USA, Spain and Turkey. GDP per capita is a measure of a country's total economic output divided by the number of the population of the country. In general, it is a reasonable and good measurement of a country's living quality and standard [37]. Therefore, the modeling of GDP per capita is crucial and predicting GDP per capita is very essential not only for researchers but also for companies, investors, manufacturers and institutions. To assess the performance of Deep Assessment in modeling, we compare the proposed model with Polynomial Regression and Fractional Model-1 [34]. Besides, in the same way, for the prediction, we compared the model with Long-Short Term Memory (LSTM), a special type of neural networks used in time series problems.

The structure of the study is the following. Section 2 explains the formulation of the problem. After that, Section 3, namely *Our Approach*, is devoted to explaining how to obtain modeling, simulation, testing and prediction. Then, in Section 4, the results are presented. Lastly, Section 5 highlights the conclusion of the study.

2. Formulation of the Problem

In this section, the mathematical foundation of the proposed method is given. Before going into the mathematical manipulations, it is better to explain the approach and the main steps for the formulation. The study aims to model and then, to predict GDP per capita data at any time t by using the previous GDP per capita values of the countries. Here, we assume that countries' historical data and the change of these data over time create an eco-genetics for the forthcoming. In other words, mathematically, GDP per capita at a time t is assumed to be the summation of both its previous values and the changes in time with unknown constant coefficients. In the second stage, we express a function for the GDP per capita as a series expansion by using Taylor expansion of a continuous and bounded function. Then, the differential equation obtained from this series expansion is defined. After that, the unknown constant coefficients are found by the least-squares method. The method aims to minimize the error between the proposed GDP per capita function and the dataset.

First, it is a reasonable idea to approximate a function $g(x)$ as the finite summation of the previous values of the same function weighted with unknown coefficients α_k and the summation of the derivatives of the previous values of the same function weighted with unknown coefficients β_k because, intuitively, the recent value of data, in general, is related to and correlated with its previous values and the change rates. The purpose is to find the upcoming values of any dataset with a minimum error by employing the previously inherited features of the dataset. As a starting point, an arbitrary function is assumed to be approximately the finite summation of the previous values and the change rates weighted with some constant coefficients. To use the heritability of fractional calculus, this presupposition for modeling of the function itself and predicting future values is done [6,28,34].

$$g(x) \cong \sum_{k=1}^{l} \alpha_k g(x-k) + \sum_{k=1}^{l} \beta_k g'(x-k). \tag{1}$$

Here, g' is the first derivative of $g(x-k)$ with respect to x. After assuming Equation (1), the function $g(x)$ can be expanded as the summation of polynomials with unknown constant coefficients, a_n as given in Equation (2). Here, $g(x)$ is assumed to be a continuous and differentiable function.

$$g(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{2}$$

Then, $g(x-k)$ becomes as Equation (3)

$$g(x-k) = \sum_{n=0}^{\infty} a_n (x-k)^n \tag{3}$$

The final form of $g(x)$ is given as Equation (4).

$$g(x) \cong \sum\nolimits_{k=1}^{l} \alpha_k \sum\nolimits_{n=0}^{\infty} a_n (x-k)^n + \sum\nolimits_{k=1}^{l} \beta_k \sum\nolimits_{n=0}^{\infty} a_n n (x-k)^{n-1}. \tag{4}$$

After combining $\alpha_k a_n$ as a_{kn}, $\beta_k a_n$ as b_{kn} and approximating Equation (4), Equation (5) is obtained. Here, truncation of ∞ to M is performed. After truncation, the first derivative of $g(x)$ is taken and given in Equation (6).

$$g(x) \cong \sum_{k=1}^{l} \sum_{n=0}^{M} a_{kn} (x-k)^n + \sum_{k=1}^{l} \sum_{n=0}^{M} b_{kn} n (x-k)^{n-1} \tag{5}$$

$$\frac{dg(x)}{dx} \cong \sum\nolimits_{k=1}^{l} \sum\nolimits_{n=1}^{M} a_{kn} n (x-k)^{n-1} + \sum\nolimits_{k=1}^{l} \sum\nolimits_{n=1}^{M} b_{kn} n (n-1)(x-k)^{n-2}. \tag{6}$$

The expression given in Equation (7) is the definition of Caputo's fractional derivative [11]. Throughout the study, Caputo's description of the fractional derivative is employed.

$$\mathfrak{D}_x^{\gamma} g(x) = \frac{d^{\gamma} g(x)}{dx^{\gamma}} = \frac{1}{\Gamma(n-\gamma)} \int_0^x \frac{g^{(n)}(k) dk}{(x-k)^{\gamma-n+1}}, \quad (n-1 < \gamma < n). \tag{7}$$

In Equation (7), $\Gamma(1-\gamma)$ is the Gamma function, the fractional derivative is taken with respect to x in the order of γ and $g^{(\mathrm{n})}$ corresponds to the n^{th} derivative again, with respect to x. In our study, $n = 1$ is assumed and the fractional-order spans between 0 and 1. Here, two expansions are done to express $g(x)$, approximately. The first one is to express the function as the finite summation of the previous values of the function. Second, expressing the function $g(x)$ as the summation of polynomials known as Taylor Expansion assuming that $g(x)$ is a continuous and differentiable function.

Finally, the mathematical background is enough to go further in the proposed methodology. As a summary, above, we mentioned three important tools. First, a function is expressed as the summation of its previous samples. Second, Taylor expansion for a continuous and differentiable function is defined. After that, the Caputo definition of the fractional derivative is given. Now, it is time to express Deep Assessment Methodology by using fractional calculus for the modeling and prediction. Apart from above, there is an assumption that the fractional derivative $f(x)$ in the order of γ is equal to Equation (8). After this assumption, it is required to find unknown $f(x)$ which satisfies the fractional differential equation below and models the discrete dataset.

$$\frac{d^{\gamma} f(x)}{dx^{\gamma}} \cong \sum\nolimits_{k=1}^{l} \sum\nolimits_{n=1}^{\infty} a_{kn} n (x-k)^{n-1} + \sum\nolimits_{k=1}^{l} \sum\nolimits_{n=1}^{\infty} b_{kn} n (n-1)(x-k)^{n-2}, \tag{8}$$

where, $f(x)$ stands for the GDP per capita of the countries and x corresponds to the time.

Note that, in (6), allowing the order of the derivation in the left-hand side of Equation (6) to be non-integer gives a more general model [28]. This generalization is employed in Deep Assessment Methodology for $f(x)$ which stands for the GDP per capita.

Here, the motivation is to find a_{kn} and b_{kn} given in Equation (8). To find the unknowns, the differential equation needs to be solved. The strategy is as follows—first, it is required to take the Laplace transform which leads to having an algebraic equation instead of a differential equation. In other words, the Laplace transform is taken for Equation (8) to reduce the differential equation to algebraic equation, then, by using inverse Laplace transform properties, the final form of $f(x)$ is obtained as Equation (9) [11].

$$\begin{array}{l} f(x,\gamma) \cong f(0) + \sum_{k=1}^{l}\sum_{n=1}^{\infty} a_{kn}C_{kn}(x,\gamma) + \sum_{k=1}^{l}\sum_{n=1}^{\infty} b_{kn}D_{kn}(x,\gamma), \\ \text{where,} \\ \qquad C_{kn}(x,\gamma) \triangleq \frac{\Gamma(n+1)}{\Gamma(n+\gamma)}(x-k)^{n+\gamma-1} \\ \qquad D_{kn}(x,\gamma) \triangleq \frac{\Gamma(n+1)}{\Gamma(n+\gamma-1)}(x-k)^{n+\gamma-2}. \end{array} \tag{9}$$

To obtain the numerical calculation, the infinite summation of polynomials is approximated as a finite summation given in Equation (10).

$$f(x,\gamma) \cong f(0) + \sum_{k=1}^{l}\sum_{n=1}^{M} a_{kn}C_{kn}(x,\gamma) + \sum_{k=1}^{l}\sum_{n=1}^{M} b_{kn}D_{kn}(x,\gamma). \tag{10}$$

Here, $f(0)$, a_{kn} and b_{kn} are unknown coefficients that need to be determined. Note that, below, properties of the Laplace transform ($\mathcal{L}$) are given to find Equations (9) and (10) [11].

$$\mathcal{L}\left[(x-k)^{n-1}\right] = \frac{\Gamma(n)}{s^n}e^{-ks} \text{ and } \mathcal{L}\left[\frac{d^{\gamma}f(x)}{dx^{\gamma}}\right] = s^{\gamma}F(s) - s^{\gamma-1}f(0) \text{ for } 0 < \gamma < 1.$$

where, $\mathcal{L}$ stands for the Laplace transform and $\mathcal{L}[f(x)] = F(s)$.

For the numerical calculation, the infinite summation is converted into a finite summation, as given in Equation (10).

3. Our Approach

3.1. Modeling with Deep Assessment

In this part, the methodology for the modeling of the problem is given in detail. To predict the upcoming years, the problem has four regions as given in Figure 1. Dataset spans in Region 1, 2 and 3. Note that, there is no data for Region 4 where the prediction is aimed. Region 1 is called "before modeling region" which consists of historical data. Each of the coefficients $(x-k)^{n+\gamma-1}$ and derivative coming from previous values of GDP per capita for different values of k and multiplication by different weights as given in Equation (10) will add the contribution to the recent data. For modeling, the historical data is employed directly for the modeling of the data located in Region 2. Region 2 and 3 are named as modeling and testing, respectively. In the modeling region, the GDP per capita is tried to be modeled and the unknown coefficients are found. Note that, the approach uses the previous l values ($P_{i-1}, P_{i-2}, \ldots, P_{i-l}$ and corresponding $f(i-1),\ f(i-2),\ \ldots f(i-l)$) for arbitrary P_i located in Region 2. The third region consists of the data used to test for upcoming predictions. Finally, Region 4 is called the "prediction region" where the aim is to find the GDP per capita values for the time that the actual values have not known yet and implement prediction. The region division is required because there are parameters given in the previous section (Equation (10)) such as $M,\ l,\ \gamma$ which need to be found before the prediction. In Region 2, the modeling is done to find the optimum values of coefficients $a_{kn},\ M,\ l,\ \gamma$ in Equation (10) for modeling. To model the data, Least Squares Method is employed, which is explained later in this section. After that, one of the purposes of the study is achieved. This is the modeling of the data using the fractional approach. Then, the second purpose comes which is to predict the values of GDP per capita for the upcoming unknown years. In order to find optimum $M,\ l,\ \gamma$ values for the prediction, Region 3, namely testing is needed. In the region, there is an iterative solution where the real discrete data is again known. For instance, in Region 3, it is required to find $f(m_1+1)$. Then, by using the proposed method employing the fractional calculus and Least Squares Method, $f(m_1+1)$ is obtained with a minimum error by optimizing $M,\ l,\ \gamma$ values for $f(m_1+1)$ itself. Then, $f(m_1+1)$ is included the dataset for the next test which is done for $f(m_1+2)$. This continues up to $f(m)$. Then, with optimized $M,\ l,\ \gamma$, the predicted $f(m_x)$ is found in Region 4.

To model the known data, $f(x)$ representing the data optimally should be obtained. In other words, the unknowns a_{kn}, b_{kn} and $f(0)$ in Equation (10) or Equation (11) should be determined. For this, the Least Squares Method is employed.

$$f(i,\gamma) = f(0) + \sum_{k=1}^{l} \sum_{n=1}^{M} a_{kn} C_{kn}(i,\gamma) + \sum_{k=1}^{l} \sum_{n=1}^{M} b_{kn} D_{kn}(i,\gamma). \tag{11}$$

In Equation (12), the squares of total error ${\epsilon_T}^2$ is given. The main purpose of the modeling region is to minimize ${\epsilon_T}^2$ by a gradient-based approach which requires minimization of the square of the total error as the following.

$${\epsilon_T}^2 = \sum_{i=l}^{m_1} (P_i - f(i,\gamma))^2 \tag{12}$$

$$\frac{\partial {\epsilon_T}^2}{\partial f(0)} = 0,$$

$$\frac{\partial {\epsilon_T}^2}{\partial a_{rt}} = 0,$$

and

$$\frac{\partial {\epsilon_T}^2}{\partial b_{rt}} = 0.$$

where, $r = 1,\ 2,\ 3,\ \ldots l$ and $t = 1,\ 2,\ 3,\ \ldots M$.

It is better to give an example of how to obtain $\frac{\partial {\epsilon_T}^2}{\partial a_{rt}} = 0$ and $\frac{\partial {\epsilon_T}^2}{\partial b_{rt}} = 0$.

$$\frac{\partial {\epsilon_T}^2}{\partial a_{rt}} = 0 \rightarrow \frac{\partial}{\partial a_{rt}} \sum_{i=l}^{m_1} (P_i - f(i,\gamma))^2 = 0$$

Then,

$$2 \sum_{i=l}^{m_1} [P_i - f(i,\gamma)] C_{rt}(i,\gamma) = 0$$

$$\sum_{i=l}^{m_1} C_{rt}(i,\gamma) P_i = f(0) \sum_{i=l}^{m_1} C_{rt}(i,\gamma) + \sum_{i=l}^{m_1} \left\{ \sum_{k=1}^{l} \sum_{n=1}^{M} a_{kn} C_{kn}(i,\gamma) \right\} C_{rt}(i,\gamma)$$

The same procedure is followed for $\frac{\partial {\epsilon_T}^2}{\partial b_{rt}} = 0$.

$$\frac{\partial {\epsilon_T}^2}{\partial b_{rt}} = 0 \rightarrow \frac{\partial}{\partial b_{rt}} \sum_{i=l}^{m_1} (P_i - f(i,\gamma))^2 = 0$$

Then,

$$\sum_{i=l}^{m_1} [P_i - f(i,\gamma)] D_{rt}(i,\gamma) = 0$$

$$\sum_{i=l}^{m_1} D_{rt}(i,\gamma) P_i = f(0) \sum_{i=l}^{m_1} D_{rt}(i,\gamma) + \sum_{i=l}^{m_1} \left\{ \sum_{k=1}^{l} \sum_{n=1}^{M} a_{kn} D_{kn}(i,\gamma) \right\} D_{rt}(i,\gamma)$$

This leads to having a system of linear algebraic equations (SLAE) as given in (13).

$$[A].[B] = [C] \tag{13}$$

where, $[A]$, $[B]$ and $[C]$ is shown in Equations (14), (15) and (16), respectively.

To find $f(x)$ the continuous curve modeling with a minimum error, the optimum fractional-order γ is inquired between (0, 1). Then, with optimum fractional-order γ, the unknown coefficients are determined. In the study, the GDP per capita of Brazil, China, the European Union, India, Italy, Japan, the UK, the USA, Spain and Turkey were used from 1960 until 2018 [38]. The dataset is shown in Tables A1 and A2.

Among them, the year 2018 is in Region 3 as testing to predict for the next years.

Here,

t (years): [1960, 1961, ..., 2018]

i (points): [1, 2, ..., 59]

P_i (value of i): $[P_1, P_2, \ldots, P_{59}]$

P_i: It shows the actual GDP per capita of each country in each i^{th} year. For example, P_2 is the GDP per capita of the country in 1961.

i: It stands for the number for each year. For example, $i = 1$ for 1960, $i = 3$ for 1962 and $i = 59$ for 2018.

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \tag{14}$$

A matrix consists of the matrix set below, where,

$C_{kn}(x, \gamma) = C_{kn}$ and $D_{kn}(x, \gamma) = D_{kn}$

$$A_{1,1} = \begin{bmatrix} m_1 - l + 1 & \sum_{i=l}^{m_1} C_{11} & \cdots & \sum_{i=l}^{m_1} C_{1M} & \sum_{i=l}^{m_1} C_{21} & \cdots & \sum_{i=l}^{m_1} C_{2M} & \cdots & \sum_{i=l}^{m_1} C_{l1} & \cdots & \sum_{i=l}^{m_1} C_{lM} \\ \sum C_{11} & \sum_{i=l}^{m_1} C_{11}C_{11} & \cdots & \sum_{i=l}^{m_1} C_{1M}C_{11} & \sum_{i=l}^{m_1} C_{21}C_{11} & \cdots & \sum_{i=l}^{m_1} C_{2M}C_{11} & \cdots & \sum_{i=l}^{m_1} C_{l1}C_{11} & \cdots & \sum_{i=l}^{m_1} C_{lM}C_{11} \\ \sum C_{12} & \sum_{i=l}^{m_1} C_{11}C_{12} & \cdots & \sum_{i=l}^{m_1} C_{1M}C_{12} & \sum_{i=l}^{m_1} C_{21}C_{12} & \cdots & \sum_{i=l}^{m_1} C_{2M}C_{12} & \cdots & \sum_{i=l}^{m_1} C_{11}C_{12} & \cdots & \sum_{i=l}^{m_1} C_{lM}C_{12} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum C_{lm} & \sum_{i=l}^{m_1} C_{11}C_{lM} & \cdots & \sum_{i=l}^{m_1} C_{1M}C_{lM} & \sum_{i=l}^{m_1} C_{21}C_{lM} & \cdots & \sum_{i=l}^{m_1} C_{2M}C_{lM} & \cdots & \sum_{i=l}^{m_1} C_{l1}C_{lM} & \cdots & \sum_{i=l}^{m_1} C_{lM}C_{lM} \end{bmatrix}$$

$$A_{2,1} = \begin{bmatrix} \sum D_{11} & \sum_{i=l}^{m_1} C_{11}D_{11} & \cdots & \sum_{i=l}^{m_1} C_{1M}D_{11} & \sum_{i=l}^{m_1} C_{21}D_{11} & \cdots & \sum_{i=l}^{m_1} C_{2M}D_{11} & \cdots & \sum_{i=l}^{m_1} C_{l1}D_{11} & \cdots & \sum_{i=l}^{m_1} C_{lM}D_{11} \\ \sum_{i=l}^{m_1} D_{12} & \sum_{i=l}^{m_1} C_{11}D_{12} & \cdots & \sum_{i=l}^{m_1} C_{1M}D_{12} & \sum_{i=l}^{m_1} C_{21}D_{12} & \cdots & \sum_{i=l}^{m_1} C_{2M}D_{12} & \cdots & \sum_{i=l}^{m_1} C_{l1}D_{12} & \cdots & \sum_{i=l}^{m_1} C_{lM}D_{12} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{i=l}^{m_1} D_{lM} & \sum_{i=l}^{m_1} C_{11}D_{lM} & \cdots & \sum_{i=l}^{m_1} C_{1M}D_{lM} & \sum_{i=l}^{m_1} C_{21}D_{lM} & \cdots & \sum_{i=l}^{m_1} C_{2M}D_{lM} & \cdots & \sum_{i=l}^{m_1} C_{l1}D_{lM} & \vdots & \sum_{i=l}^{m_1} C_{lM}D_{lM} \end{bmatrix}$$

$$A_{1,2} = \begin{bmatrix} \sum_{i=l}^{m_1} D_{11} & \cdots & \sum_{i=l}^{m_1} D_{1M} & \cdots & \sum_{i=l}^{m_1} D_{lM} \\ \sum_{i=l}^{m_1} D_{11}C_{11} & \cdots & \sum_{i=l}^{m_1} D_{1M}C_{11} & \cdots & \sum_{i=l}^{m_1} D_{lM}C_{11} \\ \sum_{i=l}^{m_1} D_{11}C_{12} & \cdots & \sum_{i=l}^{m_1} D_{1M}C_{12} & \cdots & \sum_{i=l}^{m_1} D_{lM}C_{12} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{i=l}^{m_1} D_{11}C_{lM} & \cdots & \sum_{i=l}^{m_1} D_{1M}C_{lM} & \cdots & \sum_{i=l}^{m_1} D_{lM}C_{lM} \end{bmatrix}$$

$$A_{2,2} = \begin{bmatrix} \sum_{i=l}^{m_1} D_{11}D_{11} & \cdots & \sum_{i=l}^{m_1} D_{1M}D_{11} & \cdots & \sum_{i=l}^{m_1} D_{lM}D_{11} \\ \sum_{i=l}^{m_1} D_{11}D_{12} & \cdots & \sum_{i=l}^{m_1} D_{1M}D_{12} & \cdots & \sum_{i=l}^{m_1} D_{lM}D_{12} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{i=l}^{m_1} D_{11}D_{lM} & \cdots & \sum_{i=l}^{m_1} D_{1M}D_{lM} & \cdots & \sum_{i=l}^{m_1} D_{lM}D_{lM} \end{bmatrix}$$

$$[B] = \begin{bmatrix} f(0) & a_{11} & a_{12} & \ldots & a_{1M} & a_{21} & a_{22} & \ldots & a_{2M} & \ldots a_{l1} \ldots & a_{lM} & b_{11} & b_{12} & \cdots & b_{1M} & b_{21} & \ldots & b_{2M} & \ldots & b_{l1} & b_{l2} & \ldots & b_{lM} \end{bmatrix}^T \tag{15}$$

$$[C] = \left[\sum_{i=l}^{m_1} P_i \quad \sum_{i=l}^{m_1} P_i C_{11} \quad \sum_{i=l}^{m_1} P_i C_{12} \quad \ldots \quad \sum_{i=l}^{m_1} P_i C_{lM} \quad \sum_{i=l}^{m_1} P_i D_{11} \quad \sum_{i=l}^{m_1} P_i D_{12} \quad \ldots \quad \sum_{i=l}^{m_1} P_i D_{lM} \right]^T \tag{16}$$

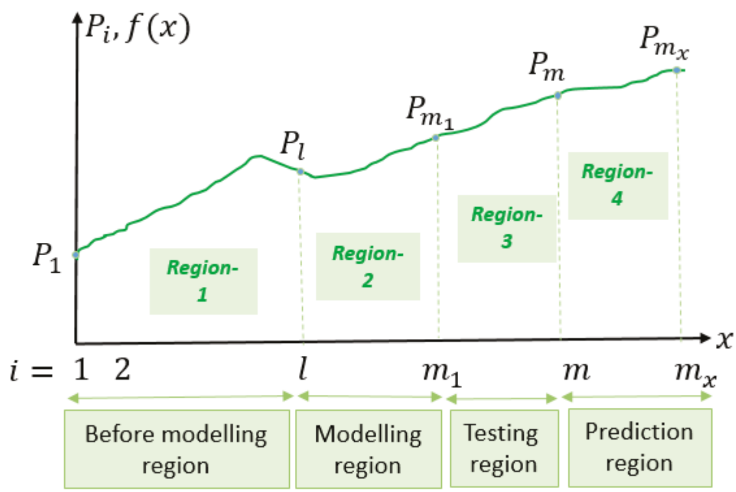


Figure 1. The regions of the dataset.

3.2. Prediction with Deep Assessment

To find the optimized values of the unknowns for the prediction, the testing region (3rd region) is required. The predictions obtained in the test region $(m_1 < i < m)$ are also given in Table 1. For testing, the data up to $m_1 = 58$ have been taken into consideration in the operations. The $f(m_1 + 1)$ value was found from the obtained modeling. Then, the value is kept, and the next step was started again for $(f(m_1 + 2))$. These operations are done until the last value of the test zone. In our case, $m = 59$.

Table 1. Comparison of modeling results (γ, M and MAPE values) of countries for $l = 10$.

Country	γ Deep Assessment	γ Fractional Model-1	Deep Assessment * ($l<i<m$)	Polynomial Model * ($l<i<m$)	Fractional Model-1 * ($l<i<m$)	M
US	0.44	0.54	0.81%	1.01%	1.06%	15
UK	0.14	0.85	5.38%	7.03%	6.61%	15
Brazil	0.06	0.58	7.26%	7.13%	9.00%	17
China	0.03	0.95	2.84%	5.62%	5.67%	11
India	0.15	0.02	3.09%	2.51%	4.10%	16
Japan	0.26	0.69	4.45%	4.64%	5.82%	20
EU	0.06	0.89	4.02%	3.41%	5.71%	20
Italy	0.39	1	4.70%	8.81%	8.81%	9
Spain	0.22	0.58	4.44%	6.49%	6.36%	13
Turkey	0.71	0.01	6.09%	11.81%	8.93%	10

* $MAPE_{Modeling}$ values.

The last region is called the "Prediction Region." Here, using Region 1, 2 and 3, the prediction for the upcoming years is obtained. After having modeled and tested regions, the unknowns in Equation (11) have already found in an optimal manner. After testing, Region 4 is started. In the region, the first prediction $f(m + 1)$ is found by using the coefficients and unknowns found by the testing region. After that, the first predicted value ($f(m+1)$) is included in Region 3 (testing) for the consecutive prediction $f(m+2)$. This procedure is reiterated and recycled up to $f(m_x)$.

The prediction results for 2019 are given in Table 2. For example, as of the end of 2019 ($f(m+2)$), Brazil, China, European Union, India, Italy, Japan, the UK, the USA, Spain and Turkey's GDP per capita values are expected as listed.

Table 2. Test ($m_1 < i < m$) results (γ, l, M and MAPE) of GDP per capita for corresponding countries.

Country	γ	l	M	γ Interpolation	Deep Assessment *	Deep Learning *
Brazil	0.18	24	3	0.32	0.1303%	0.4728%
China	0.97	11	3	0.5	0.7147%	1.6365%
India	0.96	3	2	0.99	0.3379% 5	0.7203%
Italy	0.43	20	4	0.43	0.1048%	3.0796%
Japan	0.57	4	3	1	0.3499%	1.1091%
Spain	0.99	2	3	0.99	0.0560%	1.5683%
Turkey	0.39	17	4	0.39	0.1167%	2.3691%
EU	0.32	20	5	0.22	0.1044%	0.2522%
US	0.39	25	2	0.18	0.1081%	0.8424%
UK	0.18	18	7	0.05	0.9129%	3.0508%

* $MAPE_{Prediction}$ values.

In Figure 2, the algorithm for prediction with DAM is illustrated. The first step of the algorithm is to initialize the parameters (l, M, x_1, x_2, $\ldots x_m$ and P_1, P_2, $\ldots$ P_m). Then, the counter variable N is introduced, which counts the number of prediction steps. The total number of required predicted steps is denoted as n_0. As an initial value, the fractional-order γ is assigned 0 and the increment is 0.01 for each loop to find the optimized value. For each value of γ between 0 and 1, matrix A given as Equation (14) is created and then, the unknown coefficients given in Equation (10) are calculated. After that, using the actual data in Region 1 and Region 2, the modeling of data between P_l and P_m is actualized for Region 2. Then, the error defined in Equation (12) is calculated. The value of the error is analyzed and compared to previously obtained values. If it is smaller than the previous one, the corresponding fractional-order value is memorized. At the end of Loop II, the optimal value of the fractional-order, which coincides with the optimal modeling is found and corresponding coefficients given in Equation (10) is determined. Then, the prediction for the next forthcoming value is made with Equation (10). After that, all the procedures starting from the increment of N is repeated so that the previously predicted value is added to the initial data for the next step prediction. This process is repeated up to the termination of Loop I. Finally, n_0 the number of predictions is obtained. Keep in mind that, for the parameters l and M, there exist two loops starting from 1 to L_0 and 1 to M_0 searching the optimum values of the parameters in order to get the outcomes with a minimum error for the testing region, respectively. Here, L_0 and M_0 are pre-defined some constant values.

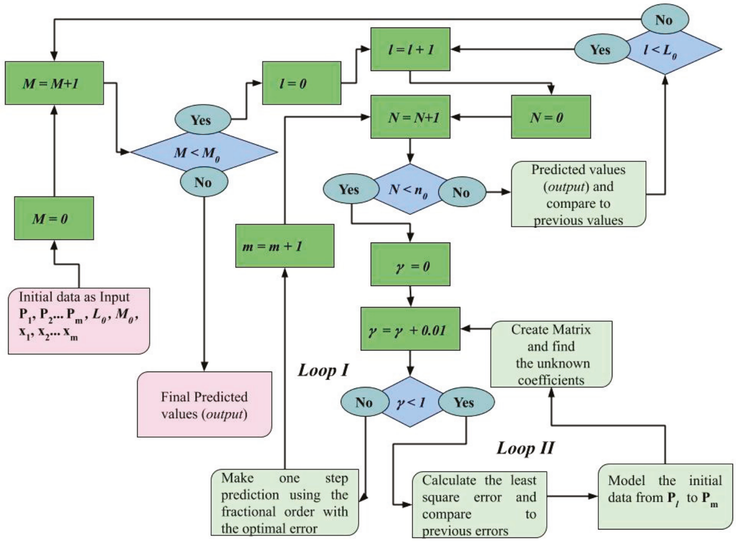


Figure 2. The algorithm for the prediction.

3.3. Long Short-Term Memory

In our study, we compare the modeling with the polynomial curve fitting method and in the prediction, we compare Deep Assessment with the LSTM method. Conventional neural networks are insufficient for modeling the content of temporal data. Recursive neural networks (RNN) model the sequential structure of data by feeding itself with the output of the previous time step. LSTMs are special types of RNNs that operate over sequences and are used in time series analysis [39]. An LSTM cell has four gates: input, forget, output and gate. With these gates, LSTMs optionally inherit the information from the previous time steps. Forget gate (f), input gate (i) and output gate (o) are sigmoid functions (σ) and they take values between 0 and 1. Gate g has hyperbolic tangent $(tanh)$ activation and is between -1 and 1. The Gate and forward propagation equations are listed below as Equations (17)–(22). Here c_t^l and h_t^l refer to cell state and hidden state of layer l at time step t, respectively. Each gate takes input from the previous time step (h_{t-1}^l) and previous layer (h_t^{l-1}) and has its own set of learnable parameters W's and b's.

$$f_t = \sigma\Big(W_f[h_{t-1}^l, h_t^{l-1}] + b_f\Big) \tag{17}$$

$$i_t = \sigma\Big(W_i[h_{t-1}^l, h_t^{l-1}] + b_i\Big) \tag{18}$$

$$o_t = \sigma\Big(W_o[h_{t-1}^l, h_t^{l-1}] + b_o\Big) \tag{19}$$

$$g_t = tanh\Big(W_g[h_{t-1}^l, h_t^{l-1}] + b_g\Big) \tag{20}$$

$$c_t^l = f \odot c_{t-1}^l + i \odot g \tag{21}$$

$$h_t^l = o \odot \tanh\Big(c_t^l\Big) \tag{22}$$

Here, $\odot$ is the Hadamard product. Each LSTM neuron in a network may consist of one or more cells. In every time step, every cell updates its own cell state, c_t^l. Equation (22) describes how these cells get updated with forget gate and input gate; f gate decides how much of previous cell state that cell should remember while i gate decides how much it should consider the new input from the previous layer. Then, LSTM neuron updates its internal hidden state by multiplying output and squashed version of

c_t^l. An LSTM neuron gives outputs only in hidden state information to another LSTM neuron. Gate o and c_t are used internally in the computation of forward time steps [40]. To forecast time series and compare our proposed approach to neural networks, we employed a stacked LSTM model with 2 layers of LSTMs (each having 50 hidden units) and a linear prediction layer. LSTM model is trained with the Adam optimizer [40].

4. Numerical Results

In this section, we report the modeling and prediction performance of the Deep Assessment methodology. Further, we compare the proposed method to other modeling and prediction approaches such as Polynomial Model, Fractional Model-1 [34,35] and LSTM. In this section, results are reported with the Mean Average Precision Error (MAPE) metric and calculated as follows:

$$MAPE = \frac{1}{k}\sum_{i=1}^{k}\left|\frac{v(i) - \tilde{\mathrm{v}}(i)}{v(i)}\right| \times 100, \tag{23}$$

where k is the total number of samples, $v(i)$ is the actual value and $\tilde{\mathrm{v}}(i)$ is the predicted value for i^{th} sample.

Before presenting the results, it is important to highlight that for modeling, M_0 and l_0 are taken 20 and 10, respectively whereas for prediction, M_0 and l_0 are taken 8 and 25, respectively. The number of prediction, n_0 is equal to 1.

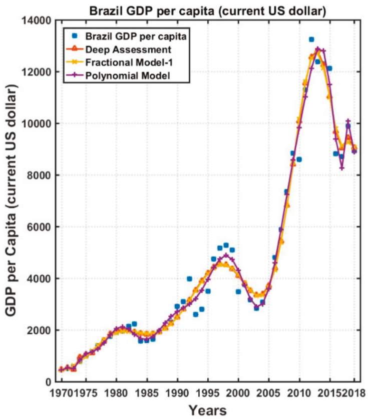
4.1. Modeling Results

In this part, we compare the modeling performance with Polynomial, Fractional Model-1 and Deep Assessment models.

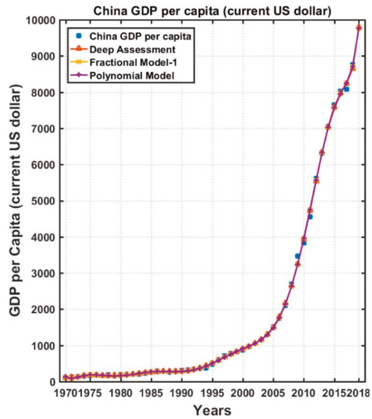
To achieve modeling, l value needs to be investigated. For the modeling of the GDP per capita of each country, the required previous data l of past years used in the algorithm differs after optimization. In order to make a fair evaluation, l value is fixed among all countries to 10. Modeling results for Deep Assessment, Polynomial Model and Fractional Model-1 are shown in Table 1. Optimized M values after processing can be seen in the last column. The Deep Assessment model has a %4.308 average MAPE and outperforms Polynomial and Fractional Model-1 by %1.538 and %1.899 average error rates. All three methods model the US best with %0.81, %1.01 and %1.06 error. Further, in the case of Italy, Fractional Model-1 uses the fractional-order value of 1 and produces %8.81 MAPE, equal to the Polynomial method as expected because for the fractional-order value of 1 is the same with the Polynomial Method. However, DAM yields fractional order of 0.39, decreasing the error to 4.70%, justifying the advantage of employing fractional calculus and previous values of the data itself.

$$MAPE_{Modeling} = \frac{1}{m-l+1}\sum_{i=l}^{m}\left|\frac{P(i) - f(i,\gamma)}{P(i)}\right| \times 100. \tag{24}$$

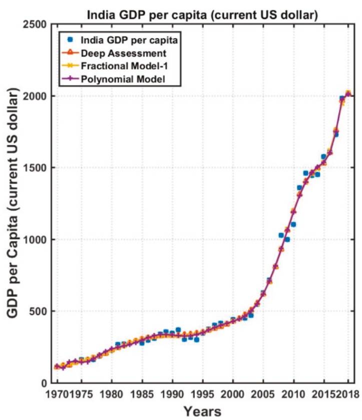
GDP per capita data, Deep Assessment, Polynomial and Fractional Model-1 modeling results are shown in Figure 1 for each country. One can conclude that when data points have high variance all models produce high error rates, as in Turkey and Italy. For Japan and Brazil, DAM (Deep Assessment Method) and Polynomial models produce similar results. Also, it can be seen from the Figure 3, both Deep Assessment and Fractional Model-1 have a low bias when compared to the Polynomial model and overfits to dataset less. This is possible because of the memory property of the proposed approach. Except for Brazil, India and the EU, the proposed method yields superior results compared to other models.



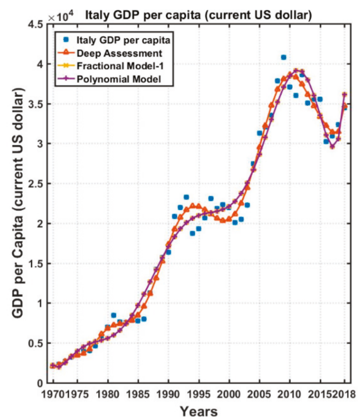
(**a**): Modelling results of Brazil).



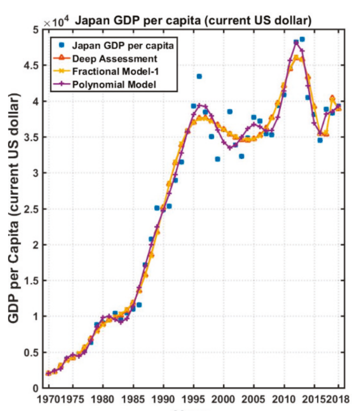
(**b**): Modelling results of China.



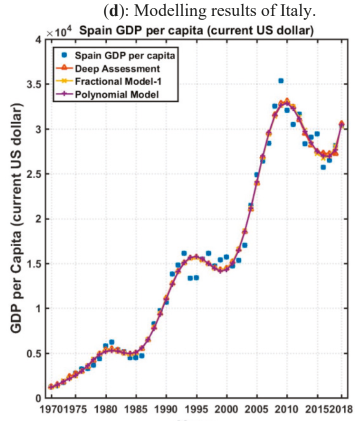
(**c**): Modelling results of India.



(**d**): Modelling results of Italy.

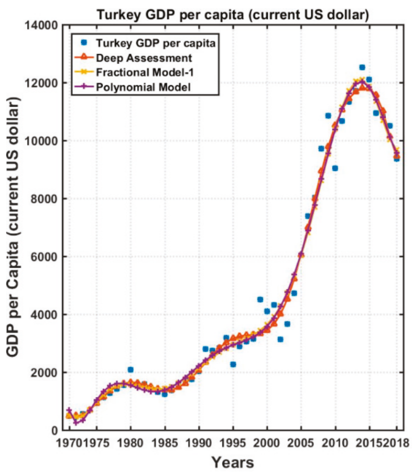


(**e**): Modelling results of Japan .

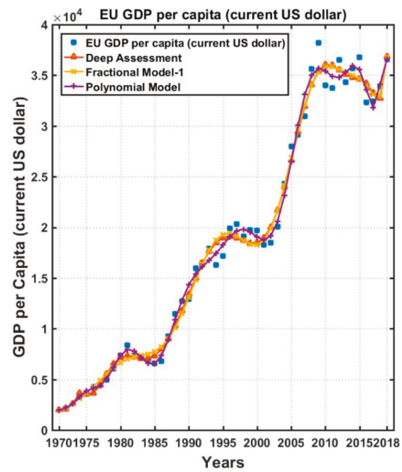


(**f**): Modelling results of Spain.

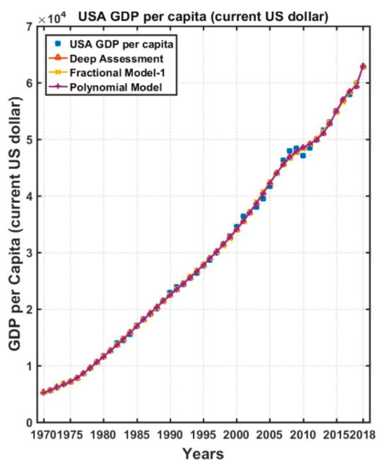
Figure 3. *Cont.*



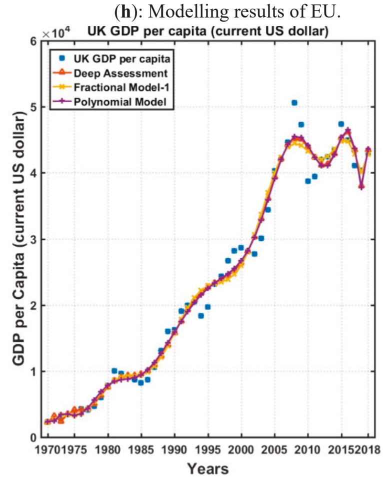
(**g**): Modelling results of Turkey .



(**h**): Modelling results of EU.



(**i**): Modelling results of US.



(**j**): Modelling results of UK.

Figure 3. Modelling results of the countries (Brazil, China, India, Italy, Japan, the UK, the USA, Spain and Turkey) and the European Union or Deep Assessment (Blue), Fractional Model 1 (Yellow), Polynomial Model (Purple).

4.2. Prediction Results

In this section, we compare the accuracy rate of the prediction of Deep Assessment and Deep Learning models. As in modeling, the GDP per capita dataset is used to assess the performance of the proposed method. Table 2 illustrates optimized γ**,** l, M values and the corresponding performance of DAM and LSTM. Here, column 6 reports the performance of DAM while column 7 represents LSTM. Column 5 shows that the Deep Assessment methodology predicts GDP per capita with an average 0.29% error with predicting all countries with 1.< (less than 1 percent) of error. The best-predicted country is Spain while UK's prediction is the least accurate with 0.91% error. On the other hand, LSTM yields 1.51% error on average. For both DAM and LSTM, UK yields the highest error. Table 2 demonstrates that in the implemented setting, DAM outperforms LSTM by 1.21% average error and produces fair results.

$$MAPE_{Prediction} = \frac{1}{m_1 - l + 1} \sum_{i=m_1}^{m} \left| \frac{P(i) - f(i,\gamma)}{P(i)} \right| \times 100. \quad (25)$$

Table 3 reports the prediction of GDP per capita for the year 2019 is illustrated in Table 2 for both DAM and LSTM methods. For countries Brazil, China, India, Turkey, the UK and the US, predictions obtained by the two models are similar. On the other hand, Italy and Spain yield different results.

Table 3. GDP per Capita Prediction of Countries for 2019 (US dollars).

Country	Deep Assessment	Deep Learning
Brazil	7932	8013
China	10,312	10,273
India	2154	1967
Italy	39,028	35,141
Japan	34,421	37,994
Spain	30,385	35,372
Turkey	8260	8920
US	65,767	63,844
UK	44,897	44,702
EU	40,487	36,487

5. Conclusions

In this study, a model called "Deep Assessment" is introduced which employs Fractional Calculus to model discrete data as the summation of previous values and derivatives. Different to the literature and our previous work, the proposed approach also predicts the incoming values of the discrete data in addition to modeling. The method is evaluated on modeling and predicting GDP per capita, using a dataset including the period of 1960–2018 for nine countries (Brazil, China, European Union, India, Italy, Japan, UK, the USA, Spain and Turkey) and the European Union. Using the fractional differential equation and the summation of previous values for the modeling of GDP per capita at a specific time instant bring non-locality, memory and generalization of the problem for different fractional order. In experiments, first, GDP per capita is modeled. The Deep Assessment model has a 4.308% average MAPE and outperforms Polynomial and Fractional Model-1 by 1.538% and 1.899% average error rates for modeling. For prediction, LSTM, a special type of neural network is used to assess the performance of the model. In the selected test region, it is shown that Deep Assessment is superior to LSTM by 1.51% average error. Results illustrate that the proposed method yields promising results and demonstrates the benefits of combining fractional calculus and differential equations. Evaluation of multivariable and multifunctional problems, analyzing time windows, randomness, noise and error changes are left to future work.

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Appendix A

Table A1. GDP per capita (US dollars) values of the countries.

i	Years	Brazil	China	EU	India	Italy
1	1960	210.1099	89.52054	890.4056	82.1886	804.4926
2	1961	205.0408	75.80584	959.71	85.3543	887.3367
3	1962	260.4257	70.90941	1037.326	89.88176	990.2602
4	1963	292.2521	74.31364	1135.194	101.1264	1126.019
5	1964	261.6666	85.49856	1245.499	115.5375	1222.545
6	1965	261.3544	98.48678	1346.058	119.3189	1304.454
7	1966	315.7972	104.3246	1448.551	89.99731	1402.442
8	1967	347.4931	96.58953	1546.804	96.33914	1533.693
9	1968	374.7868	91.47272	1602.06	99.87596	1651.939
10	1969	403.8843	100.1299	1762.472	107.6223	1813.388
11	1970	445.0231	113.163	1950.732	112.4345	2106.864
12	1971	504.7495	118.6546	2195.145	118.6032	2305.61
13	1972	586.2144	131.8836	2611.729	122.9819	2671.137
14	1973	775.2733	157.0904	3296.935	143.7787	3205.252
15	1974	1004.105	160.1401	3685.596	163.4781	3621.146
16	1975	1153.831	178.3418	4274.046	158.0362	4106.994
17	1976	1390.625	165.4055	4406.238	161.0921	4033.099
18	1977	1567.006	185.4228	4968.988	186.2135	4603.6
19	1978	1744.257	156.3964	6064.883	205.6934	5610.498
20	1979	1908.488	183.9832	7377.165	224.001	6990.286
21	1980	1947.276	194.8047	8384.718	266.5778	8456.919
22	1981	2132.883	197.0715	7391.077	270.4706	7622.833
23	1982	2226.767	203.3349	7093.702	274.1113	7556.523
24	1983	1570.54	225.4319	6859.966	291.2381	7832.575
25	1984	1578.926	250.714	6572.019	276.668	7739.715
26	1985	1648.082	294.4588	6775.647	296.4352	7990.687
27	1986	1941.491	281.9281	9265.924	310.4659	11,315.02
28	1987	2087.308	251.812	11,432.23	340.4168	14,234.73
29	1988	2300.377	283.5377	12,711.96	354.1493	15,744.66
30	1989	2908.496	310.8819	12,936.46	346.1129	16,386.66
31	1990	3100.28	317.8847	15,989.22	367.5566	20,825.78
32	1991	3975.39	333.1421	16,496.51	303.0556	21,956.53
33	1992	2596.92	366.4607	17,919.02	316.9539	23,243.47
34	1993	2791.209	377.3898	16,256.42	301.159	18,738.76
35	1994	3500.611	473.4923	17,194.12	346.103	19,337.63
36	1995	4748.216	609.6567	19,898.44	373.7665	20,664.55
37	1996	5166.164	709.4138	20,295.17	399.9501	23,081.6
38	1997	5282.009	781.7442	19,121.21	415.4938	21,829.35
39	1998	5087.152	828.5805	19,763.51	413.2989	22,318.14
40	1999	3478.373	873.2871	19,698.89	441.9988	21,997.62
41	2000	3749.753	959.3725	18,261.97	443.3142	20,087.59
42	2001	3156.799	1053.108	18,457.89	451.573	20,483.22
43	2002	2829.283	1148.508	20,055.33	470.9868	22,270.14
44	2003	3070.91	1288.643	24,310.25	546.7266	27,465.68
45	2004	3637.462	1508.668	27,960.05	627.7742	31,259.72
46	2005	4790.437	1753.418	29,115.63	714.861	32,043.14
47	2006	5886.464	2099.229	30,960.56	806.7533	33,501.66
48	2007	7348.031	2693.97	35,630.94	1028.335	37,822.67
49	2008	8831.023	3468.304	38,185.62	998.5223	40,778.34
50	2009	8597.915	3832.236	34,019.28	1101.961	37,079.76
51	2010	11,286.24	4550.454	33,740.65	1357.564	36,000.52
52	2011	13,245.61	5618.132	36,506.64	1458.104	38,599.06
53	2012	12,370.02	6316.919	34,328.82	1443.88	35,053.53
54	2013	12,300.32	7050.646	35,683.86	1449.606	35,549.97
55	2014	12,112.59	7651.366	36,787.23	1573.881	35,518.42
56	2015	8814.001	8033.388	32,319.45	1605.605	30,230.23
57	2016	8712.887	8078.79	32,425.13	1729.268	30,936.13
58	2017	9880.947	8759.042	33,908	1981.269	32,326.84
59	2018	8920.762	9770.847	36,569.73	2009.979	34,483.2

Table A2. GDP per capita (US dollars) values of the countries.

i	Years	Japan	Spain	UK	US	Turkey
1	1960	478.9953	396.3923	1397.595	3007.123	509.4239
2	1961	563.5868	450.0533	1472.386	3066.563	283.8283
3	1962	633.6403	520.2061	1525.776	3243.843	309.4467
4	1963	717.8669	609.4874	1613.457	3374.515	350.6629
5	1964	835.6573	675.2416	1748.288	3573.941	369.5834
6	1965	919.7767	774.7616	1873.568	3827.527	386.3581
7	1966	1058.504	889.6599	1986.747	4146.317	444.5494
8	1967	1228.909	968.3068	2058.782	4336.427	481.6937
9	1968	1450.62	950.5457	1951.759	4695.923	526.2135
10	1969	1669.098	1077.679	2100.668	5032.145	571.6178
11	1970	2037.56	1212.289	2347.544	5234.297	489.9303
12	1971	2272.078	1362.166	2649.802	5609.383	455.1049
13	1972	2967.042	1708.809	3030.433	6094.018	558.421
14	1973	3997.841	2247.553	3426.276	6726.359	686.4899
15	1974	4353.824	2749.925	3665.863	7225.691	927.7991
16	1975	4659.12	3209.837	4299.746	7801.457	1136.375
17	1976	5197.807	3279.313	4138.168	8592.254	1275.956
18	1977	6335.788	3627.591	4681.44	9452.577	1427.372
19	1978	8821.843	4356.439	5976.938	10,564.95	1549.644
20	1979	9105.136	5770.215	7804.762	11,674.19	2079.22
21	1980	9465.38	6208.578	10,032.06	12,574.79	1564.247
22	1981	10,361.32	5371.166	9599.306	13,976.11	1579.074
23	1982	9578.114	5159.709	9146.077	14,433.79	1402.406
24	1983	10,425.41	4478.5	8691.519	15,543.89	1310.256
25	1984	10,984.87	4489.989	8179.194	17,121.23	1246.825
26	1985	11,584.65	4699.656	8652.217	18,236.83	1368.401
27	1986	17,111.85	6513.503	10,611.11	19,071.23	1510.677
28	1987	20,745.25	8239.614	13,118.59	20,038.94	1705.895
29	1988	25,051.85	9703.124	15,987.17	21,417.01	1745.365
30	1989	24,813.3	10,681.97	16,239.28	22,857.15	2021.859
31	1990	25,359.35	13,804.88	19,095.47	23,888.6	2794.35
32	1991	28,925.04	14,811.9	19,900.73	24,342.26	2735.708
33	1992	31,464.55	16,112.19	20,487.17	25,418.99	2842.37
34	1993	35,765.91	13,339.91	18,389.02	26,387.29	3180.188
35	1994	39,268.57	13,415.29	19,709.24	27,694.85	2270.338
36	1995	43,440.37	15,471.96	23,123.18	28,690.88	2897.866
37	1996	38,436.93	16,109.08	24,332.7	29,967.71	3053.947
38	1997	35,021.72	14,730.8	26,734.56	31,459.14	3144.386
39	1998	31,902.77	15,394.35	28,214.27	32,853.68	4496.497
40	1999	36,026.56	15,715.33	28,669.54	34,513.56	4108.123
41	2000	38,532.04	14,713.07	28,149.87	36,334.91	4316.549
42	2001	33,846.47	15,355.7	27,744.51	37,133.24	3119.566
43	2002	32,289.35	17,025.53	30,056.59	38,023.16	3659.94
44	2003	34,808.39	21,463.44	34,419.15	39,496.49	4718.2
45	2004	37,688.72	24,861.28	40,290.31	41,712.8	6040.608
46	2005	37,217.65	26,419.3	42,030.29	44,114.75	7384.252
47	2006	35,433.99	28,365.31	44,599.7	46,298.73	8035.377
48	2007	35,275.23	32,549.97	50,566.83	47,975.97	9711.874
49	2008	39,339.3	35,366.26	47,287	48,382.56	10,854.17
50	2009	40,855.18	32,042.47	38,713.14	47,099.98	9038.52
51	2010	44,507.68	30,502.72	39,435.84	48,466.82	10,672.39
52	2011	48,168	31,636.45	42,038.5	49,883.11	11,335.51
53	2012	48,603.48	28,324.43	42,462.71	51,603.5	11,707.26
54	2013	40,454.45	29,059.55	43,444.56	53,106.91	12,519.39
55	2014	38,109.41	29,461.55	47,417.64	55,032.96	12,095.85
56	2015	34,524.47	25,732.02	44,966.1	56,803.47	10,948.72
57	2016	38,794.33	26,505.62	41,074.17	57,904.2	10,820.63
58	2017	38,331.98	28,100.85	40,361.42	59,927.93	10,513.65
59	2018	39,289.96	30,370.89	42,943.9	62,794.59	9370.176

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