Mathematics for Elementary Teachers

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Contents

Part VIII. Voyaging on Hōkūle`a

This book will help you to understand elementary mathematics more deeply, gain facility with creating and using mathematical notation, develop a habit of looking for reasons and creating mathematical explanations, and become more comfortable exploring unfamiliar mathematical situations.

The primary goal of this book is to help you learn to think like a mathematician in some very specific ways. You will:

- **Make sense of problems and persevere in solving them.** You will develop and demonstrate this skill by working on difficult problems, making incremental progress, and revising solutions to problems as you learn more.
- **Reason abstractly and quantitatively.** You will demonstrate this skill by learning to represent situations using mathematical notation (abstraction) as well as creating and testing examples (making situations more concrete).
- **Construct viable arguments and critique the reasoning of others.** You will be expected to create both written and verbal explanations for your solutions to problems. The most important questions in this class are "*Why*?" and "*How do you know you're right*?" Practice asking these questions of yourself, of your professor, and of your fellow students.
- **Model with mathematics.** You will demonstrate this skill by inventing mathematical notation and drawings to represent physical situations and solve problems.
- **Use appropriate tools strategically.** You will be expected to use computers, calculators, measuring devices, and other mathematical tools when they are helpful.
- **Attend to precision.** You will write and express mathematical ideas clearly, using mathematical terms properly, providing clear definitions and descriptions of your ideas, and distinguishing between similar ideas (for example "factor" versus "multiple".)

• **Look for and make use of mathematical structure.** You will find, describe, and most importantly explain patterns that come up in various situations including problems, tables of numbers, and algebraic expressions.

• **Look for and express regularity in repeated reasoning.** You will demonstrate this skill by recognizing (and expressing) when calculations or ideas are repeated, and how that can be used to draw mathematical conclusions (for example why a decimal must repeat) or develop shortcuts to calculations.

Throughout the book, you will **learn how to learn** mathematics on you own by reading, working on problems, and making sense of new ideas on your own and in collaboration with other students in the class.

This book was developed at the University of Hawai`i at Mānoa for the Math 111 and 112 (Mathematics for Elementary Teachers I and II) courses. The materials were written by Prof. Michelle Manes with tremendous assistance from lots of people.

I owe a huge debt to Dr. Tristan Holmes, who has taught the courses for years and assisted greatly on the revision and current format of the textbook. I also thank the graduate students who helped to design and develop the original iBook version of these materials: Amy Brandenberg, Jon Brown, Jessica Delgado, Paul Nguyen, Geoff Patterson, and especially Ryan Felix. Thanks to Monique Chyba, PI of the SUPER-M project (NSF grant DGE-0841223), for supporting this work, and to the UH Mānoa College of Natural Sciences and College of Education for their support as well.

Thanks also to the hundreds of Math 111 and 112 students I've taught over the past ten years. Your enthusiasm, energy, joy, and humor is what keeps me going.

I am grateful to all of my colleagues and professors, past and present, from whom I have learned so much about mathematics and

about education. Special thanks to Dr. Carol Findell and Dr. Suzanne Chapin at Boston University, who gave me an entirely new perspective on mathematics teaching and learning.

I can never thank Dr. Al Cuoco enough for his support and intellectual leadership. I owe him more than I can say.

Unless otherwise noted, images were created by Michelle Manes using LaTeX, Mathematica, or Geometer's Sketchpad.

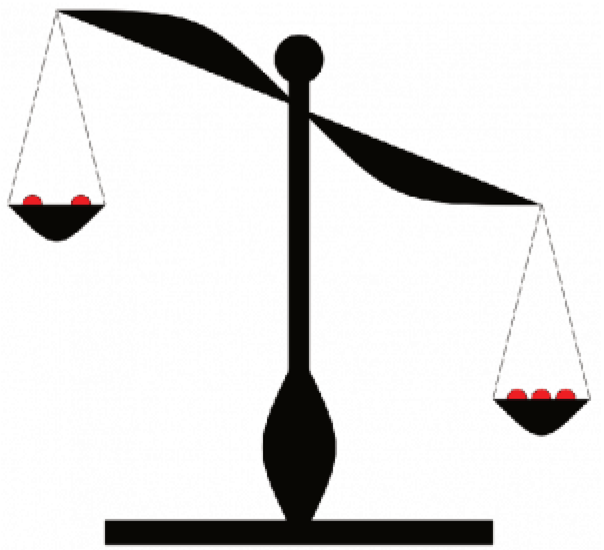
Michelle Manes

Honolulu, HI

December, 2017

PART I

PROBLEM SOLVING



Solving a problem for which you know there's an answer is like climbing a mountain with a guide, along a trail someone else has laid. In mathematics, the truth is somewhere out there in a place no one knows, beyond all the beaten paths.

– Yoko Ogawa

1. Introduction

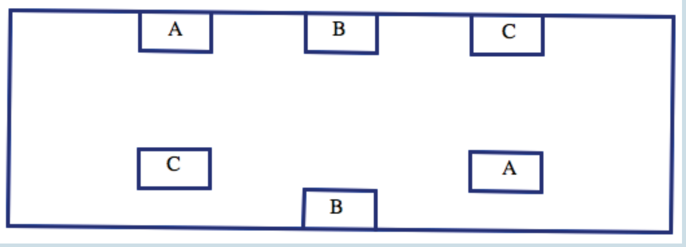
The Common Core State Standards for Mathematics (http://www.corestandards.org/Math/Practice) identify eight "Mathematical Practices" – the kinds of expertise that all teachers should try to foster in their students, but they go far beyond any particular piece of mathematics content. They describe what mathematics is really about, and why it is so valuable for students to master. The very first Mathematical Practice is:

> Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary.

This chapter will help you develop these very important mathematical skills, so that you will be better prepared to help your future students develop them. Let's start with solving a problem!

Problem 1 (ABC)

Draw curves connecting A to A, B to B, and C to C. Your curves cannot cross or even touch each other,they cannot cross through any of the lettered boxes, and they cannot go outside the large box or even touch it's sides.



Think / Pair / Share

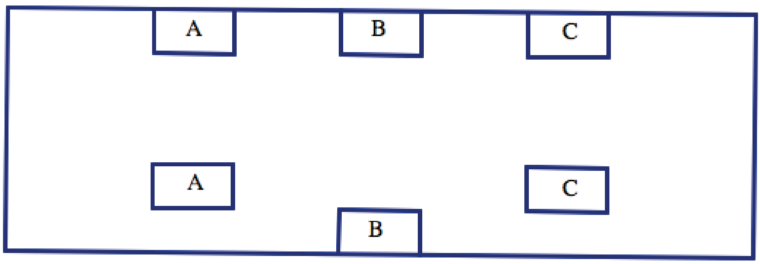
After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it).

- What did you try?
- What makes this problem difficult?

- Can you change the problem slightly so that it would be easier to solve?

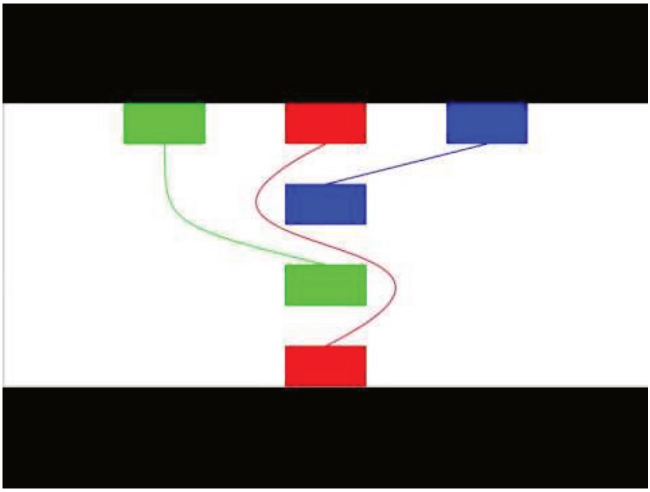
Problem Solving Strategy 1 (Wishful Thinking). *Do you wish something in the problem was different? Would it then be easier to solve the problem?*

For example, what if ABC problem had a picture like this:



Can you solve this case and use it to help you solve the original case? Think about moving the boxes around once the lines are already drawn.

Here is one possible solution.



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2. Problem or Exercise?

The main activity of mathematics is solving problems. However, what most people experience in most mathematics classrooms is practice exercises. An exercise is different from a problem.

In a **problem**, you probably don't know at first how to approach solving it. You don't know what mathematical ideas might be used in the solution. Part of solving a problem is understanding what is being asked, and knowing what a solution should look like. Problems often involve false starts, making mistakes, and lots of scratch paper!

In an **exercise**, you are often practicing a skill. You may have seen a teacher demonstrate a technique, or you may have read a worked example in the book. You then practice on very similar assignments, with the goal of mastering that skill.

Note: What is a problem for some people may be an exercise for other people who have more background knowledge! For a young student just learning addition, this might be a problem:

Fill in the blank to make a true statement: $\underline{\qquad} + 4 = 7$.

But for you, that is an exercise!

Both problems and exercises are important in mathematics learning. But we should never forget that the ultimate goal is to develop more and better skills (through exercises) so that we can solve harder and more interesting problems.

Learning math is a bit like learning to play a sport. You can practice a lot of skills:

- hitting hundreds of forehands in tennis so that you can place them in a particular spot in the court,
- breaking down strokes into the component pieces in swimming so that each part of the stroke is more efficient,
- keeping control of the ball while making quick turns in soccer,
- shooting free throws in basketball,

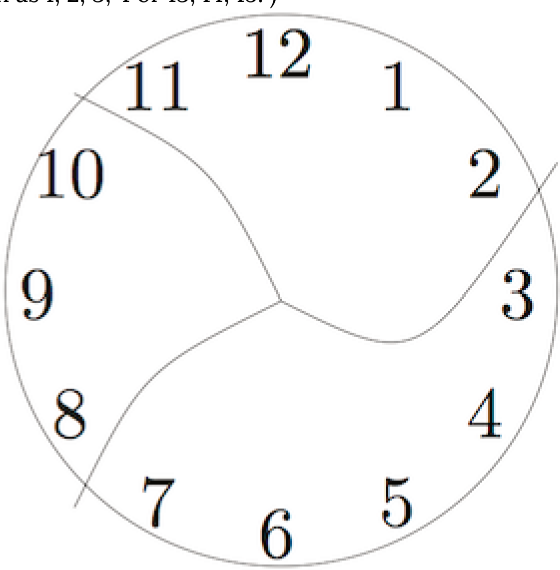
- catching high fly balls in baseball,
- and so on.

But the point of the sport is to play the game. You practice the skills so that you are better at playing the game. In mathematics, solving problems is playing the game!

On Your Own

For each question below, decide if it is a *problem* or an *exercise*. (You do not need to solve the problems! Just decide which category it fits for you.) After you have labeled each one, compare your answers with a partner.

1. This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers.(Note: **Consecutive numbers** are whole numbers that appear one after the other, such as 1, 2, 3, 4 or 13, 14, 15.)



Can you break another clock into a different number of pieces so that the sums are consecutive numbers? Assume that each piece has at least two numbers and that no number is damaged (e.g. 12 isn't split into two digits 1 and 2).

2. A soccer coach began the year with a \$500 budget. By the end of December, the coach spent \$450. How much money in the budget was not spent?

3. What is the product of 4,500 and 27?

4. Arrange the digits 1–6 into a "difference triangle" where each number in the row below is the difference of the two numbers above it.

5. Simplify the following expression:

$$\frac{2+2(5^3-4^2)^5-2^2}{2(5^3-4^2)}.$$

6. What is the sum of $\frac{5}{2}$ and $\frac{3}{13}$?

7. You have eight coins and a balance scale. The coins look alike, but one of them is a counterfeit. The counterfeit coin is lighter than the others. You may only use the balance scale two times. How can you find the counterfeit coin?

8. How many squares, of any possible size, are on a standard 8 × 8 chess board?

9. What number is 3 more than half of 20?

10. Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1's are separated by one digit, the 2's are separated by two digits, the 3's by three digits, and the 4's by four digits.

3. Problem Solving Strategies

Think back to the first problem in this chapter, the ABC Problem. What did you do to solve it? Even if you did not figure it out completely by yourself, you probably worked towards a solution and figured out some things that *did not* work.

Unlike exercises, there is never a simple recipe for solving a problem. You can get better and better at solving problems, both by building up your background knowledge and by simply practicing. As you solve more problems (and learn how other people solved them), you learn strategies and techniques that can be useful. But no single strategy works every time.

Pólya's *How to Solve It*

> George Pólya was a great champion in the field of *teaching* effective problem solving skills. He was born in Hungary in 1887, received his Ph.D. at the University of Budapest, and was a professor at Stanford University (among other universities). He wrote many mathematical papers along with three books, most famously, "How to Solve it." Pólya died at the age 98 in 1985.[1]

1. Image of Pólya by Thane Plambeck from Palo Alto, California (Flickr) [CC BY

George Pólya, circa 1973

In 1945, Pólya published the short book *How to Solve It*, which gave a four-step method for solving mathematical problems:

1. First, you have to understand the problem.
2. After understanding, then make a plan.
3. Carry out the plan.
4. Look back on your work. How could it be better?

This is all well and good, but how do you actually do these steps?!?! Steps 1. and 2. are particularly mysterious! How do you "make a plan?" That is where you need some tools in your toolbox, and some experience to draw upon.

Much has been written since 1945 to explain these steps in more detail, but the truth is that they are more art than science. This is where math becomes a creative endeavor (and where it becomes so much fun). We will articulate some useful problem solving strategies, but no such list will ever be complete. This is really just a start to help you on your way. The best way to become a skilled problem solver is to learn the background material well, and then to solve a lot of problems!

We have already seen one problem solving strategy, which we call "Wishful Thinking." Do not be afraid to change the problem! Ask yourself "what if" questions:

- What if the picture was different?
- What if the numbers were simpler?
- What if I just made up some numbers?

You need to be sure to go back to the original problem at the end, but wishful thinking can be a powerful strategy for getting started.

This brings us to the most important problem solving strategy of all:

Problem Solving Strategy 2 (Try Something!). *If you are really trying to solve a problem, the whole point is that you do not know what to do right out of the starting gate. You need to just try something! Put pencil to paper (or stylus to screen or chalk to board or whatever!) and try something. This is often an important step in understanding the problem; just mess around with it a bit to understand the situation and figure out what is going on.*

And equally important: If what you tried first does not work, try something else! Play around with the problem until you have a feel for what is going on.

Problem 2 (Payback)

Last week, Alex borrowed money from several of his friends. He finally got paid at work, so he brought cash to school to pay back his debts. First he saw Brianna, and he gave her 1/4 of the money he had brought to school. Then Alex saw Chris and gave him 1/3 of what he had left after paying Brianna. Finally, Alex saw David and gave him 1/2 of what he had remaining. Who got the most money from Alex?

Think/Pair/Share

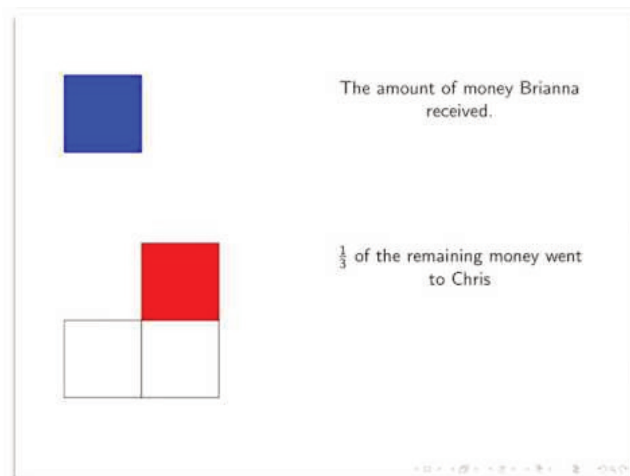
After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What did you try? What did you figure out about the problem?

This problem lends itself to two particular strategies. Did you try either of these as you worked on the problem? If not, read about the strategy and then try it out before watching the solution.

Problem Solving Strategy 3 (Draw a Picture). *Some problems are obviously about a geometric situation, and it is clear you want to draw a picture and mark down all of the given information before you try to solve it. But even for a problem that is not geometric, like this* *one, thinking visually can help! Can you represent something in the situation by a picture?*

Draw a square to represent all of Alex's money. Then shade 1/4 of the square — that's what he gave away to Brianna. How can the picture help you finish the problem?

After you have worked on the problem yourself using this strategy (or if you are completely stuck), you can watch someone else's solution.



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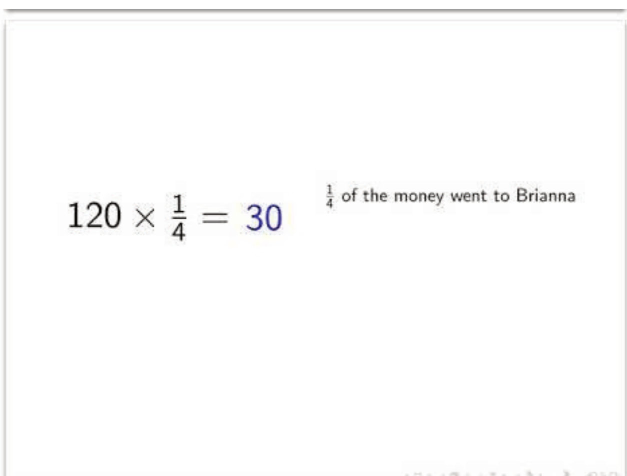
Problem Solving Strategy 4 (Make Up Numbers). *Part of what makes this problem difficult is that it is about money, but there are no numbers* *given. That means the numbers must not be important. So just make them up!*

You can work forwards: Assume Alex had some specific amount of money when he showed up at school, say $100. Then figure out how much he gives to each person. Or you can work backwards: suppose he has some specific amount left at the end, like $10. Since he gave Chris half of what he had left, that means he had $20 before running into Chris. Now, work backwards and figure out how much each person got.

Watch the solution only after you tried this strategy for yourself.



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If you use the "Make Up Numbers" strategy, it is really important to remember what the original problem was asking! You do not want to answer something like "Everyone got $10." That is not true in the original problem; that is an artifact of the numbers you made up. So after you work everything out, be sure to re-read the problem and answer what was asked!

Problem 3 (Squares on a Chess Board)

How many squares, of any possible size, are on a 8 × 8 chess board? (The answer is not 64... It's a lot bigger!)

Remember Pólya's first step is to understand the problem. If you are not sure what is being asked, or why the answer is not just 64, be sure to ask someone!

Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What did you try? What did you figure out about the problem, even if you have not solved it completely?

It is clear that you want to draw a picture for this problem, but even with the picture it can be hard to know if you have found the correct answer. The numbers get big, and it can be hard to keep track of your work. Your goal at the end is to be absolutely positive that you found the right answer. You should never ask the teacher, "Is this right?" Instead, you should declare, "Here's my answer, and here is why I know it is correct!"

Problem Solving Strategy 5 (Try a Simpler Problem). *Pólya suggested this strategy: "If you can't solve a problem, then there is an easier problem you can solve: find it." He also said: "If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem?" In this case, an 8 × 8 chess board is pretty big. Can you solve the problem for smaller boards? Like 1 × 1? 2 × 2? 3 × 3?*

Of course the ultimate goal is to solve the original problem. But working with smaller boards might give you some insight and help you devise your plan (that is Pólya's step (2)).

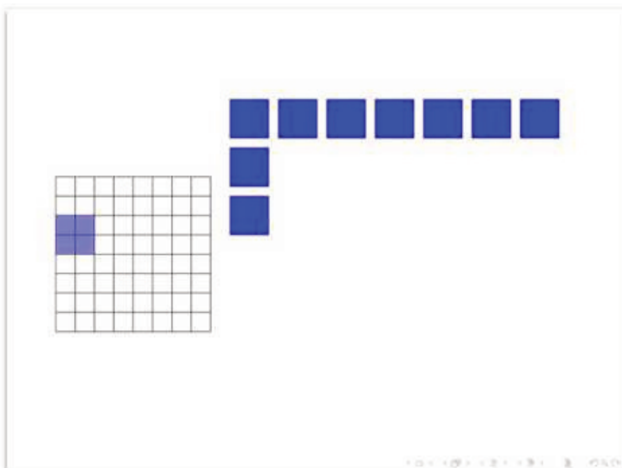
Problem Solving Strategy 6 (Work Systematically). *If you are working on simpler problems, it is useful to keep track of what you have figured out and what changes as the problem gets more complicated.*

For example, in this problem you might keep track of how many 1 × 1 squares are on each board, how many 2 × 2 squares on are each board, how many 3 × 3 squares are on each board, and so on. You could keep track of the information in a table:

size of board	# of 1 × 1 squares	# of 2 × 2 squares	# of 3 × 3 squares	# of 4 × 4 squares	…
1 by 1	1	0	0	0	
2 by 2	4	1	0	0	
3 by 3	9	4	1	0	
…					

Problem Solving Strategy 7 (Use Manipulatives to Help You Investigate). *Sometimes even drawing a picture may not be enough to help you investigate a problem. Having actual* *materials that you move around can sometimes help a lot!*

For example, in this problem it can be difficult to keep track of which squares you have already counted. You might want to cut out 1 × 1 squares, 2 × 2 squares, 3 × 3 squares, and so on. You can actually move the smaller squares across the chess board in a systematic way, making sure that you count everything once and do not count anything twice.



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Problem Solving Strategy 8 (Look for and Explain Patterns). *Sometimes the numbers in a problem are so big, there is no way you will actually count everything up by hand. For example, if the problem* *in this section were about a 100 × 100 chess board, you would not want to go through counting all the squares by hand! It would be much more appealing to find a pattern in the smaller boards and then extend that pattern to solve the problem for a 100 × 100 chess board just with a calculation.*

Think / Pair / Share

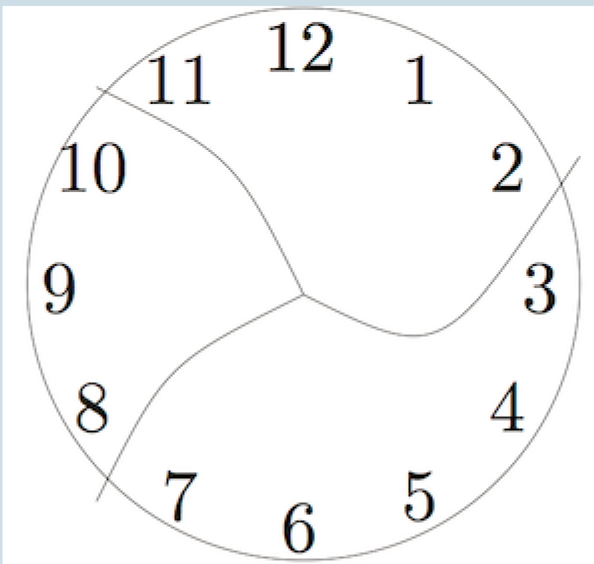
If you have not done so already, extend the table above all the way to an 8 × 8 chess board, filling in all the rows and columns. Use your table to find the total number of squares in an 8 × 8 chess board. Then:

- Describe all of the patterns you see in the table.
- Can you explain and justify any of the patterns you see? How can you be sure they will continue?
- What calculation would you do to find the total number of squares on a 100 × 100 chess board?

(We will come back to this question soon. So if you are not sure right now how to explain and justify the patterns you found, that is OK.)

Problem 4 (Broken Clock)

This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. (**Consecutive numbers** are whole numbers that appear one after the other, such as 1, 2, 3, 4 or 13, 14, 15.)



Can you break another clock into a different number of pieces so that the sums are consecutive numbers? Assume that each piece has at least two numbers and that no number is damaged (e.g. 12 isn't split into two digits 1 and 2.)

Remember that your first step is to understand the problem. Work out what is going on here. What are the sums of the numbers on each piece? Are they consecutive?

Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What did you try? What progress have you made?

Problem Solving Strategy 9 (Find the Math, Remove the Context). *Sometimes the problem has a lot of details in it that are unimportant, or at least unimportant for getting started. The goal is to find the underlying math problem, then come back to the original question and see if you can solve it using the math.*

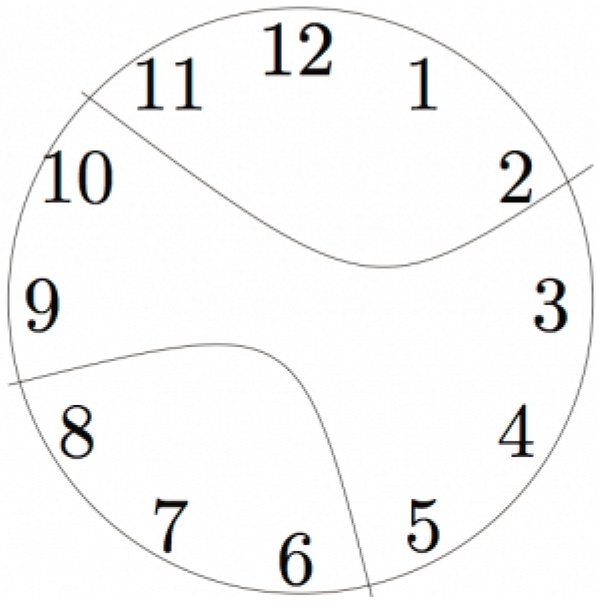
In this case, worrying about the clock and exactly how the pieces break is less important than worrying about finding consecutive numbers that sum to the correct total. Ask yourself:

- What is the sum of all the numbers on the clock's face?
- Can I find two consecutive numbers that give the correct sum? Or four consecutive numbers? Or some other amount?
- How do I know when I am done? When should I stop looking?

Of course, solving the question about consecutive numbers is not the same as solving the original problem. You have to go back and see if the clock can actually break apart so that each piece gives you one of those consecutive numbers. Maybe you can solve the math problem, but it does not translate into solving the clock problem.

Problem Solving Strategy 10 (Check Your Assumptions). *When* *solving problems, it is easy to limit your thinking by adding extra assumptions that are not in the problem. Be sure you ask yourself: Am I constraining my thinking too much?*

In the clock problem, because the first solution has the clock broken radially (all three pieces meet at the center, so it looks like slicing a pie), many people assume that is how the clock must break. But the problem does not require the clock to break radially. It might break into pieces like this:



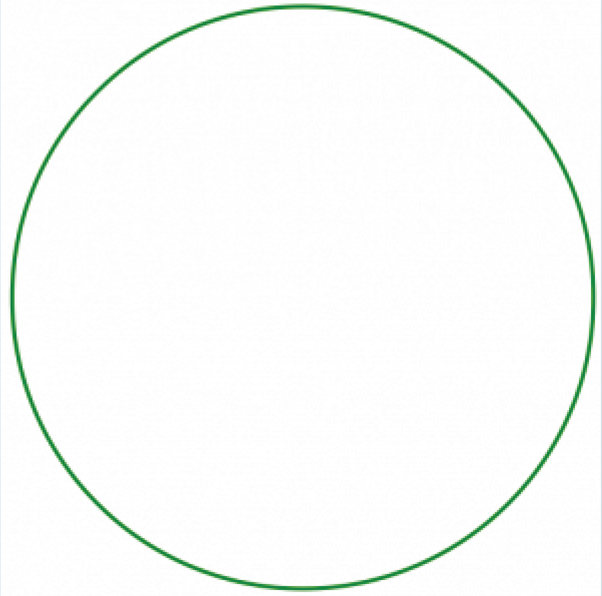
Were you assuming the clock would break in a specific way? Try to solve the problem now, if you have not already.

4. Beware of Patterns!

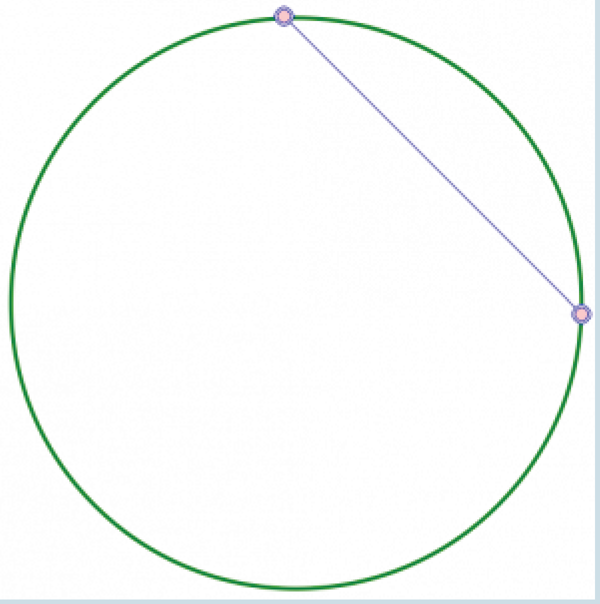
The “Look for Patterns” strategy can be particularly appealing, but you have to be careful! Do not forget the “and Explain” part of the strategy. Not all patterns are obvious, and not all of them will continue.

***Problem 5** (Dots on a Circle)*

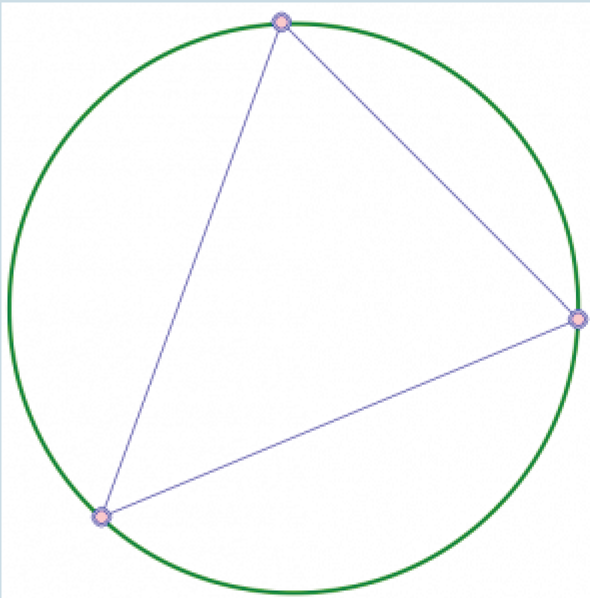
Start with a circle.



If I put two dots on the circle and connect them, the line divides the circle into two pieces.



If I put three dots on the circle and connect each pair of dots, the lines divides the circle into four pieces.



Suppose you put one hundred dots on a circle and connect each pair of dots, meaning every dot is connected to 99 other dots. How many pieces will you get? Lines may cross each other, but assume the points are chosen so that three or more lines never meet at a single point.

Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What strategies did you try? What did you figure out? What questions do you still have?

The natural way to work on this problem is to use smaller numbers of dots and look for a pattern, right? If you have not already, try it. How many pieces when you have four dots? Five dots? How would you describe the pattern?

Now try six dots. You will want to draw a big circle and space out the six dots to make your counting easier. Then carefully count up how many pieces you get. It is probably a good idea to work with a partner so you can check each other's work. Make sure you count every piece once and do not count any piece twice. How can you be sure that you do that?

Were you surprised? For the first several steps, it seems to be the case that when you add a dot you double the number of pieces. But that would mean that for six dots, you should get 32 pieces, and you only get 30 or 31, depending on how the dots are arranged. No matter what you do, you cannot get 32 pieces. The pattern simply does not hold up.

Mathematicians love looking for patterns and finding them. We get excited by patterns. But we are also very skeptical of patterns! If we cannot explain why a pattern would occur, then we are not willing to just believe it.

For example, if my number pattern starts out: 2, 4, 8, … I can find lots of ways to continue the pattern, each of which makes sense in some contexts. Here are some possibilities:

- 2, 4, 8, 2, 4, 8, 2, 4, 8, 2, 4, 8, …

This is a a repeating pattern, cycling through the numbers 2, 4, 8 and then starting over with 2.

- 2, 4, 8, 32, 256, 8192, …

To get the next number, multiply the previous two numbers together.

- 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, …
- 2, 4, 8, 14, 22, 32, 44, 58, 74 …

Think / Pair / Share

- For the last two patterns above, describe in words how the number sequence is being created.
- Find at least two other ways to continue the sequence 2, 4, 8, . . . that looks different from all the ones you have seen so far. Write your rule in words, and write the next five terms of the number sequence.

So how can you be sure your pattern fits the problem? You have to tie them together! Remember the "Squares on a Chess Board" problem? You might have noticed a pattern like this one:

If the chess board has 5 squares on a side, then there are

- $5 \times 5 = 25$ squares of size 1×1.
- $4 \times 4 = 16$ squares of size 2×2.
- $3 \times 3 = 9$ squares of side 3×3.
- $2 \times 2 = 4$ squares of size 4×4.
- $1 \times 1 = 1$ squares of size 5×5.

So there are a total of

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

squares on a 5×5 chess board. You can probably guess how to continue the pattern to any size board, but how can you be absolutely sure the pattern continues in this way? What if this is like "Dots on a Circle," and the obvious pattern breaks down after a few steps? You have to tie the pattern to the problem, so that it is clear why the pattern must continue in that way.

The first step in explaining a pattern is writing it down clearly. This brings us to another problem solving strategy.

Problem Solving Strategy 11 (Use a Variable!). *One of the most powerful tools we have is the use of a variable. If you find yourself doing calculations on things like "the number of squares," or "the number of dots," give those quantities a name! They become much easier to work with.*

Think/Pair/Share

For now, just work on describing the pattern with variables.

- Stick with a 5×5 chess board for now, and consider a small square of size $k \times k$. Describe the pattern: How many squares of size $k \times k$ fit on a chess board of size $5 \times$

5?

- What if the chess board is bigger? Based on the pattern above, how many squares of size $k \times k$ should fit on a chess board of size 10×10?
- What if you do not know how big the chess board is? Based on the pattern above, how many squares of size $k \times k$ should fit on a chess board of size $n \times n$?

Now comes the tough part: explaining the pattern. Let us focus on an 8×8 board. Since it measures 8 squares on each side, we can see that we get $8 \times 8 = 64$ squares of size 1×1. And since there is just a single board, we get just one square of size 8×8. But what about all the sizes in-between?

Think/Pair/Share

Using the Chess Board video in the previous chapter as a model, work with a partner to carefully explain why the number of 3×3 squares will be $6 \cdot 6 = 36$, and why the number of 4×4 squares will be $5 \cdot 5 = 25$.

There are many different explanations other than what is found in the video. Try to find your own explanation.

Here is what a final justification might look like (*watch the Chess Board video as a concrete example of this solution*):

Solution (Chess Board Pattern). Let n be the side of the chess board and let k be the side of the square. If the square is going to fit on the chess board at all, it must be true that $k \leq n$. Otherwise, the square is too big.

If I put the $k \times k$ square in the upper left corner of the chess board, it takes up k spaces across and there are $(n - k)$ spaces to the right of it. So I can slide the $k \times k$ square to the right $(n - k)$ times, until it hits the top right corner of the chess board. The square is in $(n - k + 1)$ different positions, counting the starting position.

If I move the $k \times k$ square back to the upper left corner, I can shift it down one row and repeat the whole process again. Since there are $(n - k)$ rows below the square, I can shift it down $(n - k)$ times until it hits the bottom row. This makes $(n - k + 1)$ total rows that the square moves across, counting the top row.

So, there are $(n - k + 1)$ rows with $(n - k + 1)$ squares in each row. That makes $(n - k + 1)^2$ total squares.

Thus, the solution is the sum of $(n - k + 1)^2$ for all $k \leq n$. In symbols:

$$\text{number of squares on an } n \times n \text{ board} = \sum_{k=1}^{n} (n - k + 1)^2.$$

Once we are sure the pattern continues, we can use it to solve the problem. So go ahead!

- How many squares on a 10 × 10 chess board?
- What calculation would you do to solve that problem for a 100 × 100 chess board?

There is a number pattern that describes the number of pieces you get from the "Dots on a Circle" problem. If you want to solve the problem, go for it! Think about all of your problem solving strategies. But be sure that when you find a pattern, you can explain *why it is the right pattern for this problem*, and not just another pattern that seems to work but might not continue.

5. Problem Bank

You have several problem solving strategies to work with. Here are the ones we have described so far (and you probably came up with even more of your own strategies as you worked on problems).

1. Wishful Thinking.
2. Try Something!
3. Draw a Picture.
4. Make Up Numbers.
5. Try a Simpler Problem.
6. Work Systematically.
7. Use Manipulatives to Help you Investigate.
8. Look for and Explain Patterns.
9. Find the Math, Remove the Context.
10. Check Your Assumptions.
11. Use a Variable.

Try your hand at some of these problems, keeping these strategies in mind. If you are stuck on a problem, come back to this list and ask yourself which of the strategies might help you make some progress.

Problem 6

You have eight coins and a balance scale. The coins look alike, but one of them is a counterfeit. The counterfeit coin is lighter than the others. You may only use the balance scale two times. How can you find the counterfeit coin?

Problem 7

You have five coins, no two of which weigh the same. In seven weighings on a balance scale, can you put the coins in order from lightest to heaviest? That is, can you determine which coin is the lightest, next lightest, . . . , heaviest.

Problem 8

You have ten bags of coins. Nine of the bags contain good coins weighing one ounce each. One bag contains counterfeit coins weighing 1.1 ounces each. You have a regular (digital) scale, not a balance scale. The scale is correct to one-tenth of an ounce. In one weighing, can you determine which bag contains the bad coins?

Problem 9

Suppose you have a balance scale. You have three different weights, and you are able to weigh every whole number from 1 gram to 13 grams using just those three weights. What are the three weights?

Problem 10

There are a bunch of coins on a table in front of you. Your friend tells you how many of the coins are heads-up. You are blindfolded and cannot see a thing, but you can move the coins around, and you can flip them over. However, you cannot tell just by feeling them if the coins are showing heads or tails. Your job: separate the coins into two piles so that the same number of heads are showing in each pile.

Problem 11

The digital root of a number is the number obtained by repeatedly adding the digits of the number. If the answer is not a one-digit number, add those digits. Continue until a one-digit sum is reached. This one digit is the digital root of the number.

For example, the digital root of 98 is 8, since 9 + 8 = 17 and 1 + 7 = 8.

Record the digital roots of the first 30 integers and find as many patterns as you can. Can you explain any of the patterns?

Problem 12

If this lattice were continued, what number would be directly to the right of 98? How can you be sure you're right?

	3		6		9		12		...
1	2	4	5	7	8	10	11	13	...

Problem 13

Arrange the digits 0 through 9 so that the first digit is divisible by 1, the first two digits are divisible by 2, the first three digits are divisible by 3, and continuing until you have the first 9 digits divisible by 9 and the whole 10-digit number divisible by 10.

Problem 14

There are 25 students and one teacher in class. After an exam, everyone high-fives everyone else to celebrate how well they did. How many high- fives were there?

Problem 15

In cleaning out your old desk, you find a whole bunch of 3¢ and 7¢ stamps. Can you make exactly 11¢ of postage? Can you make exactly 19¢ of postage? What is the largest amount of postage you cannot make?

Problem 16

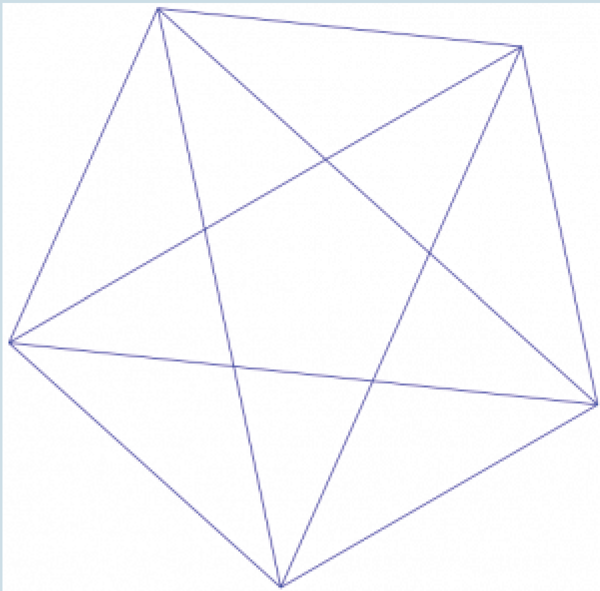
Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1's are separated by one digit, the 2's are separated by two digits, the 3's by three digits, and the 4's by four digits.

Problem 17

Kami has ten pockets and 44 dollar bills. She wants to have a different amount of money in each pocket. Can she do it?

Problem 18

How many triangles of all possible sizes and shapes are in this picture?



Problem 19

Arrange the digits 1–6 into a “difference triangle” where each number in the row below is the difference of the two numbers above it.

Example: Below is a difference triangle, but it does not work because it uses 1 twice and does not have a 6:

4 5 3

1 2

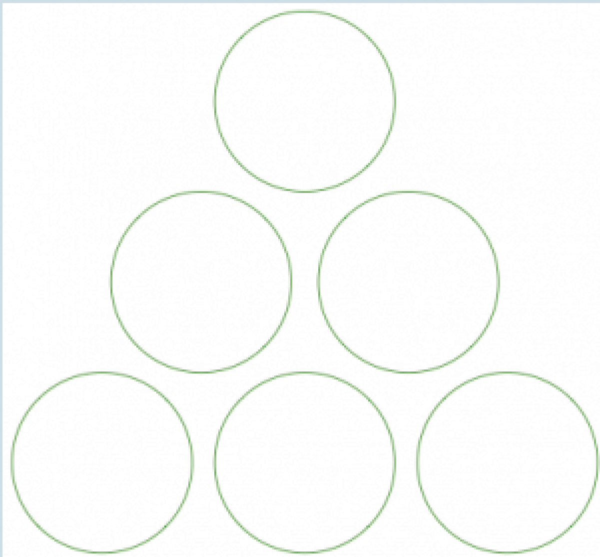
1

Problem 20

Certain pipes are sold in lengths of 6 inches, 8 inches, and 10 inches. How many different lengths can you form by attaching three sections of pipe together?

Problem 21

Place the digits 1, 2, 3, 4, 5, 6 in the circles so that the sum on each side of the triangle is 12. Each circle gets one digit, and each digit is used exactly once.



Problem 22

Find a way to cut a circular pizza into 11 pieces using just four straight cuts.

6. Careful Use of Language in Mathematics

This section might seem like a bit of a sidetrack from the idea of problem solving, but in fact it is not. Mathematics is a social endeavor. We do not just solve problems and then put them aside. Problem solving has (at least) three components:

1. Solving the problem. This involves a lot of scratch paper and careful thinking.
2. Convincing yourself that your solution is complete and correct. This involves a lot of self-check and asking yourself questions.
3. Convincing someone else that your solution is complete and correct. This usually involves writing the problem up carefully or explaining your work in a presentation.

If you are not able to do that last step, then you have not really solved the problem. We will talk more about how to write up a solution soon. Before we do that, we have to think about how mathematicians use language (which is, it turns out, a bit different from how language is used in the rest of life).

Mathematical Statements

Definition

A mathematical statement is a complete sentence that is either true or false, but not both at once.

So a "statement" in mathematics cannot be a question, a command, or a matter of opinion. It is a complete, grammatically correct sentence (with a subject, verb, and usually an object). It is important that the statement is either true or false, though you may not know which! (Part of the work of a mathematician is figuring out which sentences are true and which are false.)

Think / Pair / Share

For each English sentence below, decide if it is a mathematical statement or not. If it is, is the statement true or false (or are you unsure)? If it is not a mathematical statement, in what way does it fail?

1. Blue is the prettiest color.
2. 60 is an even number.
3. Is your dog friendly?

4. Honolulu is the capital of Hawaii.
5. This sentence is false.
6. All roses are red.
7. UH Manoa is the best college in the world.
8. 1/2 = 2/4.
9. Go to bed.
10. There are a total of 204 squares on an 8 × 8 chess board.

Now write three mathematical statements and three English sentences that fail to be mathematical statements.

Notice that "1/2 = 2/4" is a perfectly good mathematical statement. It does not look like an English sentence, but read it out loud. The subject is "1/2." The verb is "equals." And the object is "2/4." This is a very good test when you write mathematics: try to read it out loud. Even the equations should read naturally, like English sentences.

Statement (5) is different from the others. It is called a **paradox**: a statement that is self-contradictory. If it is true, then we conclude that it is false. (Why?) If it is false, then we conclude that it is true. (Why?) Paradoxes are no good as mathematical statements, because it cannot be true and it cannot be false.

And / or

Consider this sentence:

> After work, I will go to the beach, or I will do my grocery shopping.

In everyday English, that probably means that if I go to the beach, I will not go shopping. I will do one or the other, but not both activities. This is called an "exclusive or."

We can usually tell from context whether a speaker means "either one or the other or both," or whether he means "either one or the other but not both." (Some people use the awkward phrase "and/or" to describe the first option.)

Remember that in mathematical communication, though, we have to be very precise. We cannot rely on context or assumptions about what is implied or understood.

Definition

In mathematics, the word "or" always means "one or the other or both."

The word "and" always means "both are true."

Think / Pair / Share

For each sentence below:

- Decide if the choice $x = 3$ makes the statement true or false.

- Choose a different value of that makes the statement true (or say why that is not possible).
- Choose a different value of that makes the statement false (or say why that is not possible).

1. x is odd or x is even.
2. x is odd and x is even.
3. x is prime or x is odd.
4. $x > 5$ or $x < 5$.
5. $x > 5$ and $x < 5$.
6. $x + 1 = 7$ or $x - 1 = 7$.
7. $x \cdot 1 = x$ or $x \cdot 0 = x$.
8. $x \cdot 1 = x$ and $x \cdot 0 = x$.
9. $x \cdot 1 = x$ or $x \cdot 0 = 0$.

Quantifiers

Problem 23 *(All About the Benjamins)*

You are handed an envelope filled with money, and you are told "Every bill in this envelope is a $100 bill."

- What would convince you beyond any doubt that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you beyond any doubt that the sentence is false? How could you convince someone else that the sentence is false?

Suppose you were given a different sentence: “There is a $100 bill in this envelope.”

- What would convince you beyond any doubt that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you beyond any doubt that the sentence is false? How could you convince someone else that the sentence is false?

Think / Pair / Share

What is the difference between the two sentences? How does that difference affect your method to decide if the statement is true or false?

Some mathematical statements have this form:

- “Every time…”
- “For all numbers. . . ”
- “For every choice. . . ”
- “It’s always true that. . . ”

These are *universal* statements. Such statements claim that something is always true, no matter what.

- To prove a universal statement is false, you must find an example where it fails. This is called a **counterexample** to the statement.
- To prove a universal statement is true, you must either check every single case, or you must find a logical reason why it would be true. (Sometimes the first option is impossible, because there might be infinitely many cases to check. You would never finish!)

Some mathematical statements have this form:

- “Sometimes…”
- “There is some number. . . ”
- “For some choice. . . ”
- “At least once…”

These are *existential* statements. Such statements claim there is some example where the statement is true, but it may not always be true.

- To prove an existential statement is true, you may just find the example where it works.
- To prove an existential statement is false, you must either show it fails in every single case, or you must find a logical reason why it cannot be true. (Sometimes the first option is impossible!)

Think / Pair / Share

For each statement below, do the following:

- Decide if it is a universal statement or an existential statement. (This can be tricky because in some statements the quantifier is “hidden” in the meaning of the words.)
- Decide if the statement is true or false, and do your best to justify your decision.

1. Every odd number is prime.
2. Every prime number is odd.
3. For all positive numbers $x, \ x^3 > x$.
4. There is some number x such that $x^3 = x$.
5. The points (1, 1), (2, 1), and (3, 0) all lie on the same line.
6. Addition (of real numbers) is commutative.
7. Division (of real numbers) is commutative.

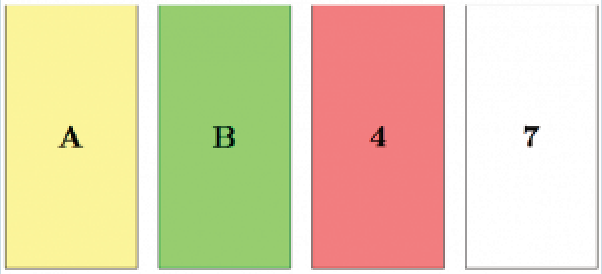
Look back over your work. you will probably find that some of your arguments are sound and convincing while others are less so. In some cases you may “know” the answer but be unable to justify it. That is okay for now! Divide your answers into four categories:

1. I am confident that the justification I gave is good.
2. I am not confident in the justification I gave.
3. I am confident that the justification I gave is not good, or I could not give a justification.
4. I could not decide if the statement was true or false.

Problem 24 (Card Logic)

You have a deck of cards where each card has a letter on one side and a number on the other side. Your friend claims: "If a card has a vowel on one side, then it has an even number on the other side."

These cards are on a table.



Which cards must you flip over to be certain that your friend is telling the truth?

Think / Pair / Share

After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Try to come to agreement on an answer you both believe.

Here is another very similar problem, yet people seem to have an easier time solving this one:

Problem 25 *(IDs at a Party)*

You are in charge of a party where there are young people. Some are drinking alcohol, others soft drinks. Some are old enough to drink alcohol legally, others are under age. You are responsible for ensuring that the drinking laws are not broken, so you have asked each person to put his or her photo ID on the table. At one table, there are four young people:

- One person has a can of beer, another has a bottle of Coke, but their IDs happen to be face down so you cannot see their ages.
- You can, however, see the IDs of the other two people. One is under the drinking age, the other is above it. They both have fizzy clear drinks in glasses, and you are not

sure if they are drinking soda water or gin and tonic.

Which IDs and/or drinks do you need to check to make sure that no one is breaking the law?

Think / Pair / Share

After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Compare these two problems. Which question is easier and why?

Definition

A **conditional statement** can be written in the form

If some statement then some statement.

Where the first statement is the **hypothesis** and the second statement is the **conclusion**.

Think / Pair / Share

These are each conditional statements, though they are not all stated in "if/then" form. Identify the hypothesis of each statement. (You may want to rewrite the sentence as an equivalent "if/then" statement.)

1. If the tomatoes are red, then they are ready to eat.
   The tomatoes are red. / The tomatoes are ready to eat.
2. An integer n is even if it is a multiple of 2.
   n is even. / n is a multiple of 2.
3. If n is odd, then n is prime.
   n is odd. / n is prime.
4. The team wins when JJ plays.
   The team wins. / JJ plays.

Remember that a mathematical statement must have a definite truth value. It is either true or false, with no gray area (even though we may not be sure which is the case). How can you tell if a conditional statement is true or false? Surely, it depends on whether the hypothesis and the conclusion are true or false. But how, exactly, can you decide?

The key is to think of a conditional statement like a promise, and ask yourself: under what condition(s) will I have broken my promise?

Examples

Example 1. Here is a conditional statement:

> If I win the lottery, then I'll give each of my students $1,000.

There are four things that can happen:

- **True hypothesis, true conclusion:** I do win the lottery, and I do give everyone in class $1,000. I kept my promise, so the conditional statement is TRUE.
- **True hypothesis, false conclusion:** I do win the lottery, but I decide not to give everyone in class $1,000. I broke my promise, so the conditional statement is FALSE.
- **False hypothesis, true conclusion:** I do not win the lottery, but I am exceedingly generous, so I go ahead and give everyone in class $1,000. I did not break my promise! (Do you see why?) So the conditional statement is TRUE.
- **False hypothesis, false conclusion:** I do not win the lottery, so I do not give everyone in class $1,000. I did not break my promise! (Do you see why?) So the conditional statement is TRUE.

What can we conclude from this? **A conditional statement is false only when the hypothesis is true and the conclusion is false.** In every other instance, the promise (as it were) has not been broken. If a mathematical statement is not false, it must be true.

Example 2. Here is another conditional statement:

If you live in Honolulu, then you live in Hawaii.

Is this statement true or false? It seems like it should depend on who the pronoun "you" refers to, and whether that person lives in Honolulu or not. Let us think it through:

- Sookim lives in Honolulu, so the hypothesis is true. Since Honolulu is in Hawaii, she does live in Hawaii. The statement is true about Sookim, since both the hypothesis and conclusion are true.
- DeeDee lives in Los Angeles. The statement is true about DeeDee since the hypothesis is false.

So in fact it does not matter! The statement is true either way. The right way to understand such a statement is as a universal statement: "Everyone who lives in Honolulu lives in Hawaii."

This statement is true, and here is how you might justify it: "Pick a random person who lives in Honolulu. That person lives in Hawaii (since Honolulu is in Hawaii), so the statement is true for that person. I do not need to consider people who do not live in Honolulu. The statement is automatically true for those people, because the hypothesis is false!"

Example 3. How do we show a (universal) conditional statement is false?

You need to give a specific instance where the hypothesis is true and the conclusion is false. For example:

If you are a good swimmer, then you are a good surfer.

Do you know someone for whom the hypothesis is true (that person is a good swimmer) but the conclusion is false (the person is not a good surfer)? Then the statement is false!

Think / Pair / Share

For each conditional statement, decide if it is true or false. Justify your answer.

1. If $2 \times 2 = 4$ then $1 + 1 = 3$.
2. If $2 \times 2 = 5$ then $1 + 1 = 3$.
3. If $\pi > 3$ then all odd numbers are prime.
4. If $\pi < 3$ then all odd numbers are prime.
5. If a number has a 4 in the one's place, then the number is even.
6. If a number is even, then the number has a 4 in the one's place.
7. If the product of two numbers is 0, then one of the numbers is 0.
8. If the sum of two numbers is 0, then one of the numbers is 0.

Think / Pair / Share *(Two truths and a lie)*

On your own, come up with two conditional statements that are true and one that is false. Share your three statements with a partner, but do not say which are true and which is false. See if your partner can figure it out!

7. Explaining Your Work

At its heart, mathematics is a social endeavor. Even if you work on problems all by yourself, you have not really solved the problem until you have explained your work to someone else, and they sign off on it. Professional mathematicians write journal articles, books, and grant proposals. Teachers explain mathematical ideas to their students both in writing and orally. Explaining your work is really an essential part of the problem-solving process, and probably should have been Pólya's step 5.

Writing in mathematics is different from writing poetry or an English paper. The goal of mathematical writing is not florid description, but clarity. If your reader does not understand, then you have not done a good job. Here are some tips for good mathematical writing.

Do Not Turn in Scratch Work: When you are solving problems and not exercises, you are going to have a lot of false starts. You are going to try a lot of things that do not work. You are going to make a lot of mistakes. You are going to use scratch paper. At some point (hopefully!) you will scribble down an idea that actually solves the problem. Hooray! That paper is not what you want to turn in or share with the world. Take that idea, and write it up carefully, neatly, and clearly. (The rest of these tips apply to that write-up.)

(Re)state the Problem: Do not assume your reader knows what problem you are solving. (Even if it is the teacher who assigned the problem!) If the problem has a very long description, you can summarize it. You do not have rewrite it word-for-word or give all of the details, but make sure the question is clear.

Clearly Give the Answer: It is not a bad idea to state the answer right up front, then show the work to justify your answer. That way, the reader knows what you are trying to justify as they read. It makes their job much easier, and making the reader's job easier should be one of your primary goals! In any case, the answer should be clearly stated somewhere in the writeup, and it should be easy to find.

Be Correct: Of course, everyone makes mistakes as they are working on a problem. But we are talking about after you have solved the problem, when you are writing up your solution to share with someone else. The best writing in the world cannot save a wrong approach and a wrong answer. Check your work carefully. Ask someone else to read your solution with a critical eye.

Justify Your Answer: You cannot simply give an answer and expect your reader to "take your word for it." You have to explain how you know your answer is correct. This means "showing your work," explaining your reasoning, and justifying what you say. You need to answer the question, "How do you know your answer is right?"

Be Concise: There is no bonus prize for writing a lot in math class. Think clearly and write clearly. If you find yourself going on and on, stop, think about what you really want to say, and start over.

Use Variables and Equations: An equation can be much easier to read and understand than a long paragraph of text describing a calculation. Mathematical writing often has a lot fewer words (and a lot more equations) than other kinds of writing.

Define your Variables: If you use variables in the solution of your problem, always say what a variable stands for before you use it. If you use an equation, say where it comes from and why it applies to this situation. Do not make your reader guess!

Use Pictures: If pictures helped you solve the problem, include nice versions of those pictures in your final solution. Even if you did not draw a picture to solve the problem, it still might help your reader understand the solution. And that is your goal!

Use Correct Spelling and Grammar: Proofread your work. A good test is to read your work aloud (this includes reading the equations and calculations aloud). There should be complete, natural-sounding sentences. Be especially careful with pronouns. Avoid using "it" and "they" for mathematical objects; use the names of the objects (or variables) instead.

Format Clearly: Do not write one long paragraph. Separate your thoughts. Put complicated equations on a single displayed line rather than in the middle of a paragraph. Do not write too small. Do not make your reader struggle to read and understand your work.

Acknowledge Collaborators: If you worked with someone else on solving the problem, give them credit!

Here is a problem you've already seen:

Problem 16

Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1's are separated by one digit, the 2's are separated by two digits, the 3's by three digits, and the 4's by four digits.

Think / Pair / Share

Below you will find several solutions that were turned in by students. Using the criteria above, how would you score these solutions on a scale of 1 to 5? Give reasons for your answers.

Solution (Solution 1).

41312432

This is the largest eight-digit b/c the #s 1, 2, 3, 4 & all separated by the given amount of spaces.

Solution (Solution 2).

41312432

You have to have the 4 in the highest place and work down from there. However unable to follow the rules the 2 and the 1 in the 10k and 100k place must switch.

Solution (Solution 3).

41312432

First, I had to start with the #4 because that is the largest digit I could start with to get the largest #. Then I had to place the next 4 five spaces away because I knew there had to be four digits separating the two 4's. Next, I place 1 in the second digit spot because 2 or 3 would interfere with the rule of how many digits could separate them, which allowed me to also place where the next 1 should be. I then placed the 3 because opening spaces showed me that I could fit three digits in between the two 3’s. Lastly, I had to input the final 2’s, which worked out because there were two digits separating them.

Solution (Solution 4).

1×1

2xx2

3xxx3

4xxxx4
Answer: 41312432

Solution (Solution 5).

4 3 2 4 3 2

4 2 2 4

4 1 3 1 4 3

*4 1 3 1 2 4 3 2

4 needs to be the first # to make it the biggest. Then check going down from next largest to smallest. Ex:

4 3 ___________ ×

4 2 ___________×

4 1 ___________✓

Solution (Solution 6).

41312432

I put 4 at the 10,000,000 place because the largest # should be placed at the highest value. Numbers 2 & 3 could not be placed in the 1,000,000 place because I wasn't able to separate the digits properly. So I ended up placing the #1 there. In the 100,000 place I put the #3 because it was the second highest number.

Solution (Solution 7).

41312432

Since the problem asks you for the largest 8 digit #, I knew 4 had to be the first # since it's the greatest # of the set. To solve the rest of the problem, I used the guess and test method. I tried many different combinations. First using the #3 as the second digit in the sequence, but came to no answer. Then the #2, but no combination I found correctly finished the sequence.I then finished with the #1 in the second digit in the sequence and was able to successfully fill out the entire #.

Solution (Solution 8).

4 _ _ _ _ 4 _ _

4 has to be the first digit, for the number to be the largest possible. That means the other 4 has to be the 6th digit in the number, because 4's have to be separated by four digits.

4 _ 3 _ _ 4 3 _

3 must be the third digit, in order for the number to be largest possible. 3 cannot be the second digit because the other 3 would have to be the 6th digit in the number, but 4 is already there.

4 1 3 1 _ 4 3 _

1's must be separated by one digit, so the 1's can only be the 2nd and 4th digit in the number.

4 1 3 1 2 4 3 2

This leaves the 2s to be the 5th and 8th digits.

Solution (Solution 9).

With the active rules, I tried putting the highest numbers as far left as possible. Through trying different combinations, I figured out that no two consecutive numbers can be touching in the first two digits. So I instead tried starting with the 4 then 1 then 3, since I'm going for the highest # possible.

My answer: 41312432

8. The Last Step

A lot of people – from Polya to the writers of the Common Core State Standards and a lot of people in between – talk about problem solving in mathematics. One fact is rarely acknowledged, except by many professional mathematicians: Asking good questions is as valuable (and as difficult) as solving mathematical problems.

After solving a mathematical problem and explaining your solution to someone else, it is a very good mathematical habit to ask yourself: What other questions can I ask?

Example: Squares on a Chess Board

Recall Problem 3, "Squares on a Chess Board":

> How many squares of any possible size are on a standard 8 × 8 chess board? (The answer is not 64! It's a lot bigger!)

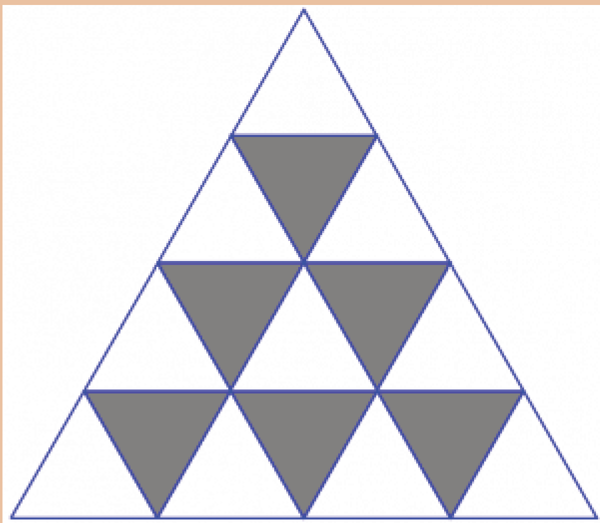
We have already talked about some obvious follow-up questions like "What about a 10 × 10 chess board? Or 100 × 100? Or $n \times n$?"

But there are a lot of interesting (and less obvious . . . and harder) questions you might ask:

- How many rectangles of any size and shape can you find on a standard 8 × 8 chess board? (This is a lot harder, because the rectangles come in all different sizes, like 1 ×

2 and 5 × 3. How could you possibly count them all?)

- How many triangles of any size and shape can you find in this picture?

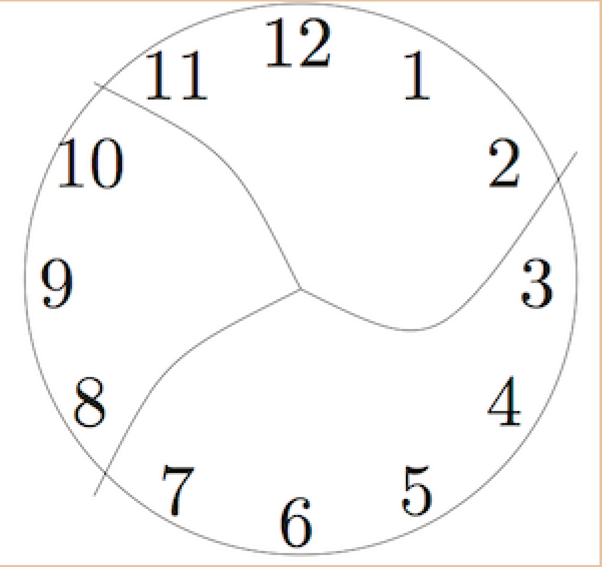


Example: Broken Clock

Recall Problem 4, “Broken Clock”:

> This clock has been broken into three pieces. If you

add the numbers in each piece, the sums are consecutive numbers. Can you break another clock into a different number of pieces so that the sums are consecutive numbers?

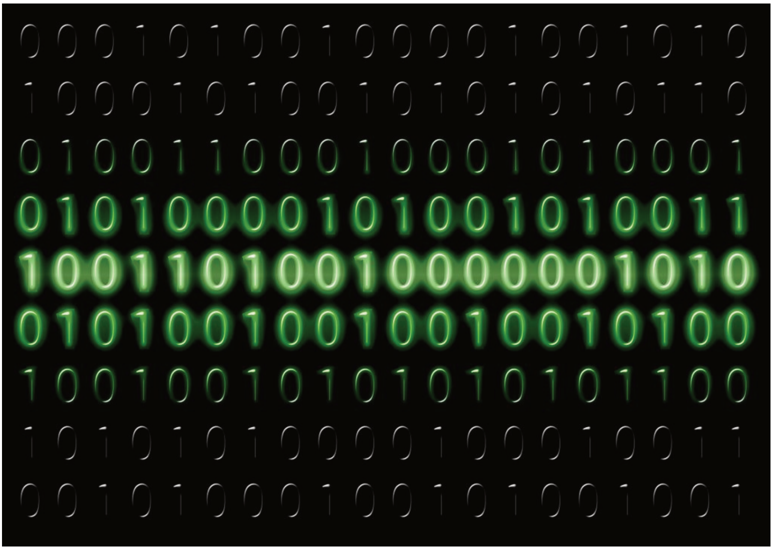


The original problem only asks if you can find one other way. The obvious follow-up question: “Find every possibly way to break the clock into some number of pieces so that the sums of the numbers on each piece are consecutive numbers. Justify that you have found every possibility.”

Think / Pair / Share

Choose a problem from the Problem Bank (preferably a problem you have worked on, but that is not strictly necessary). What follow-up or similar questions could you ask?

PART II
PLACE VALUE



Binary numbers, using just 0's and 1's, are the language of computers.

1

> The idea of expressing all quantities by nine digits

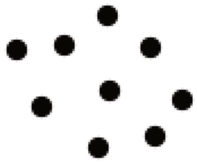
1. Images and Videos on Pixabay are released under Creative Commons CC0.

> whereby is imparted to them both an absolute value and a value of position is so simple that this very simplicity is the very reason for our not being sufficiently aware how much admiration it deserves.
> -Laplace

The “Dots and Boxes” approach to place value used in this part (and throughout this book) comes from James Tanton, and is used with his permission. See his development of these and other ideas at http://gdaymath.com/.

9. Dots and Boxes

Here are some dots; in fact there are nine of them:



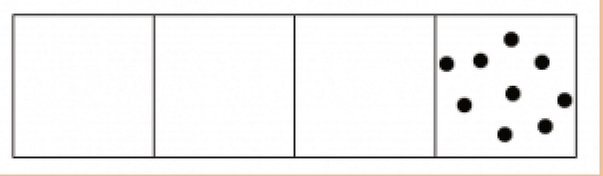
We're going to a play an "exploding dots" game. Here's the only rule for the game:

The 1←2 Rule

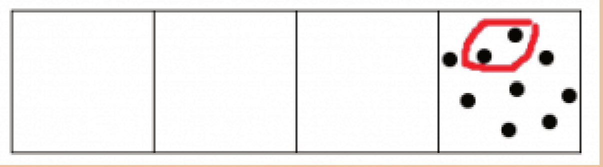
Whenever there are two dots in single box, they "explode," disappear, and become one dot in the box to the left.

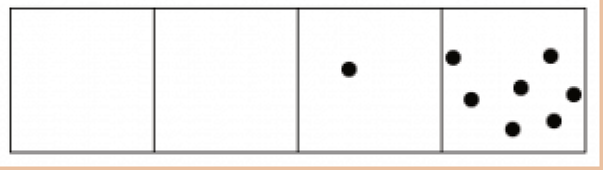
Example: Nine dots 1←2 in the system

We start by placing nine dots in the rightmost box.

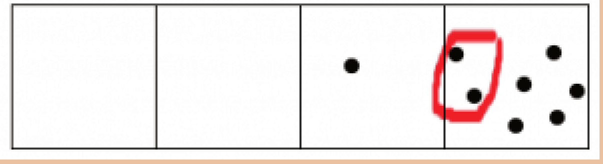


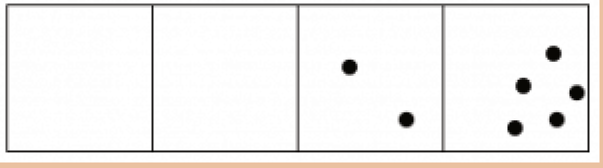
Two dots in that box explode and become one dot in the box to the left.



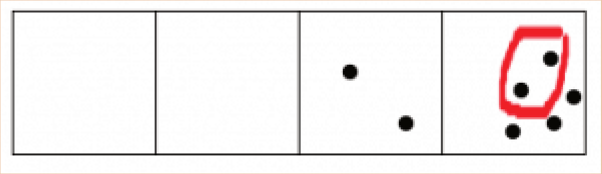


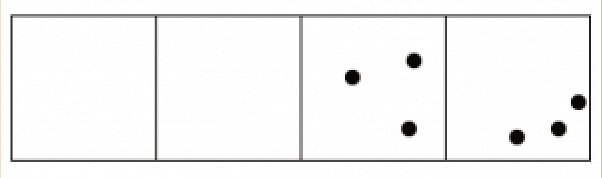
Once again, two dots in that box explode and become one dot in the box to the left.



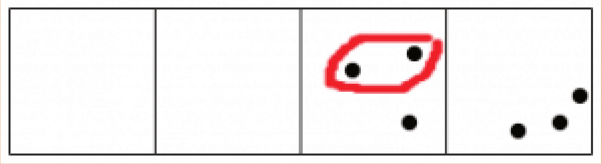


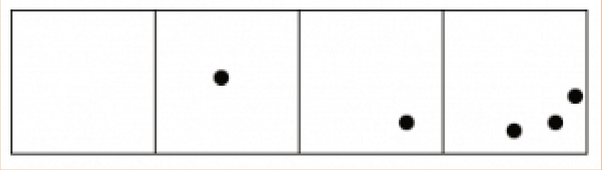
We do it again!



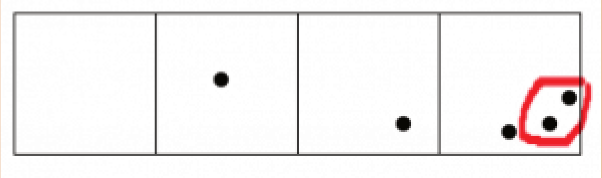


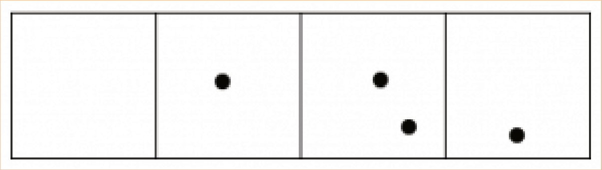
Hey, now we have more than two dots in the second box, so those can explode and move!



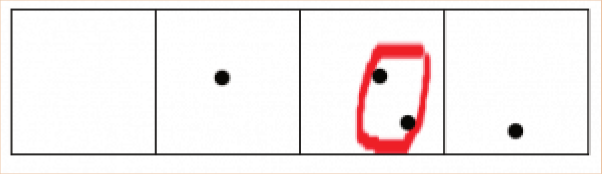


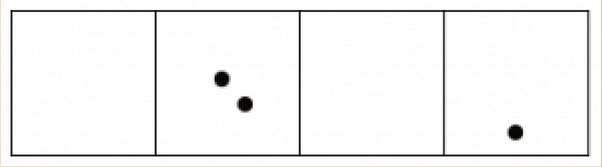
And the rightmost box still has more than two dots.

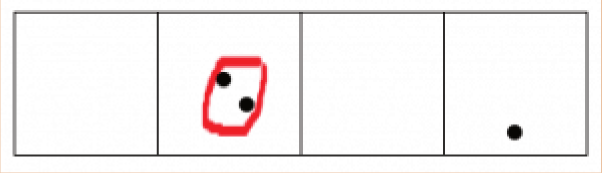


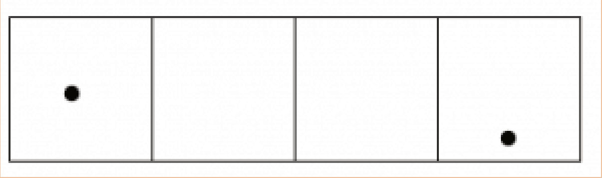


Keep going, until no box has two dots.





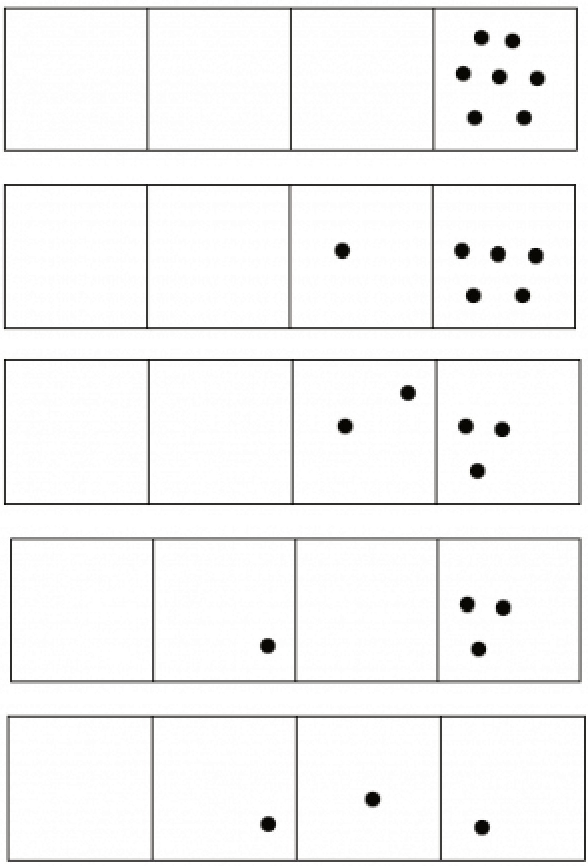




After all this, reading from left to right we are left with one dot, followed by zero dots, zero dots, and one final dot.

Solution: The 1←2 code for nine dots is: 1001.

On Your Own. Here's a diagram showing what happens for seven dots in a 1←2 box. Trace through the diagram, and circle the pairs of dots that "exploded" at each step.



Solution: The 1←2 code for seven dots is: 111.

Problem 1

Note: In solving this problem, you don't need to draw on paper; that can get tedious! Maybe you could use buttons or pennies for dots and do this by hand.

1. Draw 10 dots in the right-most box and perform the explosions. What is the 1←2 code for ten dots?
2. Find the 1←2 code for eighteen dots.
3. What number of dots has 1←2 code 101?

Think / Pair / Share

After you worked on the problem, compare your answer with a partner. Did you both get the same code? Did you have the same process?

10. Other Rules

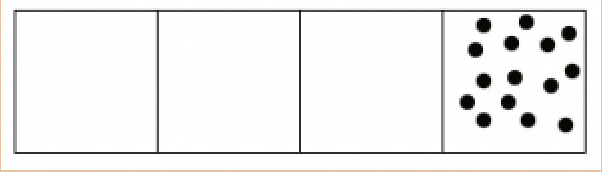
Let's play the dots and boxes game, but change the rule.

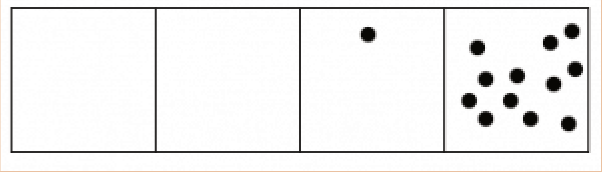
The 1←3 Rule

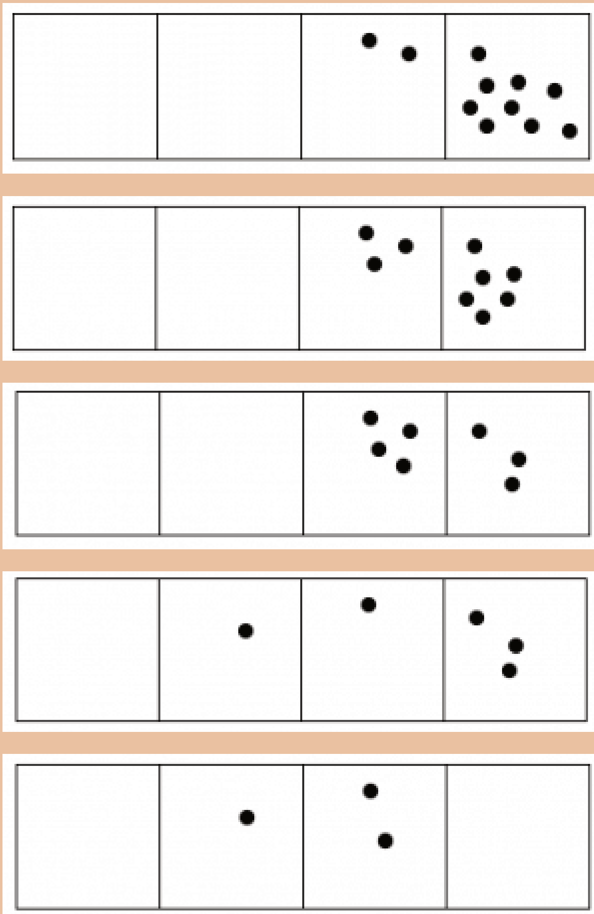
Whenever there are three dots in single box, they "explode," disappear, and become one dot in the box to the left.

Example: Fifteen dots in the 1←3 system

Here's what happens with fifteen dots:







Solution: The 1←3 code for fifteen dots is: 120.

Problem 2

1. Show that the 1←3 code for twenty dots is 202.
2. What is the 1←3 code for thirteen dots?
3. What is the 1←3 code for twenty-five dots?
4. What number of dots has 1←3 code 1022?
5. Is it possible for a collection of dots to have 1←3 code 2031? Explain your answer.

Problem 3

1. Describe how the 1←4 rule would work.
2. What is the 1←4 code for thirteen dots?

Problem 4

1. What is the 1←5 code for the thirteen dots?
2. What is the 1←5 code for five dots?

Problem 5

1. What is the 1←9 code for thirteen dots?
2. What is the 1←9 code for thirty dots?

Problem 6

a. What is the 1←10 code for thirteen dots?
b. What is the 1←10 code for thirty-seven dots?
c. What is the 1←10 code for two hundred thirty-eight dots?

d. What is the 1←10 code for five thousand eight hundred and thirty-three dots?

Think / Pair / Share

After you have worked on the problems on your own, compare your ideas with a partner. Can you describe what's going on in Problem 6 and why?

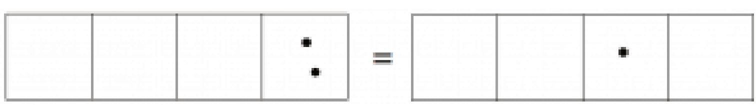
11. Binary Numbers

Let's go back to the 1←2 rule and examine what's really going on.

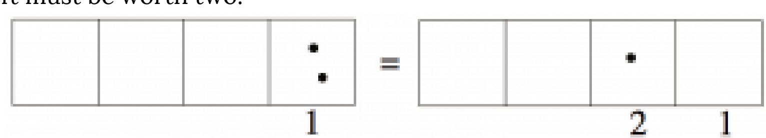
The 1←2 Rule

Whenever there are two dots in single box, they "explode," disappear, and become one dot in the box to the left.

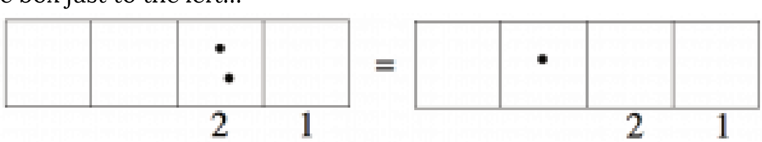
Two dots in the right-most box is worth one dot in the next box to the left.



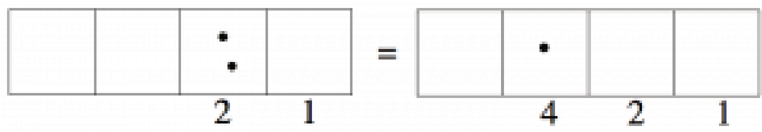
If each of the original dots is worth "one," then the single dot on the left must be worth two.



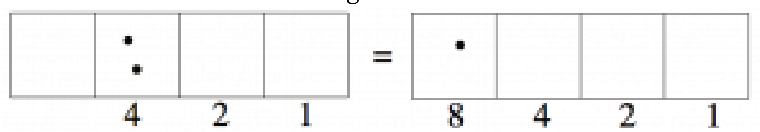
But we also have two dots in the box of value 2 is worth one dot in the box just to the left…



So that next box must be worth two 2's, which is four!



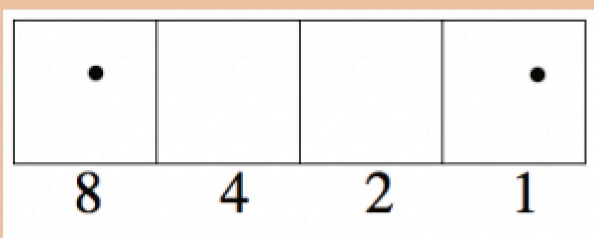
And two of these fours make eight.



Examples

Example 1: Nine dots in the 1←2 system revisited.

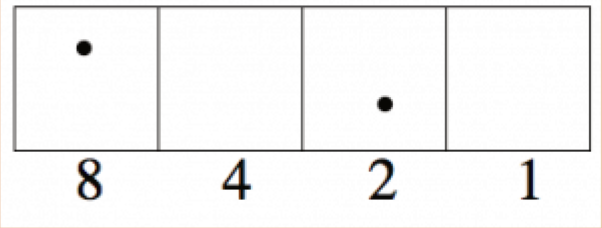
We said earlier that the 1←2 code for nine dots was 1001. Let's check:



$8+1=9$ so this works!

Example 2: Ten dots in the 1←2 system revisited.

You should have found that ten dots has 1←2 code 1010.



Yep! $8 + 2 = 10$.

Problem 7

1. If there were a box to the left of the 8 box, what would the value of that box be?
2. What would be the value of a box *two* spots to the left of the 8 box? Three spots to the left?
3. What number has 1←2 code 100101?
4. What is the 1←2 code for two hundred dots?

Definition and Notation

Numbers written in the 1←2 code are called **binary numbers** or **base two numbers**. (The prefix "bi" means "two.")

From now on, when we want to indicate that a number is written in base two, we will write a subscript "two" on the number.

So 1001_{two} means "the number of dots that has 1←2 code 1001," which we already saw was nine.

Important! When we read we say "one zero zero one base two." We don't say "one thousand and one," because "thousand" is not a binary number.

Think / Pair / Share

1. Your first goal: come up with a *general method* to find the number of dots represented by any binary number. Clearly describe your method. Test your method out on these numbers, and check your work by actually "unexploding" the dots.

   1_{two} 101_{two} 1011_{two} 11111_{two}
2. Explain why binary numbers only contain the

digits 0 and 1.

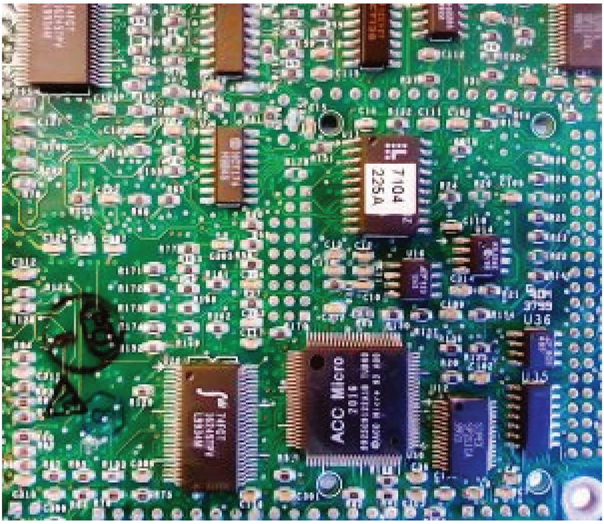
3. Here is a new (harder) goal: come up with a *general method* to find the binary number related to any number of dots *without actually going through the "exploding dot" process*. Clearly describe your method. Test your method out on these numbers, and find a way to check your work.

two dots = $\_\_\__{two}$ seventeen dots = $\_\_\__{two}$

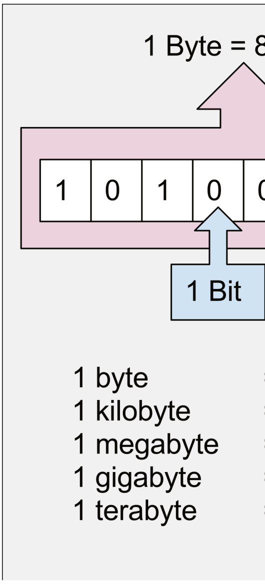
sixty-three dots = $\_\_\__{two}$ one hundred dots = $\_\_\__{two}$

History

You probably realize by now that a number is an abstract concept with many representations. The standard decimal representation of a number is only one of these. For computers, numbers are always represented in binary. The basic units are transistors which are either on (1) or off (0).



Electronic circuit board from Samsung Galaxy S III.



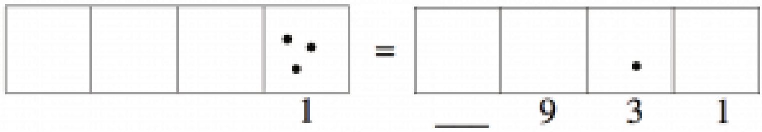
A transistor[1] is said to store ***one bit*** of information. Eight bits make a byte and a typical home computer's central processing unit performs computations on registries that are each 8 bytes (64-bits).

Using the 1←2 rule we can represent the numbers 0 through 18,446,744,073,709,551,615 with 64 bits.

1. Image of circuit board from http://www.publicdomainpictures.net/, licensed under CC0 Public Domain.

12. Other Bases

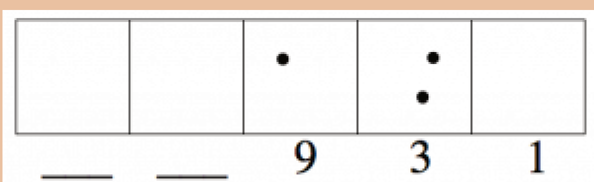
In the 1←3 system, three dots in one box is worth one dot in the box one spot to the left. This gives a new picture:



Each dot in the second box from the left is worth three ones. Each dot in the third box is worth three 3's, which is nine, and so on.

Example

We said that the 1←3 code for fifteen is 120. We see that this is correct because:



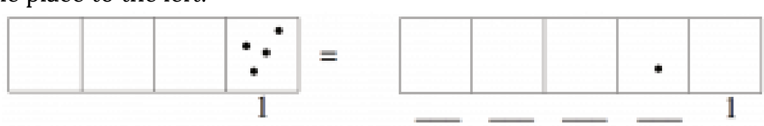
$$1 \cdot 9 + 2 \cdot 3 + 0 \cdot 1 = 15.$$

Problem 8

Answer these questions about the 1←3 system.

1. What label should go on the box to the left of the 9 box?
2. What would be the value of a box two spots to the left of the 9 box?
3. What number has 1←3 code 21002?
4. What is the 1←3 code for two hundred dots?

In the 1←4 system, four dots in one box are worth one dot in the box one place to the left.



Problem 9

Answer these questions about the 1←4 system.

1. What is the value of each box in the picture above?
2. What is the 1←4 code for twenty-nine dots?
3. What number has 1←4 code 132?

Problem 10

In the 1←10 system, ten dots in one box are worth one dot in the box one place to the left.

1. Draw a picture of the 1←10 and label the first four boxes with their values.
2. What is the 1←10 code for eight thousand four hundred and twenty-two?
3. What number has 1←10 code 95,753?
4. When we write the number 7,842, what does the "7" represent?
   The "4" is four groups of what value?
   The "8" is eight groups of what value?
   The "2" is two groups of what value?
5. Why do you think we use the 1 ← 10 system for writing numbers?

Definition

Recall that numbers written in the $1 \leftarrow 2$ system are called **binary** or **base two** numbers.

Numbers written in the $1 \leftarrow 3$ system are called **base three** numbers.

Numbers written in the $1 \leftarrow 4$ system are called **base four** numbers.

Numbers written in the $1 \leftarrow 10$ system are called **base ten** numbers.

In general, numbers written in the $1 \leftarrow b$ system are called **base *b*** numbers.

In a base b number system, each place represents a power of b, which means b^n for some whole number n. Remember this means b multiplied by itself n times:

- The right-most place is the units or ones place. (Why is this a power of b?)
- The second spot is the "b" place. (In base ten, it's the tens place.)
- The third spot is the "b^2" place. (In base ten, that's the hundreds place. Note that $10^2 = 100$.)
- The fourth spot is the "b^3" place. (In base ten, that's the thousands place, since $10^3 = 1000$.)
- And so on.

Notation

Whenever we're dealing with numbers written in different bases, we use a subscript to indicate the base so that there can be no confusion. So:

- 102_{three} is a base three number (read it as "one-zero-two base three"). This is the base three code for the number eleven.
- 222_{four} is a base four number (read it as "two-two-two base four"). This is the base four code for the number forty-two.
- 54321_{ten} is a base ten number. (It's ok to say "fifty-four thousand three hundred and twenty-one." Why?)

If the base is not written, we assume it's base ten.

Remember: when you see the subscript, you are seeing the **code** for some number of dots.

Think / Pair / Share

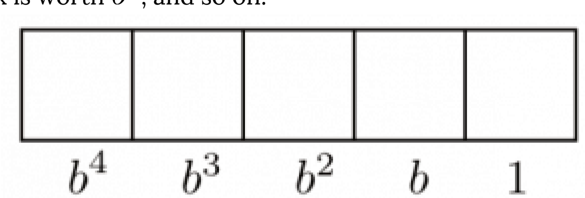
1. Find the number of dots represented by each of these:
   222_{three} $\quad$ 310_{four} $\quad$ 54321_{ten}.

2. Represent nine dots in each base:
   three, five, eight, nine, and eleven.
3. Which digits are used in the base two system? The base three system? The base four system? The base five system? The base six system? The base ten system?
4. What does the *base* tell you about the number system? (Think of as many answers as you can!)

Base b to Base Ten

We're now going to describe some general methods for converting from base b to base ten, where b can represent any whole number bigger than one.

If the base is b, that means we're in a $1 \leftarrow b$ system. A dot in the right-most box is worth 1. A dot in the second box is worth b. A dot in the third box is worth b^2, and so on.



So, for example, the number 10123_b represents

$$1 \cdot b^4 + 0 \cdot b^3 + 1 \cdot b^2 + 2 \cdot b + 3 \cdot 1,$$

because we imagine three dots in the right-most box (each worth one), two dots in the second box (each representing b dots), one dot in the third box (representing b^2 dots), and so on. That means we can just do a short calculation to find the total number of dots, without going through all the trouble of drawing the picture and "unexploding" the dots.

Examples

Example 1: Consider the number 123_{five}.

This represents the number

$$1 \cdot 5^2 + 2 \cdot 5 + 3 = 25 + 10 + 3 = 38.$$

Example 2: Consider the number 123_{seven}.

$$1 \cdot 7^2 + 2 \cdot 7 + 3 = 49 + 14 + 3 = 66.$$

Base Ten to Base b

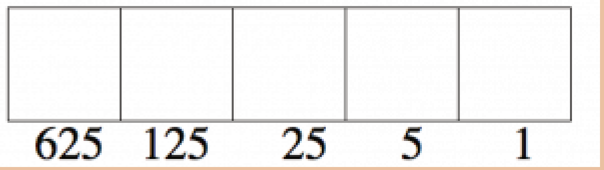
We're now going to describe some general methods for converting from base ten to base *b*, where *b* can represent any whole number bigger than one.

There are two general methods for doing these conversions. For each method, we'll work out an example, and then describe the general method. The first method we describe fills in the boxes from left to right.

Example: Method 1 (left-to-right)

To convert 321_{ten} to a base five number (without actually going through the tedious process of exploding dots in groups of five).

Find the largest power of five that is smaller than 321. We'll just list powers of five:

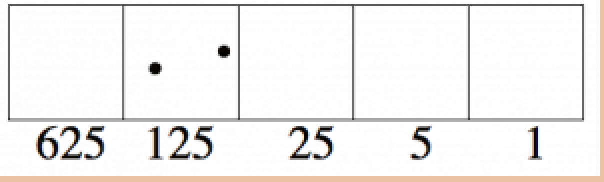


So we know that the left-most box we'll use is the 125 box, because 625 is too big.

How many dots will be in the 125 box? That's the same as asking how many 125's are in 321. Since

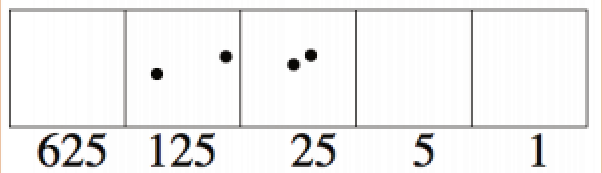
$$2 \cdot 125 = 250 \quad \text{and} \quad 3 \cdot 125 = 375,$$

we should put two dots in the 125 box. Three dots would be too much.



How many dots are left unaccounted for? $321 - 250 = 71$ dots are left.

Now repeat the process: The largest power of five that's less than 71 is $5^2 = 25$. If we put two in the 25 box, that takes care of 50 dots. (Three dots would be 75, which is too much.)

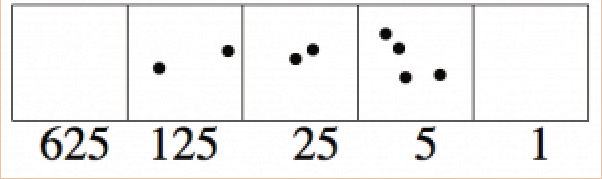


So far we have two dots in the 5^3 box and two dots in the 5^2 box, so that's a total of

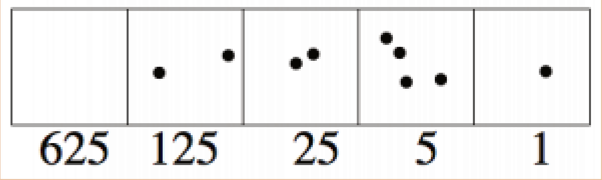
$$2 \cdot 125 + 2 \cdot 25 = 300 \text{ dots.}$$

We have $321 - 300 = 21$ dots left to account for.

Repeat the process again: The biggest power of 5 that's less than 21 is 5. How many dots can go in the 5 box? $5 \cdot 4 = 20$, so we can put four dots in the 5 box.



We have one dot left to account for. If we put one dot in the 1 box, we're done.



$$2 \cdot 125 + 2 \cdot 25 + 4 \cdot 5 + 1 = 250 + 50 + 20 + 1 = 321.$$

So $321_{\text{ten}} = 2241_{\text{five}}$.

The general algorithm to convert from base ten to base :

1. Start with your base ten number *n*. Find the largest *power of b* that's less than *n*, say that power is b^k.
2. Figure out how many dots can go in the b^k box without going over *n*. Say that number is *a*. Put the digit *a* in the b^k box, and then subtract $n - a \cdot b^k$ to figure out how many dots are left.
3. If your number is now zero, you've accounted for all the dots. Put zeros in any boxes that remain, and you have the number. Otherwise, start over at step (1) with the number of dots you have left.

The method is a little tricky to describe in complete generality. It's probably better to try a few examples on your own to get the hang of it.

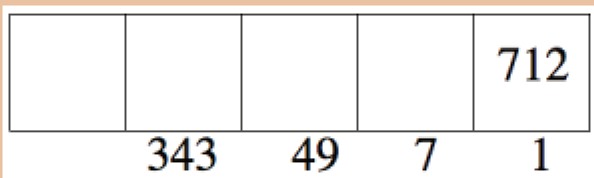
Think / Pair / Share

Use the method above to convert 99_{ten} to base three, to base four, and to base five.

Here's another method to convert base ten numbers to another base, and this method fills in the digits from right to left. Again, we'll start with an example and then describe the general method.

Example: Method 2 (right-to-left)

To convert 712_{ten} to a base seven number, imagine there are 712 dots in the ones box. We'll write the number, but imagine it as dots.

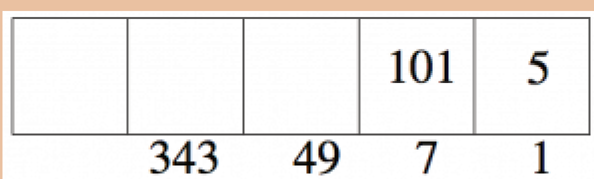


Find out how many groups of 7 you can make, and how many dots would be left over.

$$712 \div 7 = 101 \text{ R}5; \quad \text{that is, } 712 = 101 \cdot 7 + 5.$$

That means we have 101 groups of 7 dots, with 5 dots left over.

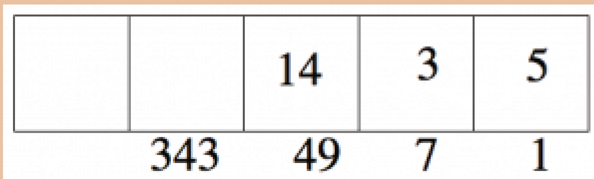
"Explode" the groups of 7 one box to the left, and leave the 5 dots behind.



Now repeat the process: How many groups of 7 can you make from the 101 dots?

$$101 \div 7 = 14 \text{ R}3, \quad \text{meaning } 101 = 14 \cdot 7 + 3.$$

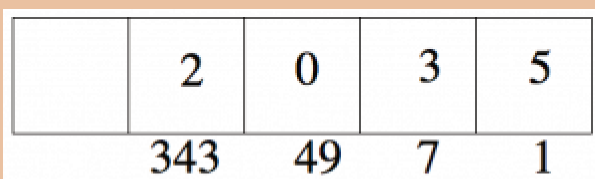
"Explode" the groups of 7 one box to the left, and leave the 3 dots behind.



Repeat:

$$14 \div 7 = 2 \text{ R}0, \quad \text{so } 14 = 2 \cdot 7 + 0.$$

"Explode" the groups of 7 one box to the left, and leave 0 dots behind.



Since there are fewer than 7 dots in each box, we're done.

$$712_{\text{ten}} = 2035_{\text{seven}}.$$

Of course, we can (and should!) check our calculation by converting the answer back to base ten:

$$2035_{\text{seven}} = 2 \cdot 7^3 + 0 \cdot 7^2 + 3 \cdot 7 + 5 = 686 + 0 + 21 + 5 = 712_{\text{ten}}.$$

So here's a second general method for converting base ten numbers to an arbitrary base *b*:

1. Divide the base ten number by b to get a quotient and a remainder.
2. Put the remainder in the right-most space in the base b number.
3. If the quotient is less than b, it goes in the space one spot to the left. Otherwise, go back to step (1) and repeat it with the quotient, filling in the remainders from right to left in the base number.

Again, the method probably makes more sense if you try it out a few times.

Think / Pair / Share

Use the method described above to convert 250_{ten} to base three, four, five, and six.

13. Number Systems

Our number system is a western adaptation of the Hindu-Arabic numeral system developed somewhere between the first and fourth centuries AD. However, numbers have been recorded with tally marks throughout history. The Ishango Bone[1] from Africa is about 25,000 years old. It's the lower leg bone from a baboon, and contains tally marks. We know the marks were used for counting because they appear in distinct groups.

1. Image of Ishango bone by Ben2 (Own work), CC-BY-SA-3.0 or CC BY-SA 2.5-2.0-1.0], via Wikimedia Commons.

This reindeer antler[2] from France is about 15,000 years old, and also shows clearly grouped tally marks.

2. Image of antler By Ryan Somma from Occoquan, USA [CC BY-SA 2.0], via Wikimedia Commons

Of course, we still use tally marks today![3]

3. Image of Hanakapiai beach warning sign by God of War at the English language Wikipedia [GFDL or CC-BY-SA-3.0], via Wikimedia Commons.

Base ten numbers (the ones you have probably been using your whole life), and base *b* numbers (the ones you've been learning about in this chapter) are both positional number systems.

Definition

A **positional number system** is one way of writing numbers. It has unique symbols for 1 through $b - 1$, where b is the base of the system. Modern positional number systems also include a symbol for 0.

The **positional value** of each symbol depends on its position in the number:

- The positional value of a symbol in the first position is just its face value.
- The positional value of a symbol in the second position is b times its value.
- The positional value of a symbol in the third position is b^2 times its value.
- And so on.

The value of a number is the sum of the positional values of its digits.

Definition

In an **additive number system,** the value of a written number is the sum of the face values of the symbols that make up the number. The only symbol necessary for an additive number system is a symbol for 1, however many additive number systems contain other symbols.

History: Roman numerals

The ancient Romans used a version of an additive number systems. The Romans represented numbers this way:

number	**Roman Numeral**
1	I
5	V
10	X
50	L
100	C
500	D
1,000	M

So the number 2013 would be represented as MMXIII. This is read as 2,000 (two M's), one ten (one X), and three ones (three I's).

For any additive number system very large numbers become impractical to write. To represent the number one million in Roman numerals it would take one thousand M's!

However, the Roman numerals did have one efficiency advantage: The order of the symbols mattered. If a symbol to the left was smaller than the symbol to the right, it would be subtracted instead of added. So for example nine is represented as IX rather than VIIII.

Think / Pair / Share

If you don't already know how to use Roman numerals, research it a little bit. Then answer these questions.

- Write the numbers 1–20 in Roman numerals.
- What is the maximum number of symbols needed to write any number between 1 and 1,000 in Roman Numerals? Justify your answer.

The earliest positional number systems are attributed to the Babylonians (base 60) and the Mayans (base 20). These positional systems were both developed before they had a symbol or a clear concept for zero. Instead of using 0, a blank space was used to indicate skipping a particular place value. This could lead to ambiguity.

Suppose we didn't have a symbol for 0, and someone wrote the number

$$2\quad 3.$$

It would be impossible to tell if they mean 23, 203, 2003, or maybe two separate numbers (two and three).

Leonardo Pisano Bigollo, more commonly known as **Fibonacci**[4], played a pivotal role in guiding Europe out of a long period in which the importance and development of math was in marked decline. He was born in Italy around 1170 CE to Guglielmo Bonacci, a successful merchant. Guglielmo brought his son with him to what is now Algeria, and Leonardo was educated in mathematics mathematics there.

Fibonacci

4. Image of Fibonacci via Wikimedia Commons, in the public domain.

At the time, Roman Numerals dominated Europe, and the official means of calculations was the abacus. Muḥammad ibn Mūsā al-Khwārizmī[5] described the use of Hindu-Arabic system in his book *On the Calculation with Hindu Numerals* in 825 CE, but it was not well-known in Europe.

5. Image of al-Khwarizmi statue by M. Tomczak [CC BY-SA 3.0], via Wikimedia Commons.

Statue of al-Khwarizmi at Amirkabir University of Technology

Fibonacci's book *Liber Abaci* described the Hindu-Arabic system and its business applications for a European readership. His book was well-received throughout Europe, and it marked the beginning of a reawakening of European mathematics.

History: Hawaiian numbers

The Hindu-Arabic number system is now used nearly exclusively throughout the globe. But many cultures had their own number systems before contact and trade with other countries spread the work of al-Khwārizmī throughout the world.

There is evidence that pre-contact Hawaiians actually used two different number systems. Depending on what they were counting, they might use base 4 instead (or a mixed base-10 and base-4 system). One theory is that certain objects (fish, taro, etc.) were often put in bundles of 4, so were more natural to count by 4's than by 10's. The number four also had spiritual significance in Hawaiian culture.

Humans have 5 fingers on
But there are 4 gaps betw
taro plants or similar obj

In the mixed base system, instead of powers of 10, numbers are broken down into sums of numbers that look like 4 times a power of 10 (40, 400, 4000, etc.).

6. Image of hand from https://pixabay.com, licensed under CC0 Creative Commons.

1	‘ekahi
2	‘elua
3	‘ekolu
4	‘ehā (or kauna)
5	‘elima
6	‘eono
7	‘ehiku
8	‘ewalu
9	‘eiwa
10	‘umi
11–19	‘umi kumamā {kahi, lua, kolu, hā, etc.}
20	iwakālua
21–29	Iwakālua kumamā {kahi, lua, kolu, hā, etc.}
30	kanakolu
31–39	kanakolu kumamā {kahi, lua, etc.}
40	kanahā
400	lau
4,000	mano
40,000	kini
400,000	lehu

Here are a few examples (refer to the table above for the Hawaiian names of the numbers):

Example

‘ekolu kini, ‘ewalu lau me ‘ekahi

translates to three 40,000’s, eight 400’s, and one;

$3 \cdot 40000 + 8 \cdot 400 + 1 = 123201$

Example

$5207 = 1 \cdot 4000 + 3 \cdot 400 + 7$

would be ‘ekahi mano, ‘ekolu lau me ‘ehiku

On Your Own

Work on the following exercises on your own or with a partner.

1. Translate this Hawaiian number to English and then write it in base ten.

‘ekahi kanahā me kanakolu kumamāiwa

2. Translate this base-ten number to Hawaiian.

1,573

Think / Pair / Share

How is learning about different number systems (including representing numbers in different bases) valuable to you as a future teacher?

14. Even Numbers

How do we know if a number is even? What does it mean?

Definition

Some number of dots is **even** if I can divide the dots into pairs, and every dot has a partner.

Some number of dots is **odd** if, when I try to pair up the dots, I always have a single dot left over with no partner.

The number of dots is either even or odd. It's a property of the quantity and is doesn't change when you represent that quantity in different bases.

Problem 13

Which of these numbers represent an even number of dots? Explain how you decide.

22_{ten} 319_{ten} 133_{five} 222_{five} 11_{seven} 11_{four}

Think / Pair / Share

Compare your answers to problem 13 with a partner. Then try these together:

1. Count by twos to 20_{ten}.
2. Count by twos to 30_{four}.
3. Count by twos to 51_{seven}.

You know that you can tell if a base ten number is even just by looking at the ones place. But why is that true? That's not the definition of an even number. There are a few key ideas behind this handy trick:

- In base ten, every number looks like

  (some multiple of ten) + (ones digit)

  $$53 = 50 + 3$$
  $$492 = 490 + 2$$
  $$45637289108 = 45637289100 + 8$$

- Every multiple of ten is an even number, since

  $$10n = 2(5n),$$

  and two times a whole number is always even.
- Your whole number looks like this:

  (some multiple of ten) + (ones digit)

  (even number) + (ones digit),
- Even plus even is even, and even plus odd is odd, so your whole number is even when the ones digit is even, and it's odd when

the ones digit is odd.

Think / Pair / Share

- Make sure you understand the explanation above. Does each piece make sense to you?
- In particular: Use the definition of even and odd above to explain the last step. Why is it true that even + even = even and even + odd = odd?
- What about odd + odd? Is that odd or even? Justify what you say.

Problem 14

1. Write the numbers zero through fifteen in base seven:

base ten	base seven
0	0_{seven}
1	1_{seven}
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	

2. Circle all of the even numbers in your list. How do you know they are even?

3. Find a rule: how can you tell if a number is even when it's written in base seven?

Problem 15

1. Write the numbers zero through fifteen in base four:

base ten	base four
0	0_{four}
1	1_{four}
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	

2. Circle all of the even numbers in your list. How do you know they are even?

3. Find a rule: how can you tell if a number is even when it's written in base four?

Think / Pair / Share

- Why are the rules for recognizing even numbers different in different bases?
- For either your base four rule or your base seven rule, can you explain *why* it works that way?

15. Problem Bank

Problem 28

1. If you were counting in base four, what number would you say just before you said 100_{four}?
2. What number is one more than 133_{four}?
3. What is the greatest three-digit number that can be written in base four? What numbers come just before and just after that number?

Problem 29

Explain what is wrong with writing 313_{two} or 28_{eight}.

Problem 30

1. Write out the base three numbers from 1_{three} to 200_{three}.
2. Write out the base five numbers from 1_{five} to 100_{five}.
3. Write the four base six numbers that come after 154_{six}.

Problem 31

Convert each base ten number to a base four number. Explain how you did it.

$$13, \qquad 8, \qquad 24, \qquad 49$$

Challenges:

$$0.125, \qquad 0.11111\cdots = 0.\overline{1}$$

Problem 32

In order to use base sixteen, we need sixteen digits – they will represent the numbers zero through fifteen. We can use our usual digits 0–9, but we need *new symbols* to represent the *digits* ten, eleven, twelve, thirteen, fourteen, and fifteen. Here's one standard convention:

base ten	base sixteen
7	7_{sixteen}
8	8_{sixteen}
9	9_{sixteen}
10	A_{sixteen}
11	B_{sixteen}
12	C_{sixteen}
13	D_{sixteen}
14	E_{sixteen}
15	F_{sixteen}
16	10_{sixteen}

1. Convert these numbers from base sixteen to base ten, and show your work:

   $6D_{\text{sixteen}}$ $\quad$ AE_{sixteen} $\quad$ $9C_{\text{sixteen}}$ $\quad$ $2B_{\text{sixteen}}$

2. Convert these numbers from base ten to base sixteen, and show your work:

97	144	203	890

Problem 33

How many different symbols would you need for a base twenty-five system? Justify your answer.

Problem 34

All of the following numbers are multiples of three.

3, 6, 9, 12, 21, 27, 33, 60, 81, 99.

1. Identify the *powers of* 3 in the list. Justify your answer.
2. Write each of the numbers above in base three.
3. In base three: how can you recognize a *multiple of* 3? Explain your answer.
4. In base three: how can you recognize a *power of* 3?

Explain your answer.

Problem 35

All of the following numbers are multiples of five.

5, 10, 15, 25, 55, 75, 100, 125, 625, 1000.

1. Identify the *powers of* 5 in the list. Justify your answer.
2. Write each of the numbers above in base five.
3. In base five: how can you recognize a *multiple of* 5? Explain your answer.
4. In base five: how can you recognize a *power of* 5? Explain your answer.

Problem 36

Convert each number to the given base.

1. 395_{ten} into base eight.
2. 52_{ten} into base two.
3. 743_{ten} into base five.

Problem 37

What bases makes theses equations true? Justify your answers.

1. $35 = 120_{\_\_\_}$
2. $41_{\text{six}} = 27_{\_\_\_}$
3. $52_{\text{seven}} = 34_{\_\_\_}$

Problem 38

What bases makes theses equations true? Justify your answers.

1. $32 = 44\_\_$
2. $57_{\text{eight}} = 10\_\_$
3. $31_{\text{four}} = 11\_\_$
4. $15_x = 30_y$

Problem 39

1. Find a base ten number that is twice the product of its two digits. Is there more than one answer? Justify what you say.
2. Can you solve this problem in any base other than ten?

Problem 40

1. I have a four-digit number written in base ten. When I multiply my number by four, the digits get reversed. Find the number.
2. Can you solve this problem in any base other than ten?

Problem 41

Convert each base four number to a base ten number. Explain how you did it.

13_{four} $\quad 322_{\text{four}}$ $\quad 101_{\text{four}}$ $\quad 1300_{\text{four}}$

Challenges:

0.2_{four} $\quad 0.111\ldots_{\text{four}} = 0.\overline{1}_{\text{four}}$

Problem 42

Consider this base ten number (I got this by writing the numbers from 1 to 60 in order next to one another):

$$12345678910111213\ldots57585960.$$

1. What is the largest number that can be produced by erasing one hundred digits of the number? (When you erase a digit it goes away. For example, if you start with the number 12345 and erase the middle digit, you produce the number 1245.) How do you *know* you got the largest possible number?
2. What is the smallest number that can be produced by erasing one hundred digits of the number? How do you *know* you got the smallest possible number?

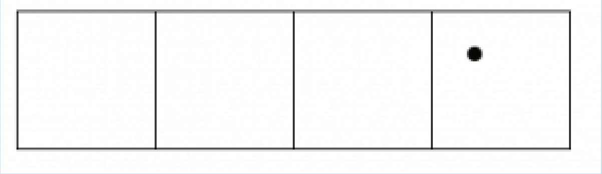
Problem 43

Can you find two different numbers (not necessarily single digits!) a and b so that $a_b = b_a$? Can you find more than one solution? Justify your answers.

16. Exploration

Problem 44

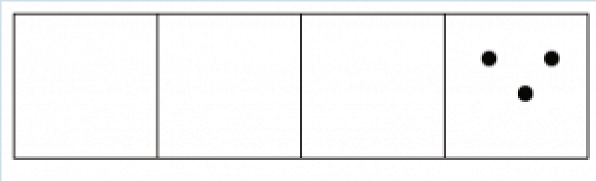
Jay decides to play with a system that follows a 1←1 rule. He puts one dot into the right-most box. What happens?



Problem 45

Poindexter decides to play with a system that follows the rule 2←3.

1. Describe what this rule does when there are three dots in the right-most box.



2. Draw diagrams or use buttons or pennies to find the codes for the following numbers:

$$1 \text{ through } 20, 24, 27, 30, 33, 36, \text{ and } 39$$

3. Can you find (and *explain*) any patterns?

Problem 46

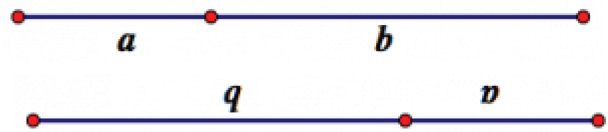
Repeat problem 45 for your own rule. Choose two numbers $a \neq 1$ and b. For each of the numbers

$$1 \text{ through } 20, 24, 27, 30, 33, 36, \text{ and } 39,$$

figure out the $a \leftarrow b$ code. Look for patterns, and explain them if you can!

PART III

NUMBER AND OPERATIONS



$a + b = b + a$

> The essence of mathematics is not to make simple things complicated, but to make complicated things simple.
> -S. Gudder

The “Dots and Boxes” approach to understanding operations used in this part (and throughout this book) comes from James Tanton, and is used with his permission. See his development of these and other ideas at http://gdaymath.com/.

17. Introduction

When learning and teaching about arithmetic, it helps to have mental and physical *models* for what the operations mean. That way, when you are presented with an unfamiliar problem or a question about why something is true, you can often work it out using the model – this might mean drawing pictures, using physical materials (manipulatives), or just thinking about the model to help you reason out the answer.

Think / Pair / Share

Write down your mental models for each of the four basic operations. What do they actually *mean*? How would you explain them to a second grader? What pictures could you draw for each operation? Think about each one separately, as well as how they relate to each other:

- addition
- subtraction
- multiplication, and
- division.

After writing down you own ideas, share them with a partner. Do you and your partner have the same models for each of the operations or do you think about them differently?

Teachers should have lots of mental models – lots of ways to explain the same concept. In this chapter, we'll look at some different ways to understand the four basic arithmetic operations. First, let's define some terms:

Definition

Counting numbers are literally the numbers we use for counting: 1, 2, 3, 4, 5… These are sometimes called the *natural numbers* by mathematicians, and they are represented by the symbol $\mathbb{N}$.

Whole numbers are the counting numbers together with zero.

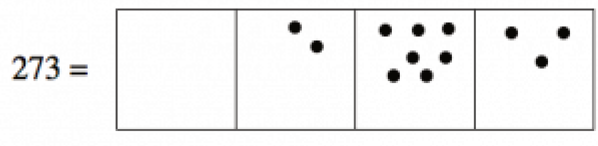
Integers include the positive and negative whole numbers, and mathematicians represent these with the symbol $\mathbb{Z}$. (This comes from German, where the word for "number" is "zählen.")

We already have a natural model for thinking about counting numbers: a number is a quantity of dots. Depending on which number system you use – Roman numerals, base ten, binary, etc. – you might write down the number in different ways. But the quantity of dots is a counting number, however you write it down.

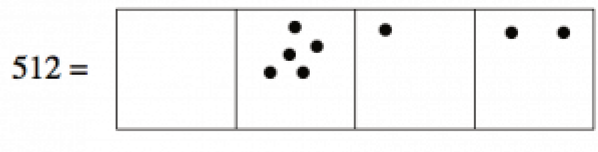
18. Addition: Dots and Boxes

Addition as combining

For now, we'll focus on the base-10 system. Here's how we think about the number 273 in that system:

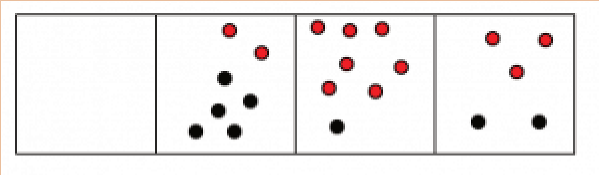


And here is the number 512:



Example: 273+512

We can add these in the natural way: just combine the piles of dots. Since they're already in place-value columns, we can combine dots from the two numbers that are in the same place-value box.



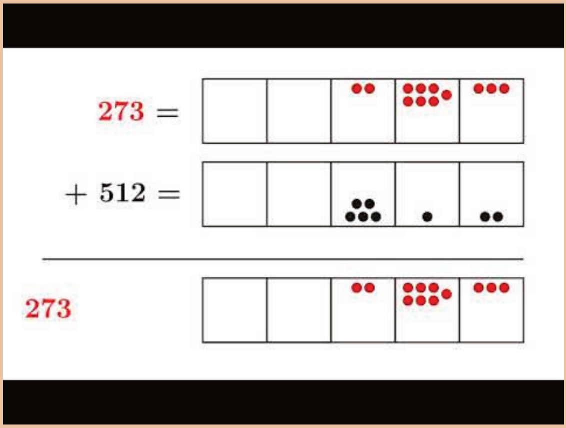
We can count up the answer: there are 7 dots in the hundreds box, 8 dots in the tens box, and 5 dots in the ones box.

$$\begin{array}{r} 273 \\ +512 \\ \hline 785 \end{array}$$

And saying out the long way we have:

- Two hundreds plus five hundreds gives 7 hundreds.
- Seven tens plus one ten gives 8 tens.
- Three ones plus two ones gives 5 ones.

This gives the answer: 785.



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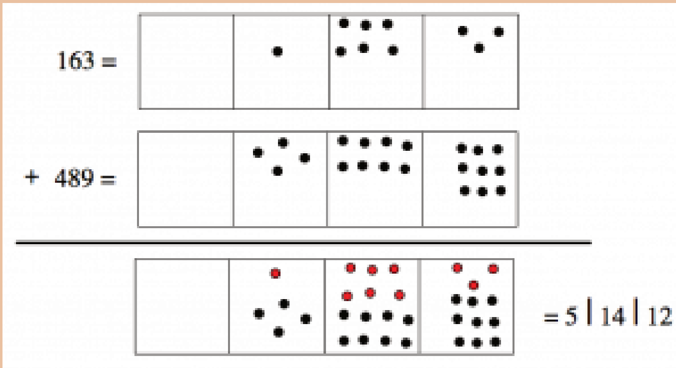
Example: 163+489

Let's do another one. Consider 163+489.



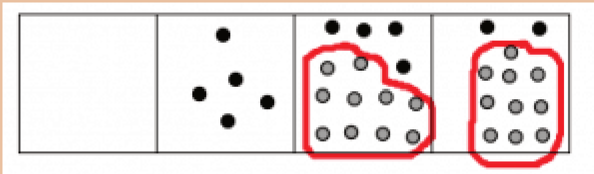
$$\begin{array}{r} 1 \quad 6 \quad 3 \\ +\ 4 \quad 8 \quad 9 \\ \hline 5\,|\,14\,|\,12 \end{array}$$

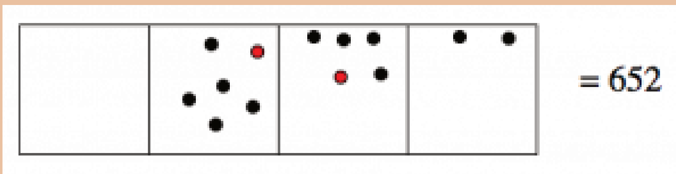
And this is absolutely correct:

- One hundred plus four hundreds is 5 hundreds.
- Six tens plus eight tens is 14 tens.
- Three ones plus nine ones is 12 ones.

The answer is 5 | 14 | 12, which we might try to pronounce as “five hundred and fourteeny-tenty twelvety.” The trouble with this answer is that most of the rest of the world wouldn’t understand what we are talking about.

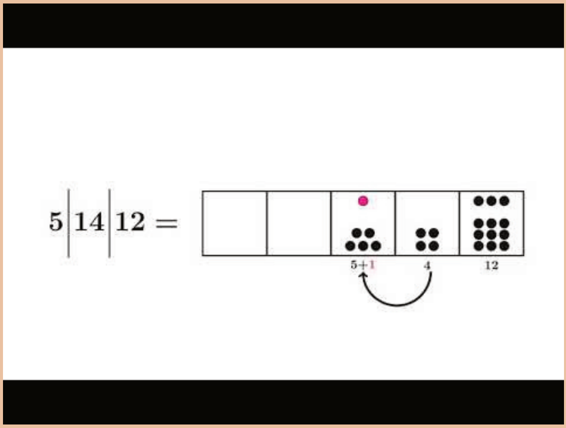
Since this is a base 10 system, we can do some explosions.





The answer is "six hundred fifty two." Okay, the world can understand this one!

$$\begin{array}{r} 1 \quad 6 \quad 3 \\ +\ 4 \quad 8 \quad 9 \\ \hline 5 \,|\, 14 \,|\, 12 = \quad 652 \end{array}$$



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Think / Pair / Share

Solve the following exercises by thinking about the dots and boxes. (You can draw the pictures, or just imagine them.)

Then translate the answer into something the rest of the world can understand.

$$\begin{array}{r} 148 \\ +323 \\ \hline \end{array} \qquad \begin{array}{r} 567 \\ +271 \\ \hline \end{array} \qquad \begin{array}{r} 310462872 \\ +389107123 \\ \hline \end{array}$$

Problem 1

Use the dots and boxes technique to solve these problems. *Do not covert to base 10! Try to work directly in the base given.* It might help to actually draw the pictures.

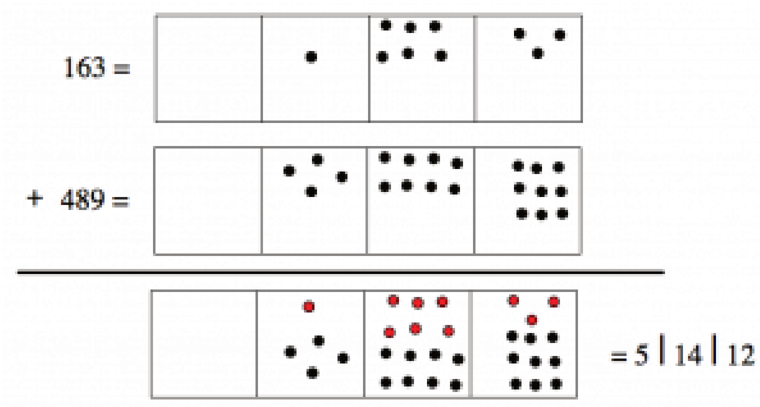
$$\begin{array}{r} 20413_{\text{five}} \\ +13244_{\text{five}} \\ \hline \end{array} \qquad \begin{array}{r} 4052_{\text{nine}} \\ +6288_{\text{nine}} \\ \hline \end{array} \qquad \begin{array}{r} 3323_{\text{seven}} \\ +3555_{\text{seven}} \\ \hline \end{array}$$

The Standard Algorithm for Addition

Let's go back to the example 163+489. Some teachers don't like writing:

$$\begin{array}{ccc} 1 & 6 & 3 \\ +4 & 8 & 9 \\ \hline \end{array}$$
$$5\,|\,14\,|\,12 = 652$$

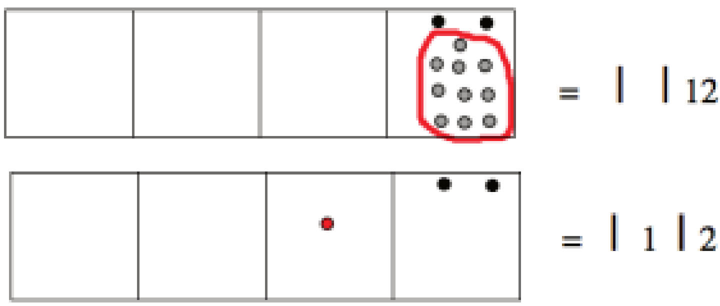
They prefer to teach their students to start with the 3 and 9 at the end and sum those to get 12. This is of course correct – we got 12 as well.



But they don't want students to write or think "twelvety," so they have their students write something like this:

$$\begin{array}{r} 1 \\ 163 \\ \underline{+489} \\ 2 \end{array}$$

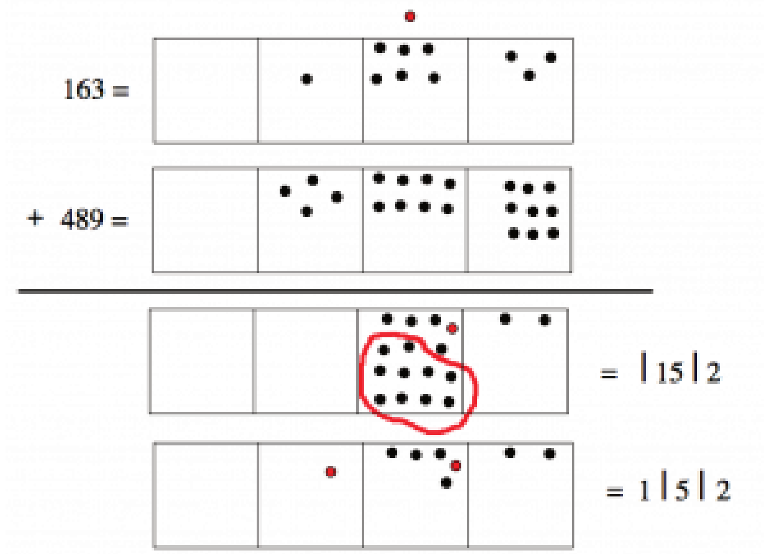
This can seem completely mysterious. What's really going on? They are exploding ten dots, of course!



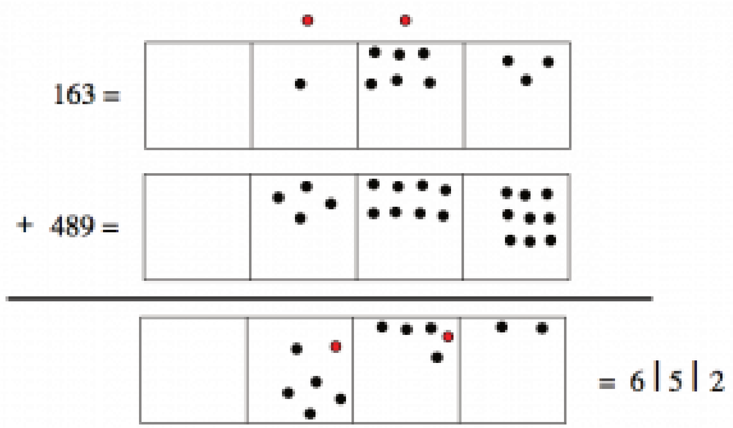
Now we carry on with the problem and add the tens. Students are taught to write:

$$\begin{array}{r} 1 \\ 163 \\ \underline{+489} \\ 52 \end{array}$$

But what this means is better shown in this next picture. Notice the "exploded" (or regrouped) dot at the very top, which is added to the tens box in the answer.



And now we finish the problem by combining the dots in the hundreds boxes:



$$\begin{array}{r} 1 \\ 163 \\ +489 \\ \hline 652 \end{array}$$

In the standard algorithm, we work from right to left, doing the "explosions" as we go along. This means that we start adding at the ones place and work towards the left-most place value, "carrying" digits that come from the explosions. (This is really not carrying; a better term for it is *regrouping*. Ten ones become one ten. Ten tens become one hundred. And so on.)

In the dots and boxes method, we add in any direction or order we like and then we do the explosions at the end.

- **Why do we like the standard algorithm?** Because it is efficient.
- **Why do we like the dots and boxes method?** Because it is easy to understand.

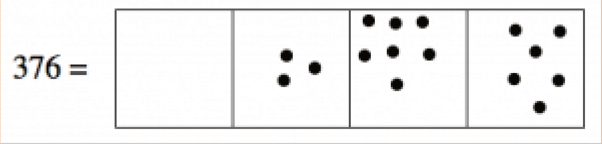
19. Subtraction: Dots and Boxes

Subtraction as Take-Away

To model addition, we started with two collections of dots (two numbers), and we *combined* them to form one bigger collection. That's pretty much the definition of addition: combining two collections of objects. In subtraction, we start with one collection of dots (one number), and we take some dots away.

Example: 376 – 125

Suppose we want to find 376-125 in the dots and boxes model. We start with the representation of 376:

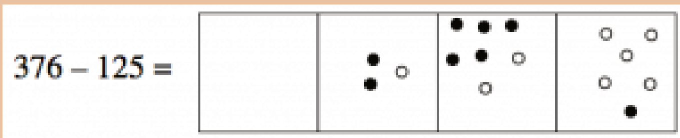


Since we want to "take away" 125, that means:

- We take away one dot from the hundreds box, leaving two dots.
- We take away two dots from the tens box, leaving five

dots.

- And we take away five dots from the ones box, leaving one dot.

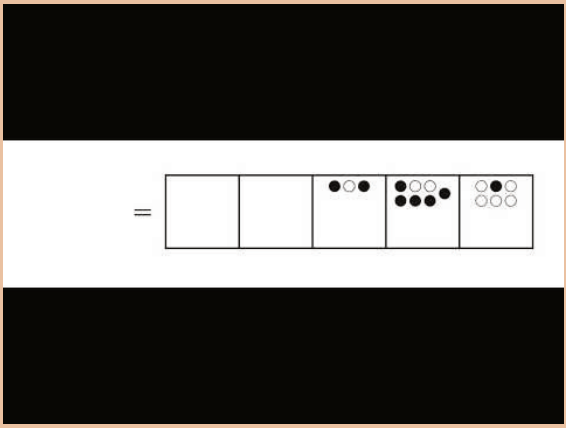


So the answer is:

$$\begin{array}{r} 376 \\ -125 \\ \hline 251 \end{array}$$

And saying it out the long way we have:

- Three hundreds take away one hundred leaves 2 hundreds.
- Seven tens take away two tens gives 5 tens.
- Six ones take away five ones gives 1 one.



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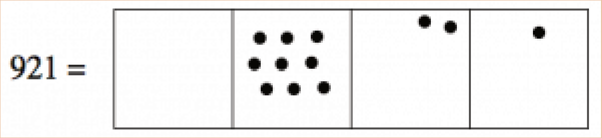
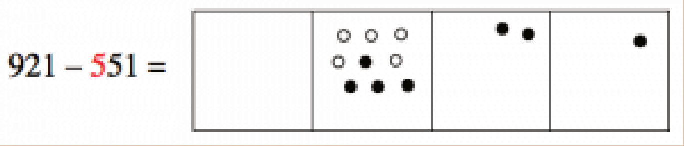
Example: 921 – 551

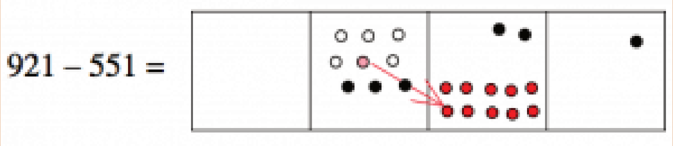
Let's try a somewhat harder example: 921-551. We start with the representation of 921:

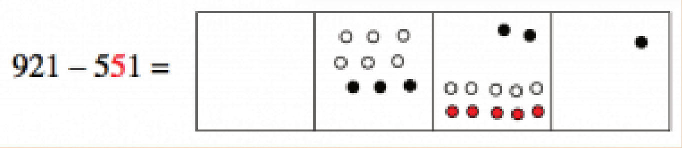


Since we want to “take away” 551, that means we take away five dots from the hundreds box, leaving four dots.



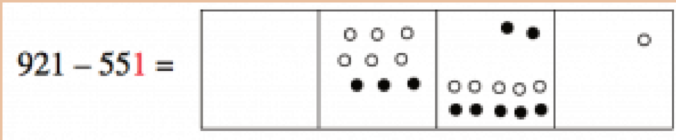
Now we want to take away five dots from the tens box, but we can’t do it! There are only two dots there. What can we do? Well, we still have some hundreds, so we can “unexplode” a hundreds dot, and put ten dots in the tens box instead. Then we’ll be able to take five of them away, leaving seven.





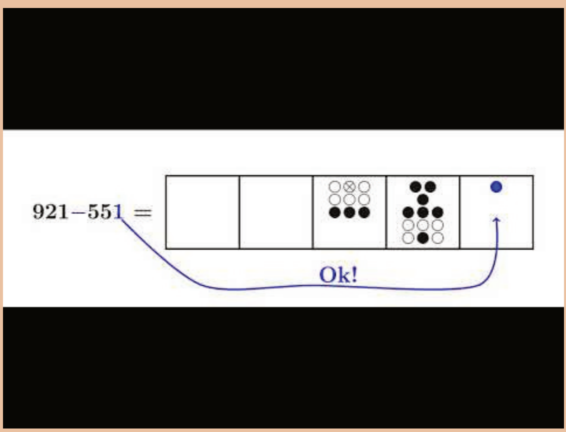
(Notice that we also have one less dot in the hundreds box; there’s only three dots there now.)

Now we want to take one dot from the ones box, and that leaves no dots there.



So the answer is:

$$\begin{array}{r} 921 \\ -551 \\ \hline 370 \end{array}$$



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Think / Pair / Share

Solve the following exercises by thinking about dots and boxes. (You can draw pictures, or just imagine them.)

$$\begin{array}{r} 323 \\ -148 \\ \hline \end{array} \qquad \begin{array}{r} 567 \\ -271 \\ \hline \end{array} \qquad \begin{array}{r} 389107123 \\ -310462872 \\ \hline \end{array}$$

Problem 2

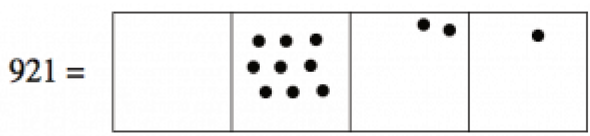
Use the dots and boxes technique to solve these problems. *Do not covert to base 10! Try to work directly in the base given.* It might help to actually draw the pictures.

$$\begin{array}{r} 20413_{\text{five}} \\ -13244_{\text{five}} \\ \hline \end{array} \qquad \begin{array}{r} 6252_{\text{nine}} \\ -4088_{\text{nine}} \\ \hline \end{array} \qquad \begin{array}{r} 4323_{\text{seven}} \\ -3524_{\text{seven}} \\ \hline \end{array}$$

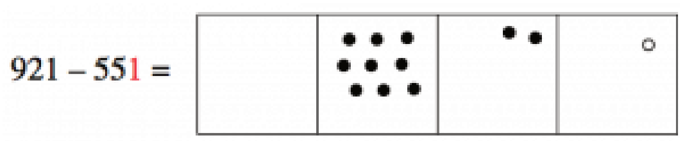
The Standard Algorithm for Subtraction

Just like in addition, the standard algorithm for subtraction requires you to work from right to left, and “borrow” (this is really *regrouping*!) whenever necessary. Notice that in the dots and boxes approach, you don’t need to go in any particular order when you do the subtraction. You just “unexplode” the dots as necessary when computing.

Here’s how the standard algorithm looks with the dots and boxes model for 921 – 551: Start with 921 dots.

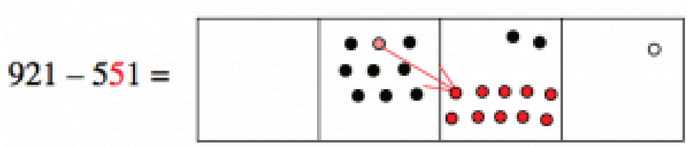


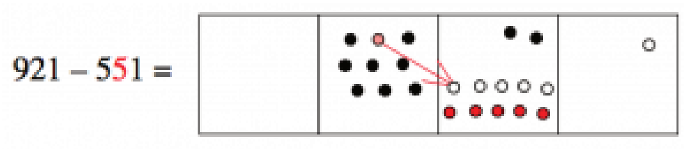
Then take away one dot from the ones box.



$$\begin{array}{r} 921 \\ -551 \\ \hline 0 \end{array}$$

Now we want to take away five dots from the tens box. But there aren’t five dots there. So we “unexplode” one of the hundreds dots to get more tens:



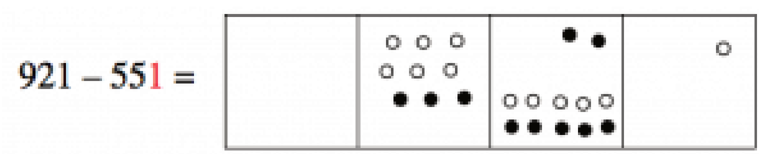


In the standard algorithm, we show the unxplosion as a regrouping, subtracting one from the hundreds place of 921 and adding ten to the tens place. So we are rewriting

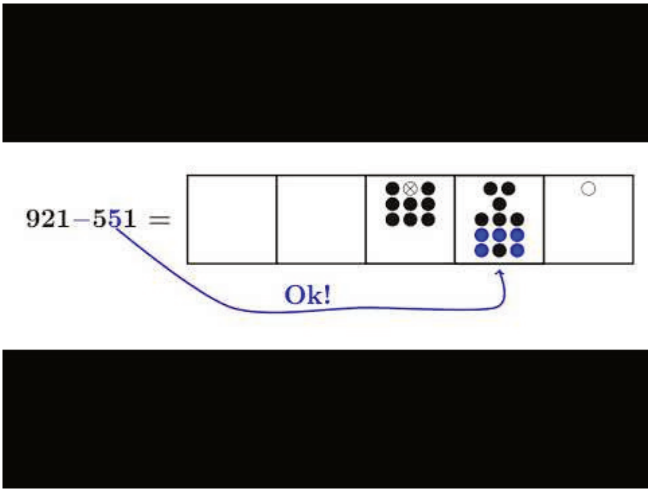
$$9 \mid 2 \mid 1 = 8 \mid 12 \mid 1.$$

$$\begin{array}{rrrr} & 8 & 12 & \\ & 9 & 2 & 1 \\ - & 5 & 5 & 1 \\ \hline & & 7 & 0 \end{array}$$

Finally, we want to take away five from the eight dots left in the hundreds column.



$$\begin{array}{rrrr} & 8 & 12 & \\ & 9 & 2 & 1 \\ - & 5 & 5 & 1 \\ \hline & 5 & 7 & 0 \end{array}$$



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20. Multiplication: Dots and Boxes

Multiplication as Repeated Addition

Problem 3

Jenny was asked to compute 243192×4. She wrote:

$$243192 \times 4 = 8 \,|\, 16 \,|\, 12 \,|\, 4 \,|\, 36 \,|\, 8.$$

1. What was Jenny thinking about? Is her answer correct?
2. Translate Jenny's answer into a number that the rest of the world can understand.
3. Use Jenny's method to find the answers to these multiplication exercises. Be sure to translate your answers into familiar base 10 numbers.

$156 \times 3 =$ $\qquad 2873 \times 2 =$ $\qquad 71181 \times 5 =$ $\qquad 3726510392 \times 2 =$

Problem 4

Can you adapt Jenny's method to solve these problems? Write your answers in base eight. Try to work directly in base eight rather than converting to base 10 and back again!

$$156_{\text{eight}} \times 3_{\text{eight}} =$$

$$2673_{\text{eight}} \times 4_{\text{eight}} =$$

$$36255772_{\text{eight}} \times 2_{\text{eight}} =$$

Jenny might have been thinking about multiplication as repeated addition. If we have some number N and we multiply that number by 4, what we mean is:

$$4 \cdot N = N + N + N + N.$$

If we take the number 243192 and add it to itself four times using the "combining method," we get

- 2 + 2 + 2 + 2 = 8 ones,
- 9 + 9 + 9 + 9 = 36 tens,
- 1 + 1 + 1 + 1 = 4 hundreds,
- and so on.

Notation

Notice that we have used both × and · to represent multiplication. It's a bit awkward to use × when you're also using variables. Is it the letter x? Or the multiplication symbol ×? It can be hard to tell! In this case, the symbol · is more clear.

We can even simplify the notation further, writing 4N instead of $4 \cdot N$. But of course we only do that when we are multiplying *variables* by some quantity. (We wouldn't want 34 to mean $3 \cdot 4$, would we?)

Problem 5

Here is a strange addition table. Use it to solve the following problems. Important: Don't try to assign numbers to A, B, and C. Solve the problems just using what you know about the operations!

+	A	B	C
A	C	A	B
B	A	B	C
C	B	C	A

A + B B + C 2A 5C 3A + 4B

Think / Pair / Share

How does an addition table help you solve multiplication problems like 5C?

21. Division: Dots and Boxes

Quotative Model of Division

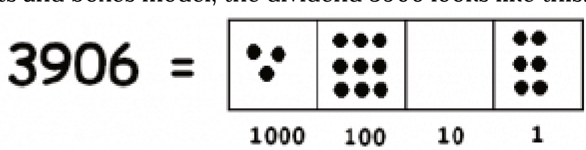
Suppose you are asked to compute 3906 : 3. One way to interpret this question (there are others) is:

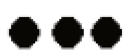
"How many groups of 3 fit into 3906?"

Definition

In the **quotative model of division**, you are given a **dividend** (here it is 3906), and you are asked to split it into equal-sized groups, where the size of the group is given by the **divisor** (here it is 3).

In our dots and boxes model, the dividend 3906 looks like this:



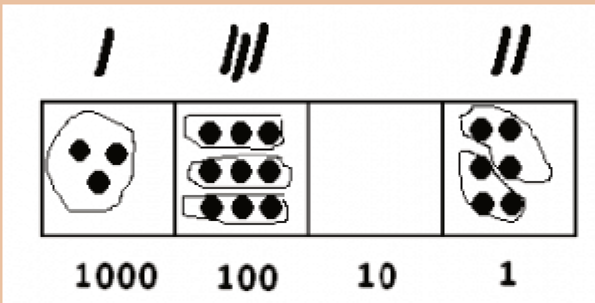
and three dots looks like this: 

So we are really asking:

"How many groups of  fit into the picture of 3906?"

Example: 3906 ÷ 3

There is one group of 3 at the thousands level, and three at the hundreds level, none at the tens level, and two at the ones level.



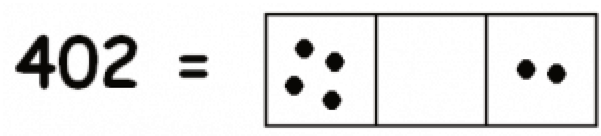
Notice what we have in the picture:

- One group of 3 in the thousands box.
- Three groups of 3 in the hundreds box.
- Zero groups of 3 in the tens box.
- Two groups of 3 in the ones box.

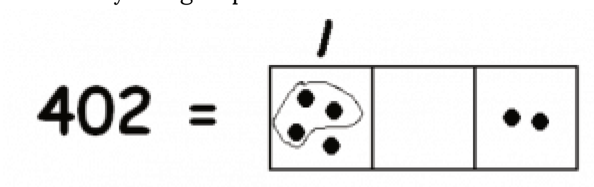
This shows that 3 goes into 3906 one thousand, three hundreds and two ones times. That is,

$$3906 \div 3 = 1302.$$

Let's try a harder one! Consider 402 : 3. Here's the picture:



We are still looking for groups of three dots: 
There is certainly one group at the 100's level.



and now it seems we are stuck there are no more groups of three!

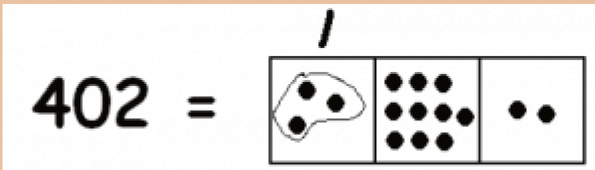
Think / Pair / Share

What can we do now? Are we really stuck? Can you finish the division problem?

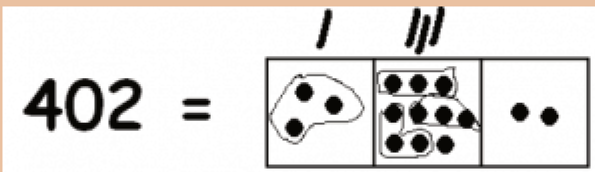
Example: 402 ÷ 3

Here are the details worked out for 402 : 3. But don't read this until you've thought about it yourself!

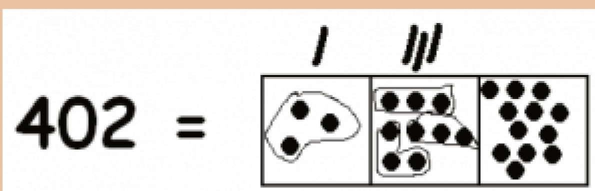
Since each dot is worth ten dots in the box to the right we can write:



Now we can find more groups of three:



There is still a troublesome extra dot. Let's unexplode it too



This gives us more groups of three:

In the picture we have:

- One group of 3 in the hundreds box.
- Three groups of 3 in the tens box.

- Four groups of 3 in the ones box.

Finally we have the answer!

$$402 \div 3 = 134.$$

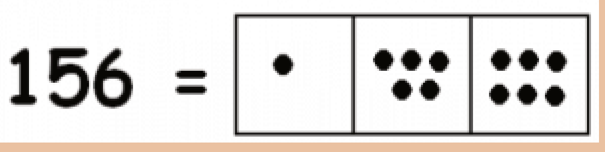
Think / Pair / Share

Solve each of these exercises using the dots and boxes method:

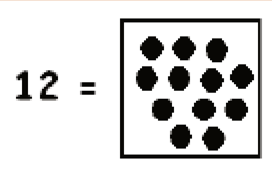
62124 : 3 61230 : 5

Example: 156 ÷ 12

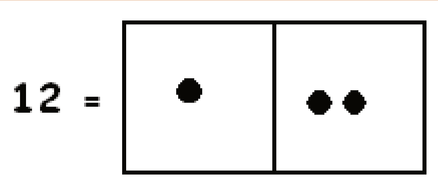
Let's turn up the difficulty a notch. Consider 156 : 12. Here we are looking for groups of 12 in this picture:



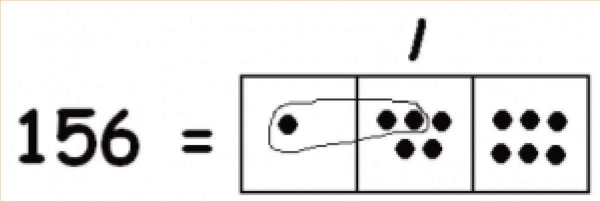
What does 12 look like? It can be twelve dots in a single box:



But most often we would write 12 this way, as a ten and 2 ones:

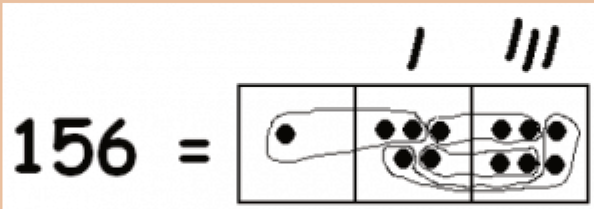


We certainly see some of these in the picture. There is certainly one at the tens level:



Note: With an unexplosion this would be twelve dots in the tens box, so we mark one group of 12 above the tens box.

We also see three groups of twelve ones:



So in the picture we have:

- One group of 12 dots in the tens box.
- Three groups of 12 dots in the ones box.

That means

$$156 : 12 = 13.$$

Problem 6

Use the dots and boxes model to compute each of the following:

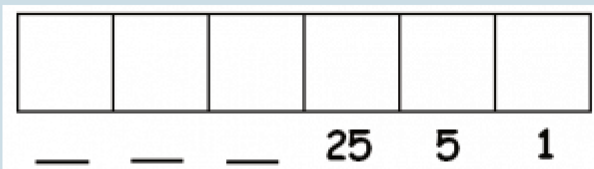
$$13453 : 11$$

$$4853 : 23$$

$$214506 : 102$$

Problem 7

Remember that base five numbers are in a $1 \leftarrow 5$ dots-and-boxes system. What are the place values in the $1 \leftarrow 5$ system? Fill in the blanks:



1. Draw a dots-and-boxes picture of the number 424_{five}.
2. Draw a dots-and-boxes picture of the number 11_{five}.
3. Use the dots and boxes method to find $424_{\text{five}} \div 11_{\text{five}}$.
4. Rewrite the division sentence $424_{\text{five}} \div 11_{\text{five}} = 34_{\text{five}}$ in base ten, and check that it's correct.
5. Use dots-and-boxes to find $2021_{\text{five}} \div 12_{\text{five}}$. *Don't convert to base 10!*

Think / Pair / Share

- Use dots and boxes to compute these.

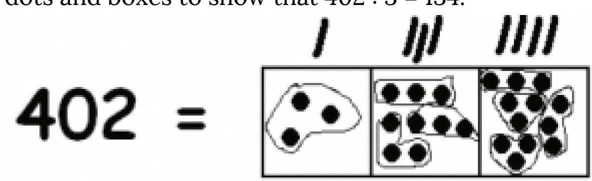
$$2130 : 10$$

$$41300 : 100$$

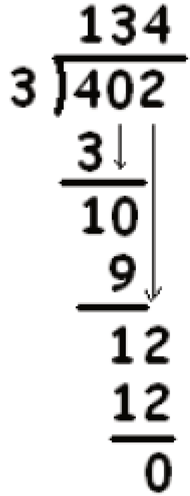
- What pictures did you use for 10 and for 100? Can you describe in words what happens when dividing by 10 and by 100 and why?

The Standard Algorithm for Division

We used dots and boxes to show that 402 : 3 = 134.

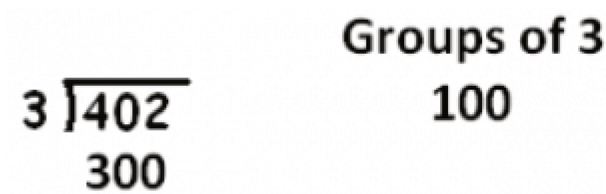


In elementary school, you might have learned to solve this division problem by using a diagram like the following:

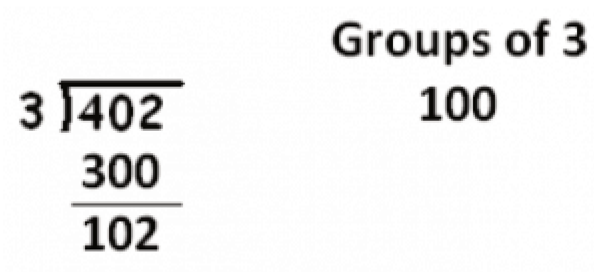


At first glance this seems very mysterious, but it is really no different from the dots and boxes method. Here is what the table means.

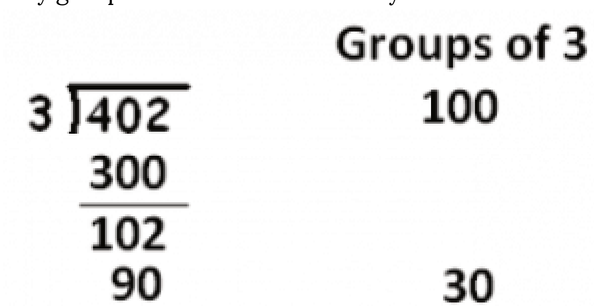
To compute 402 : 3, we first make a big estimation as to how many groups of 3 there are in 402. Let’s guess that there are 100 groups of three.



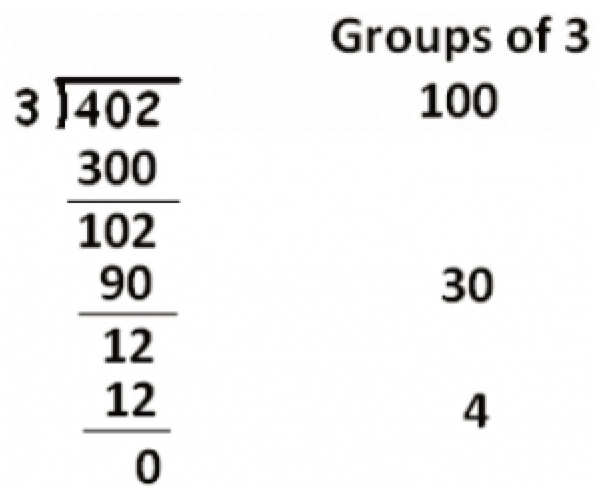
How much is left over after taking away 100 groups of 3? We subtract to find that there is 102 left.



How many groups of 3 are in 102? Let's try 30:



How many are left? There are 12 left and there are four groups of 3 in 12.

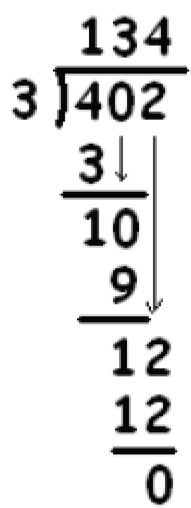


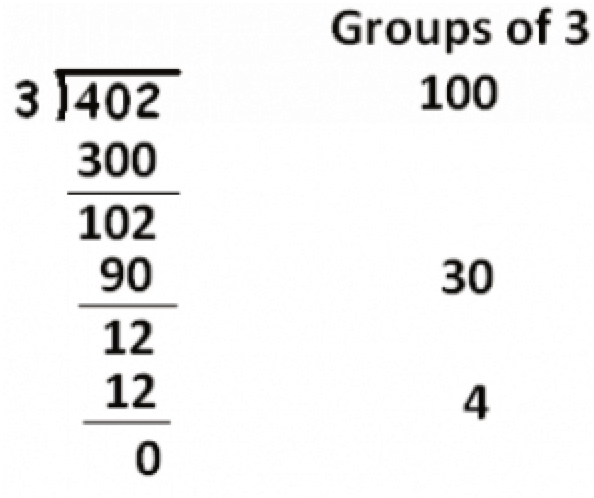
That accounts for entire number 402. And where do we find the final answer? Just add the total count of groups of three that we tallied:

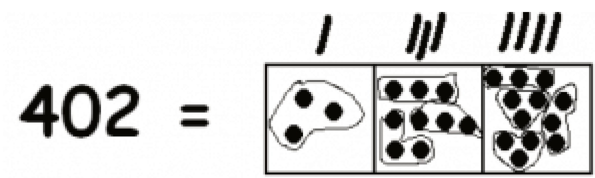
$$402 : 3 = 100 + 30 + 4 = 134.$$

Think / Pair / Share

- Compare the two division diagrams below. In what way are they the same? In what way are they different?
- Also look at the dots and boxes method. In what way is it the same or different from the two diagrams?







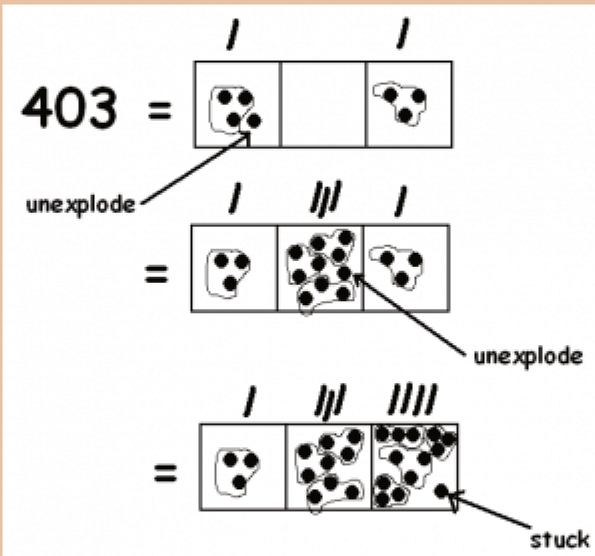
- **Why do we like the standard algorithm?** Because it is quick, not too much to write down, and it works every time.
- **Why do we like the dots and boxes method?** Because it easy to understand. (And drawing dots and boxes is kind of fun!)

Division with Remainders

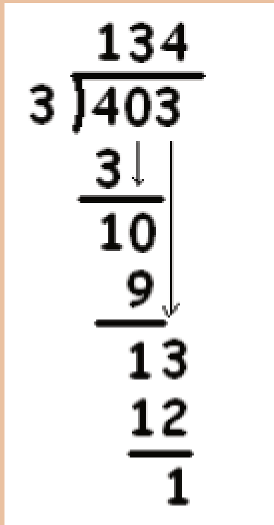
We saw that 402 is evenly divisible by 3: 402 : 3 = 134. This means that 403, one more, shouldn't be divisible by three. It should be one dot too big.

Example: 403 ÷ 3

Do we see the extra dot if we compute 402 : 3 with dots and boxes?



Yes we do! We have one dot left at the end that can't be divided. This is how it looks in the standard algorithm.



In school, we say that we have a *remainder* of one and sometimes write:

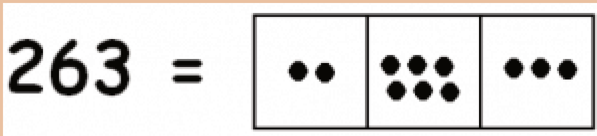
$$403 \div 3 = 134 \text{ R}1.$$

But what does that really mean? It means that we have 134 groups of three with one dot left over. So

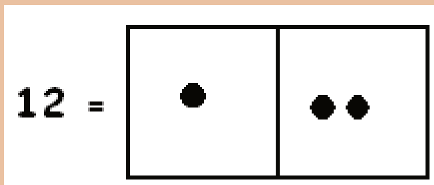
$$402 = 134 \cdot 3 + 1.$$

Example: 263 ÷ 12

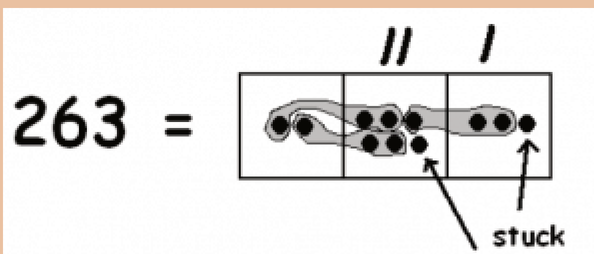
Let's try another one: 263 : 12. Here's what we have:



And we are looking for groups like this:



Here goes!



Unexploding won't help any further and we are indeed left with one remaining dot in the tens position and a dot in the ones position. We have 21 groups of twelve, and a remainder of eleven.

$$263 = 21 \cdot 12 + 11.$$

Think / Pair / Share

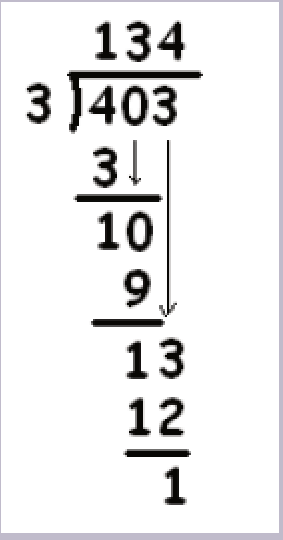
- Use the dots and boxes method to compute each quotient and remainder:

$$5210 : 4$$

$$4857 : 23$$

$$31533 : 101$$

- Now use the standard algorithm (an example is shown below) to compute each of the quotients and remainders above.



$$402 = 134 \cdot 3 + 1.$$

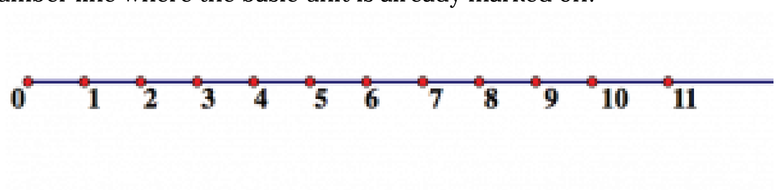
- Which method do you like better: dots and boxes or the standard algorithm method? Or does it depend on the problem you are doing?

22. Number Line Model

Another way we often think about numbers is as abstract quantities that can be measured: length, area, and volume are all examples.

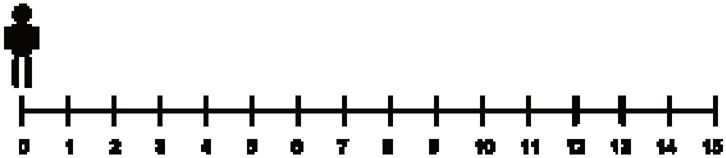
In a measurement model, you have to pick a *basic unit*. The basic unit is a quantity – length, area, or volume – that you assign to the number one. You can then assign numbers to other quantities based on how many of your basic unit fit inside.

For now, we'll focus on the quantity length, and we'll work with a number line where the basic unit is already marked off.



Addition and Subtraction on the Number Line

Imagine a person – we'll call him Zed – who can stand on the number line. We'll say that the distance Zed walks when he takes a step is exactly one unit.



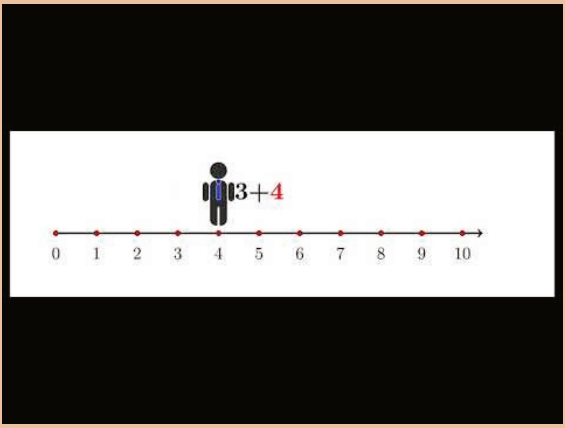
When Zed wants to add or subtract with whole numbers on the number line, he always starts at 0 and faces the positive direction (towards 1). Then what he does depends on the calculation.

If Zed wants to *add* two numbers, he walks forward (to the right of the number line) however many steps are indicated by the first number (the first *addend*). Then he walks forward (to your right on the number line) the number of steps indicated by the second number (the second *addend*). Where he lands is the *sum* of the two numbers.

Example: 3 + 4

If Zed wants to add 3 + 4, he starts at 0 and faces towards the positive numbers. He walks forward 3 steps, then he walks forward 4 more steps.

Zed ends at the number 7, so the sum of 3 and 4 is 7. 3 + 4 = 7. (But you knew that of course! The point right now is to make sense of the *number line model*.)



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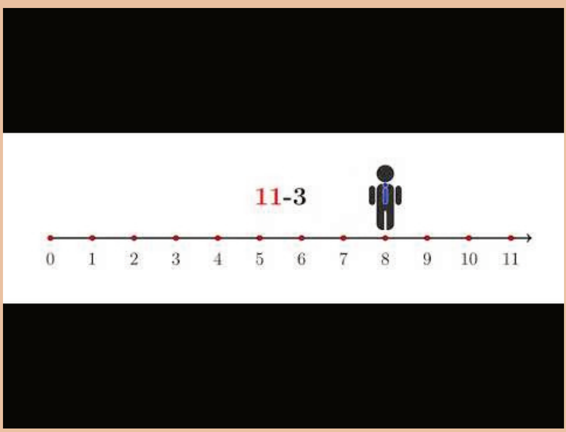
When Zed wants to *subtract* two numbers, he he walks forward (to the right on the number line) however many steps are indicated by the first number (the *minuend*). Then he walks *backwards* (to the left on the number line) the number of steps indicated by the second number (the *subtrahend*). Where he lands is the *difference* of the two numbers.

Example: 11 – 3

If Zed wants to subtract 11 – 3, he starts at 0 and faces the positive numbers (the right side of the number line). He walks forward 11 steps on the number line, then he walks backwards 3 steps.

Zed ends at the number 8, so the difference of 11 and 3 is 8. 11 – 3 = 8. (But you knew that!)



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Think / Pair / Share

- Work out each of these exercises on a number line. You can actually pace it out on a life-sized number line or draw a picture:

  $4 + 5$ $\quad$ $6 + 9$ $\quad$ $10 - 7$ $\quad$ $8 - 1$

- Why does it make sense to walk forward for addition and walk backwards for subtraction? In what way is this the same as "combining" for addition and "take away" for subtraction"?
- What happens if you do these subtraction problems on a number line? Explain your answers.

  $6 - 9$ $\quad$ $1 - 7$ $\quad$ $4 - 11$ $\quad$ $0 - 1$

- Could you do the subtraction problems above with the dots and boxes model?

Multiplication and Division on the Number Line

Since multiplication is really repeated addition, we can adapt our addition model to become a multiplication model as well. Let's think about 3 × 4. This means to add four to itself three times (that's simply the definition of multiplication!):

$$3 \times 4 = 4 + 4 + 4.$$

So to multiply on the number line, we do the process for addition several times.

To multiply two numbers, Zed starts at 0 as always, and he faces the positive direction. He walks forward the number of steps given by the second number (the second *factor*). He repeats that process the number of times given by the first number (the first *factor*). Where he lands is the *product* of the two numbers.

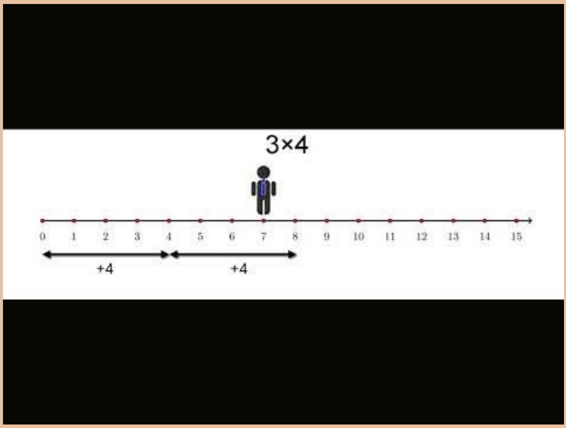
Example: 3 × 4

If Zed wants to multiply 3 × 4, he can think of it this way:

$$\underset{\swarrow \atop \text{how many times to repeat it}}{3} \quad \times \quad \underset{\searrow \atop \text{how many steps to take forward}}{4}$$

Zed starts at 0, facing the positive direction. The he repeats this three times: take four steps forward.

He ends at the number 12, so the product of 3 and 4 is 12. That is, 3 × 4 = 12.



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Remember our quotative model of division: One way to interpret 15 : 5 is:

> How many groups of 5 fit into 15?

Thinking on the number line, we can ask it this way:

> Zed takes 5 steps at a time. If Zed lands at the number 15, how many times did he take 5 steps?

To calculate a division problem on the number line, Zed starts at

0, facing the positive direction. He walks forward the number of steps given by the second number (the *divisor*). He repeats that process until he lands at the first number (the *dividend*). The number of times he repeated the process gives the *quotient* of the two numbers.

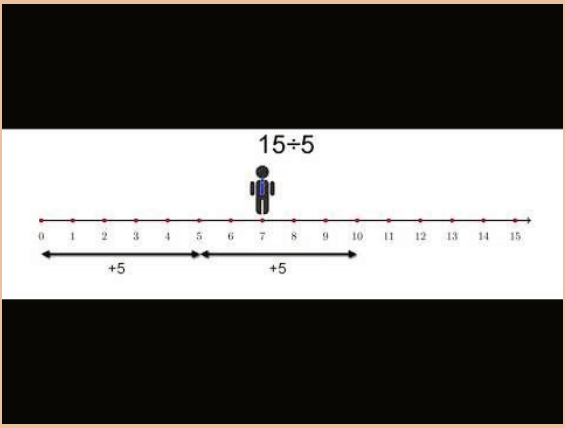
Example: 15 ÷ 5

If Zed wants to compute 15 : 5, he can think of it this way:

He starts at 0, facing the positive direction.

- Zed takes 5 steps forward. He is now at 5, not 15. So he needs to repeat the process.
- Zed takes 5 steps forward again. He is now at 10, not 15. So he needs to repeat the process.
- Zed takes 5 more steps forward. He is at 15, so he stops.

Since he repeated the process three times, we see there are 3 groups of 5 in 15. So the quotient of 15 and 5 is 3. That is, 15 : 5 = 3.



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Think / Pair / Share

- Work out each of these exercises on a number line. You can actually pace it out on a life-sized number line or draw a picture:

2×5 7×1 $10 : 2$ $6 : 1$

- Can you think of a way to interpret these multiplication problems on a number line? Explain your ideas.

4×0 0×5 $3 \times (-2)$ $2 \times (-1)$

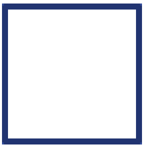
- What happens if you try to solve these division problems on a number line? Can you do it? Explain your ideas.

$0 : 2$ $0 : 10$ $3 : 0$ $5 : 0$

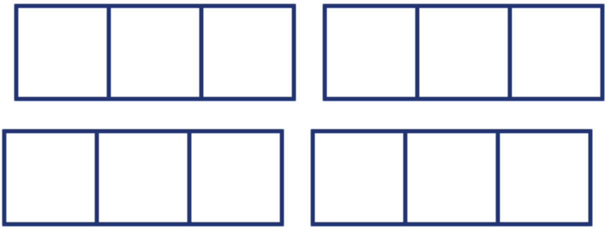
23. Area Model for Multiplication

So far we have focused on a linear measurement model, using the number line. But there's another common way to think about multiplication: using area.

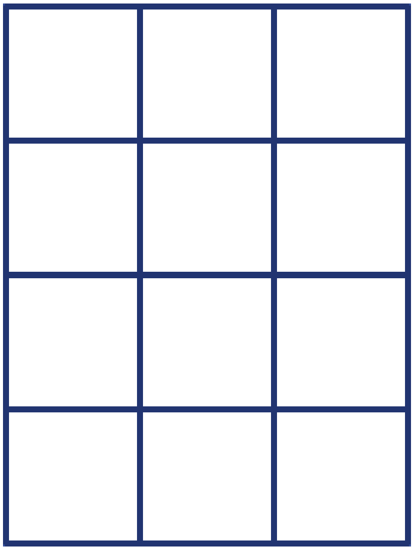
For example, suppose our basic unit is one square:



We can picture 4 × 3 as 4 groups, with 3 squares in each group, all lined up:



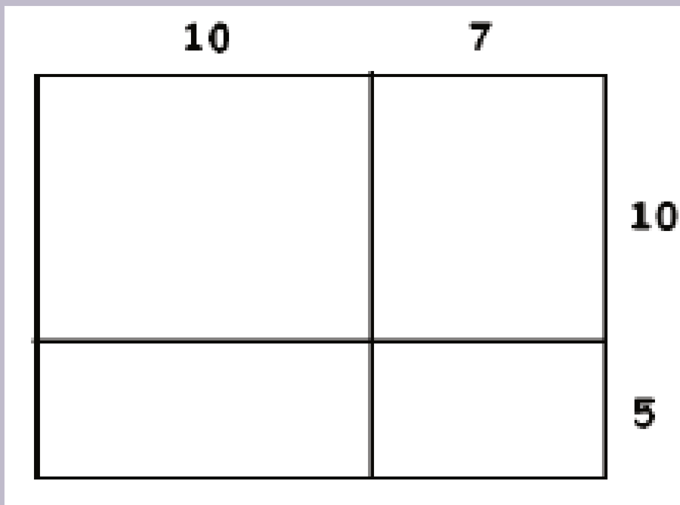
But we can also picture them stacked up instead of lined up. We would have 4 rows, with 3 squares in each row, like this:



So we can think about 4×3 as a rectangle that has length 3 and width 4. The product, 12, is the total number of squares in that rectangle. (That is also the area of the rectangle, since each square was one unit!)

Think / Pair / Share

Vera drew this picture as a model for 15×17. Use her picture to help you compute 15×17. Explain your work.



Problem 8

Draw pictures like Vera's for each of these multiplication exercises. Use your pictures to find the products without using a calculator or the standard algorithm.

23×37 8×43 371×42

The Standard Algorithm for Multiplication

How were you taught to compute 83 × 27 in school? Were you taught to write something like the following?

$$\begin{array}{lr} & 83 \\ \times & 27 \\ \hline & 21 \\ & 56 \\ & 6 \\ & 16 \\ \hline & 2241 \end{array}$$

Or maybe you were taught to put in the extra zeros rather than leaving them out?

$$
\begin{array}{r}
83 \\
\times \quad 27 \\
\hline
21 \\
560 \\
60 \\
1600 \\
\hline
2241
\end{array}
$$

This is really no different than drawing the rectangle and using Vera's picture for calculating!

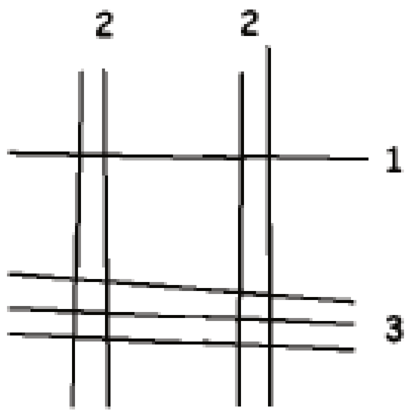
Think / Pair / Share

- Use the example above to explain why Vera's rectangle method and the standard algorithm are really the same.
- Calculate the products below using both methods. Explain where you're computing the same pieces in each algorithm.

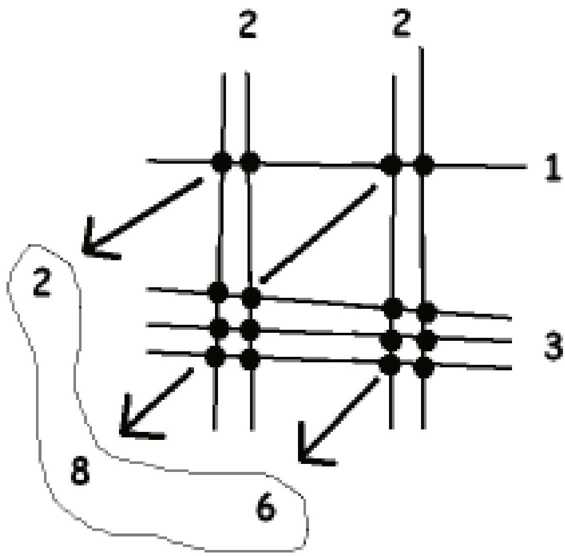
23×14 $\qquad$ 106×21 $\qquad$ 213×31

Lines and Intersections

Here's an unusual way to perform multiplication. To compute 22 × 13, for example, draw two sets of vertical lines, the left set containing two lines and the right set two lines (for the digits in 22) and two sets of horizontal lines, the upper set containing one line and the lower set three (for the digits in 13).

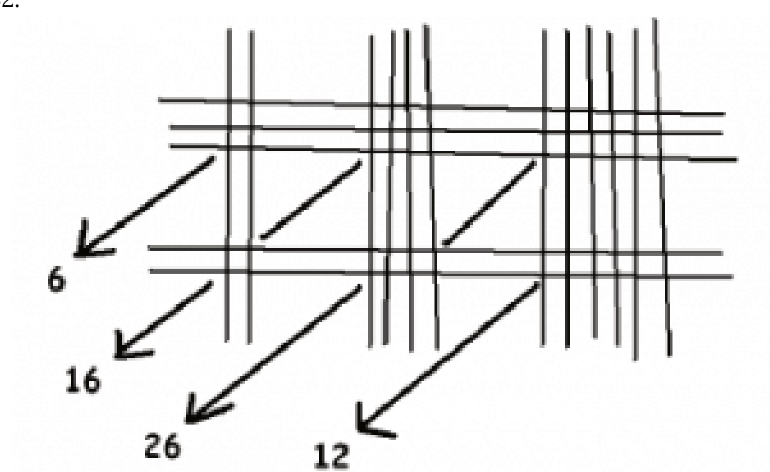


There are four sets of intersection points. Count the number of intersections in each and add the results diagonally as shown:



The answer 286 appears!

There is one possible glitch as illustrated by the computation 246 × 32:



Although the answer 6 thousands, 16 hundreds, 26 tens, and 12 ones is absolutely correct, one needs to carry digits and translate this as 7,872.

Problem 9

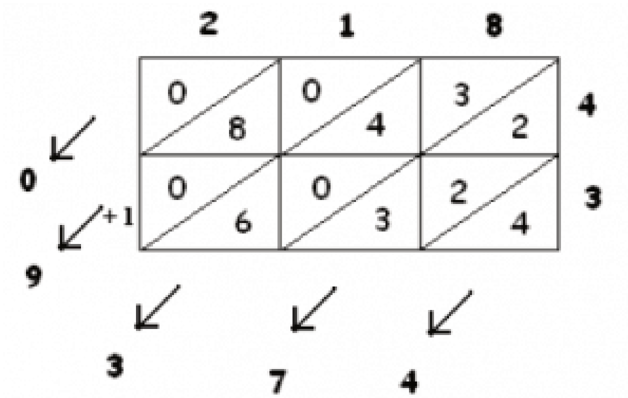
1. Compute 131×122 via this method. Check your answer using another method.
2. Compute 15×1332 via this method. Check your answer using another method.
3. Can you adapt the method to compute 102×3054? (Why is some adaptation necessary?)
4. Why does the method work in general?

Lattice Multiplication

In the 1500s in England, students were taught to compute multiplication using following galley method, now more commonly known as the *lattice method*.

To multiply 43 and 218, for example, draw a 2×3 grid of squares. Write the digits of the first number along the right side of the grid and the digits of the second number along the top.

Divide each cell of the grid diagonally and write in the product of the column digit and row digit of that cell, separating the tens from the units across the diagonal of that cell. (If the product is a one digit answer, place a 0 in the tens place.)



To get the answer, add the entries in each diagonal, carrying tens digits over to the next diagonal if necessary. In our example, we have

$$218 \times 43 = 9374.$$

Problem 10

1. Compute 5763 × 345 via the lattice method.
2. Explain why the lattice method is really the standard algorithm in disguise.
3. What is the specific function of the diagonal lines in the grid?

24. Properties of Operations

So far, you have seen a couple of different *models* for the operations: addition, subtraction, multiplication, and division. But we haven't talked much about the operations themselves – how they relate to each other, what properties they have that make computing easier, and how some special numbers behave. There's lots to think about!

The goal in this section is to use the models to understand why the operations behave according to the rules you learned back in elementary school. We're going to keep asking ourselves "Why does it work this way?"

Think / Pair / Share

Each of these models lends itself to thinking about the operation in a slightly different way. Before we really dig in to thinking about the operations, discuss with a partner:

- Of the models we discussed so far, do you prefer one of them?
- How well do the models we discussed match up with how you usually think about whole numbers and their operations?
- Which models are useful for computing? Why?
- Which models do you think will be useful for explaining how the operations work? Why?

Connections Between the Operations

We defined addition as combining two quantities and subtraction as "taking away." But in fact, these two operations are intimately tied together. These two questions are exactly the same:

$$27 - 13 = \_\_\_\_ \qquad\qquad 27 = 13 + \_\_\_\_\_.$$

More generally, for any three whole numbers a, b, and c, these two equations express the same fact. (So either both equations are true or both are false. Which is the case depends on the values you choose for a, b, and c!)

$$c - b = a \qquad\qquad c = a + b.$$

In other words, we can think of every subtraction problem as a "missing addend" addition problem. Try it out!

Problem 11

Here is a strange addition table. Use it to solve the following problems. Justify your answers. Important: Don't try to assign numbers to A, B, and C. Solve the problems just using what you know about the operations!

+	**A**	**B**	**C**
A	C	A	B
B	A	B	C
C	B	C	A

A + C B + C A − C C − A A − A B − C

Think / Pair / Share

How does an addition table help you solve subtraction problems?

We defined multiplication as repeated addition and division as forming groups of equal size. But in fact, these two operations are also tied together. These two questions are exactly the same:

$$27 : 3 = \_\_\_\_\_ \qquad\qquad 27 = \_\_\_\_\_ \times 3.$$

More generally, for any three whole numbers a, b, and c, these two equations express the same fact. (So either both equations are true or both are false. Which is the case depends on the values you choose for a, b, and c!)

$$c : b = a \qquad\qquad c = a \cdot b.$$

In other words, we can think of every division problem as a "missing factor" multiplication problem. Try it out!

Problem 12

Rewrite each of these division questions as a "missing factor" multiplication question. Which ones can you solve and which can you not solve? Explain your answers.

9 : 3   100 : 25   0 : 3   9 : 0   0 : 0

Problem 13

Here's a multiplication table.

$\times$	**A**	**B**	**C**	**D**	**E**
A	A	A	A	A	A
B	A	B	C	D	E
C	A	C	E	B	D
D	A	D	B	E	C
E	A	E	D	C	B

- Use the table to solve the problems below. Justify your answers. Important: Don't try to assign numbers to the letters. Solve the problems just using what you know about the operations!

C × D   C × A   A × A   C : D   D : C   D : E

- Can you use the table to solve these problems? Explain your answers. Recall that x^n means n copies of x multiplied together, $x \cdot x \cdot x \cdot \ldots \cdot x$

D^2 C^3 A : C   A : D   D : A   A : A

Think / Pair / Share

How does a multiplication table help you solve division (and exponentiation) problems?

Throughout this course, our focus is on explanation and justification. As teachers, you need to know what is true in mathematics, but you also need to know *why* it is true. And you will need lots of ways to explain *why*, since different explanations will make sense to different students.

Think / Pair / Share

Arithmetic Fact: $a + b = c$ and $c - b = a$ are the same mathematical fact.

Why is this *not* a good explanation?

> "I can check that this is true! For example, 2+3 = 5 and 5 −

$3 = 2$. And $3 + 7 = 10$ and $10 - 7 = 3$. It works for whatever numbers you try."

Addition and Subtraction: Explanation 1

Arithmetic Fact:

$a + b = c$ and $c - b = a$ are the same mathematical fact.

Why It's True, Explanation 1:

First we'll use the definition of the operations.

Suppose we know $c - b = a$ is true. Subtraction means "take away." So

$$c - b = a$$

means we start with quantity c and take away quantity b, and we end up with quantity a. Start with this equation, and imagine adding quantity b to both sides.

On the left, that mans we started with quantity c, took away b things, and then put those b things right back! Since we took away some quantity and then added back the exact same quantity, there's no overall change. We're left with quantity c.

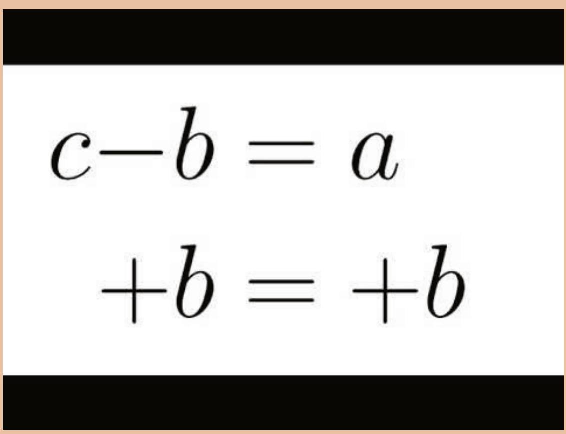
On the right, we would be combining (adding) quantity a with quantity b. So we end up with: c = a + b.

On the other hand, suppose we know the equation $a + b = c$ is true. Imagine taking away (subtracting) quantity b from both sides of this equation: a + b = c.

On the left, we started with a things and combined that with b things, but then we immediately take away those b things. So we're left with just our original quantity of a.

On the right, we start with quantity c and take away b things. That's the very definition of $c - b$. So we have the equation:

$$a = c - b.$$



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Why It's True, Explanation 2:

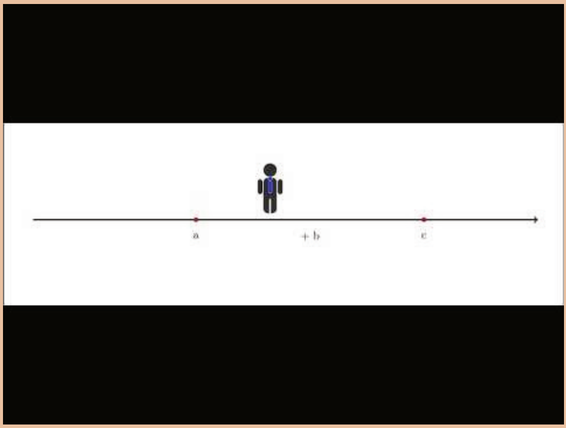
Let's use the measurement model to come up with another explanation.

The equation $a + b = c$ means Zed starts at 0, walks forward a steps, and then walks forward b steps, and he ends at c.

If Zed wants to compute $c - b$, he starts at 0, walks forward c steps, and then walks backwards b steps. But we know that to walk forward c steps, he can first walk forward a steps and then walk forward b steps. So Zed can compute $c - b$ this way:

- Start at 0.
- Walk forward a steps.
- Walk forward b steps. (Now at c, since $a + b = c$.)
- Walk backwards b steps.

The last two sets of steps cancel each other out, so Zed lands back at a. That means $c - b = a$.



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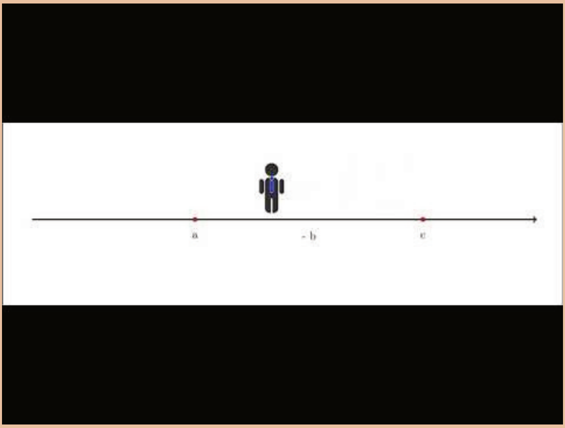
On the other hand, the equation $c - b = a$ means that Zed starts at 0, walks forward c steps, then walks backwards b steps, and he ends up at a.

If Zed wants to compute $a + b$, he starts at 0, walks forward a steps, and then walks forwards b additional steps. But we know that to walk forward a steps, he can first walk forward c steps and then walk backwards b steps. So Zed can compute $a + b$ this way:

- Start at 0.

- Walk forward c steps.
- Walk backwards b steps. (Now at a, since $c - b = a$.)
- Walk forward b steps.

The last two sets of steps cancel each other out, so Zed lands back at c. That means $a + b = c$.



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Think / Pair / Share

- Read over the two explanations in the example above. Do you think either one is more clear than the other?
- Come up with your own **explanation** (not examples!) to explain:

$$c : b = a \quad \text{is the same fact as} \quad c = a \times b.$$

Properties of Addition and Subtraction

You probably know several properties of addition, but you may never have stopped to wonder: *Why is that true*?! Now's your chance! In this section, you'll use the definition of the operations of addition and subtraction and the models you've learned to explain why these properties are always true.

Here are the three properties you'll think about:

- Addition of whole numbers is *commutative*.
- Addition of whole numbers is *associative*.
- The number 0 is an *identity* for addition of whole numbers.

For each of the properties, we don't want to confuse these three ideas:

- what the property is called and what it means (the definition),
- some examples that *demonstrate* the property, and
- an explanation for *why* the property holds.

Notice that *examples* and *explanations* are not the same! It's also very important not to confuse the *definition* of a property with the *reason* it is true!

These properties are all universal statements – statements of the form "for all," "every time," "always," etc. That means that to show they are true, you either have to check every case or find a *reason why* it must be so.

Since there are infinitely many whole numbers, it's impossible to check every case. You'd never finish! Our only hope is to look for *general explanations*. We'll work out the explanation for the first of these facts, and you will work on the others.

ADDITION IS COMMUTATIVE

Example: Commutative Law

Property:

Addition of whole numbers is commutative.

What it Means (words):

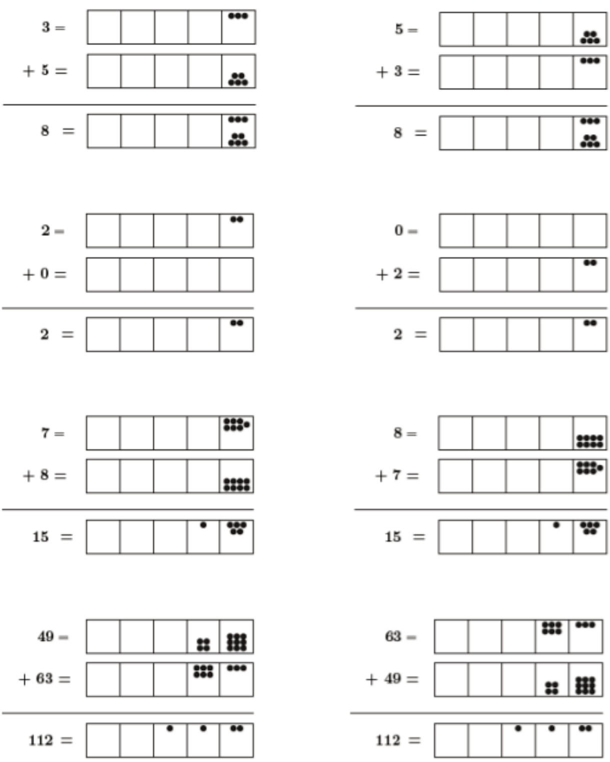
When I add two whole numbers, the order I add them doesn't affect the sum.

What it Means (symbols):

For any two whole numbers a and b,

$$a + b = b + a.$$

Examples:



Now we need a *justification*. *Why* is addition of whole numbers commutative?

Why It's True, Explanation 1:

Let's think about addition as combining two quantities of dots.

- To add $a + b$, we take a dots and b dots, and we combine them in a box. To keep things straight, lets imagine the a dots are colored red and the b dots are colored blue. So in the box we have a red dots, b blue dots and $a + b$ total dots.
- To add $b + a$, let's take b blue dots and a red dots, and put them all together in a box. We have b blue dots, a red dots and $b + a$ total dots.
- But the total number of dots are the same in the two boxes! How do we know that? Well, there are a red dots in each box, so we can match them up. There are b blue dots in each box, so we can match them up. That's it! If we can match up the dots one-for-one, there must be the same number of them!

- That means $a + b = b + a$.

Why It's True, Explanation 2:

We can also use the measurement model to explain why $a + b = b + a$ no matter what numbers we choose for a and b. Imagine taking a segment of length a and combining it linearly with a segment of length b. That's how we get a length of $a + b$.

But if we just rotate that segment so it's upside down, we see that we have a segment of length b combined with a segment of length a, which makes a length of $b + a$.

But of course it's the same segment! We just turned it upside down! So the lengths must be the same. That is, $a + b = b + a$.

ADDITION IS ASSOCIATIVE

Your turn! You'll answer the question, "Why is addition of whole numbers associative?"

Property: Addition of whole numbers is associative.

What it Means (words): When I add three whole numbers in a given order, the way I group them (to add two at a time) doesn't affect the sum.

What it Means (symbols): For any three whole numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

Problem 14

1. Come up with at least three *examples* to demonstrate associativity of addition.
2. Use our models of addition to come up with an *explanation*. Why does associativity hold in *every case*? **Note:** your explanation should not use specific numbers. It is not an example!

0 IS AN IDENTITY FOR ADDITION

Property: The number 0 is an *identity* for addition of whole numbers.

What it Means (words): When I add any whole number to 0 (in either order), the sum is the very same whole number I added to 0.

What it Means (symbols): For any whole numbers n,

$$n + 0 = n \quad \text{and} \quad 0 + n = n.$$

Problem 15

1. Come up with at least three *examples* to demonstrate that 0 is an identity for addition.
2. Use our models of addition to come up with an *explanation*. Why does this property of 0 hold in *every possible case*?

PROPERTIES OF SUBTRACTION

Since addition and subtraction are so closely linked, it's natural to wonder if subtraction has some of the same properties as addition, like commutativity and associativity.

Example: Is subtraction commutative?

Justin asked if the operation of subtraction is commutative. That would mean that the difference of two whole numbers doesn't depend on the order in which you subtract them.

In symbols: *for every choice* of whole numbers a and b we would have $a - b = b - a$.

Jared says that subtraction is *not* commutative since $4 - 3 = 1$, but $3 - 4 \neq 1$. (In fact, $3 - 4 = -1$.)

Since the statement "subtraction is commutative" is a *universal statement*, one counterexample is enough to show it's not true. So Jared's counterexample lets us say with confidence:

Subtraction is **not** commutative.

Think / Pair / Share

Can you find any examples of whole numbers a and b where $a - b = b - a$ is true? Explain your answer.

Problem 16

Lyle asked if the operation of subtraction is associative.

1. State what it would mean for subtraction to be associative. You should use words and symbols.
2. What would you say to Lyle? Decide if subtraction is associative or not. Carefully explain how you made your decision and *how you know you're right*.

Problem 17

Jess asked if the number 0 is an identity for subtraction.

1. State what it would mean for 0 to be an identity for subtraction. You should use words and symbols.
2. What would you say to Jess? Decide if 0 is an identity for subtraction or not. Carefully explain how you made your decision and *how you know you're right*

Properties of Multiplication and Division

Now we're going to turn our attention to familiar properties of multiplication and division, with the focus still on explaining why these properties are always true.

Here are the four properties you'll think about:

- Multiplication of whole numbers is *commutative*.
- Multiplication of whole numbers is *associative*.
- Multiplication of whole numbers *distributes over addition*
- The number 1 is an *identity* for multiplication of whole numbers

For each of the properties, remember to keep straight:

- what the property is called and what it means (the definition),
- some examples that *demonstrate* the property, and
- an explanation for *why* the property holds.

Once again, it's important to distinguish between *examples* and *explanations*. They are not the same! Since there are infinitely many whole numbers, it's impossible to check every case, so examples will never be enough to explain why these properties hold. You have to figure out *reasons* for these properties to hold, based on what you know about the operations.

1 IS AN IDENTITY FOR MULTIPLICATION

We'll work out the explanation for the last of these facts, and you will work on the others.

Example: 1 is an Identity for multiplication

Property:

The number 1 is an identity for multiplication of whole numbers.

What it Means (words):

When I multiply a number by 1 (in either order), the product is that number.

What it Means (symbols):

For any whole number m,

$$m \times 1 = m \quad \text{and} \quad 1 \times m = m.$$

Examples:

$$1 \times 5 = 5, \qquad 19 \times 1 = 19, \qquad \text{and } 1 \times 1 = 1.$$

Why does the number 1 act this way with multiplication?

Why It's True, Explanation 1:

Let's think first about the definition of multiplication as repeated addition:

- $m \times 1$ means to add the number one to itself m times:
$$\underbrace{1 + 1 + \cdots + 1}_{m \text{ times}}$$
So we see that $m \times 1 = m$ for any whole number m.
- On the other hand, $1 \times m$ means to add the number m to itself just one time. So $1 \times m = m$ also.

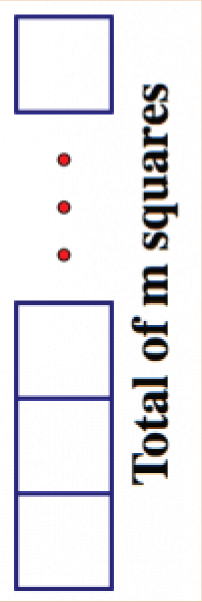
Why It's True, Explanation 2:

We can also use the number line model to create a justification. If Zed calculates $1 \times m$, he will start at 0 and face the positive direction. He will then take m steps forward, and he will do it just one time. So he lands at m, which means $1 \times m = m$.

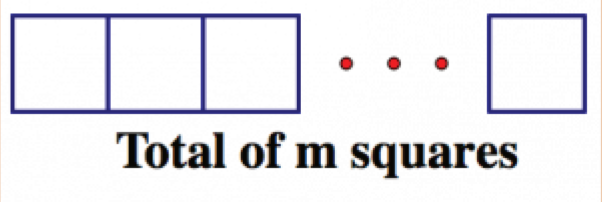
If Zed calculates $m \times 1$, he starts at 0 and faces the positive direction. Then he takes one step forward, and he repeats that m times. So he lands at m. We see that $m \times 1 = m$.

Why It's True, Explanation 3:

In the area model, $m \times 1$ represents m rows with one square in each row. That makes a total of m squares. So $m \times 1 = m$.



Similarly, $1 \times m$ represents one row of m squares. That's also a total of m squares. So $1 \times m = m$.



Think / Pair / Share

The example presented several different explanations. Do you think one is more convincing than the others? Or more clear and easier to understand?

MULTIPLICATION IS COMMUTATIVE

Property: Multiplication whole numbers is commutative.

What it Means (words): When I multiply two whole numbers, switching the order in which I multiply them does not affect the product.

What it Means (symbols): For any two whole numbers a and b,

$$a \cdot b = b \cdot a.$$

Problem 18

1. Come up with at least three *examples* to demonstrate the commutativity of multiplication.
2. Use our models of multiplication to come up with an *explanation*. Why does commutativity hold in *every case*? **Note:** Your explanation should not use particular numbers. It is not an example!

MULTIPLICATION IS ASSOCIATIVE

Property: Multiplication of whole numbers is associative.

What it Means (words): When I multiply three whole numbers in a given order, the way I group them (to multiply two at a time) doesn't affect the product.

What it Means (symbols): For any three whole numbers a, b, and c,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

Problem 19

1. Come up with at least three *examples* to demonstrate the associativity of multiplication.
2. Use our models of multiplication to come up with an *explanation*. Why does associativity hold in *every case*?

MULTIPLICATION DISTRIBUTES OVER ADDITION

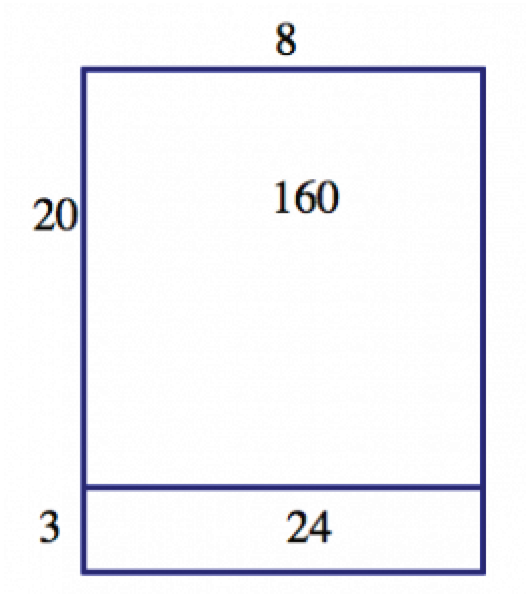
Property: Multiplication distributes over addition.

What it means: The distributive law for multiplication over addition is a little hard to state in words, so we'll jump straight to the symbols. For any three whole numbers x, y, and z:

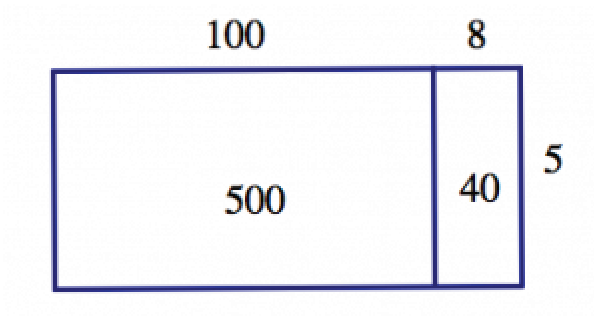
$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

Examples: We actually did calculations very much like the examples above, when we looked at the area model for multiplication.

$8 \cdot (23) = 8 \cdot (20 + 3) = 8 \cdot 20 + 8 \cdot 3 = 160 + 24 = 184$



$5 \cdot (108) = 5 \cdot (100 + 8) = 5 \cdot 100 + 5 \cdot 8 = 500 + 40 = 540$



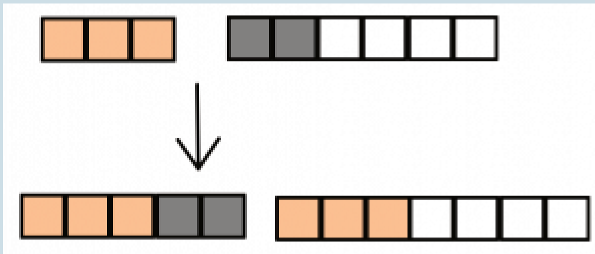
Problem 20

Which of the following pictures best represents the distributive law in the equation

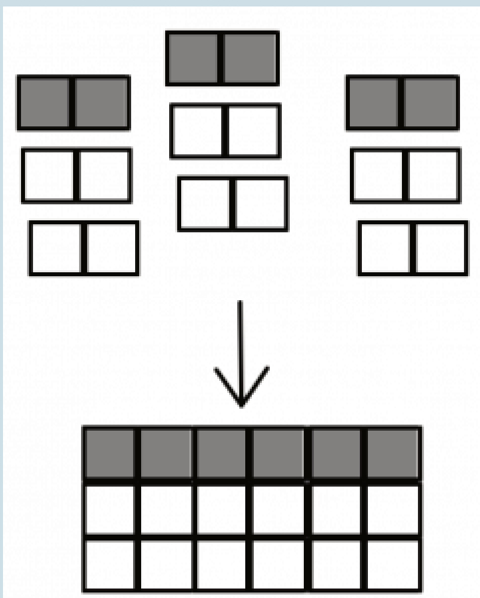
$$3 \cdot (2 + 4) = 3 \cdot 2 + 3 \cdot 4?$$

Explain your choice.

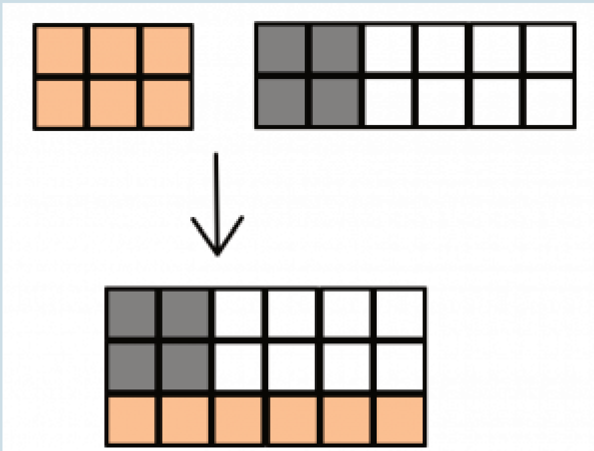
(a)



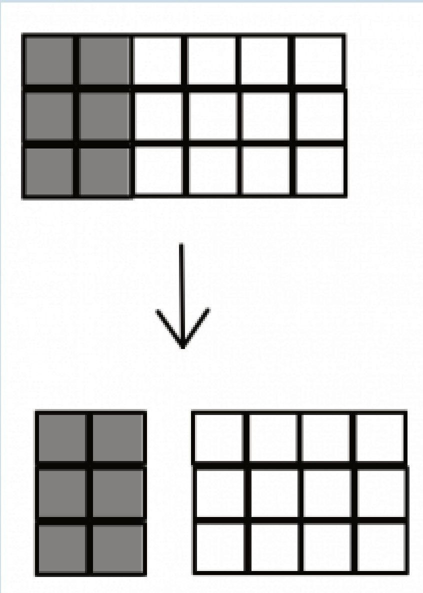
(b)

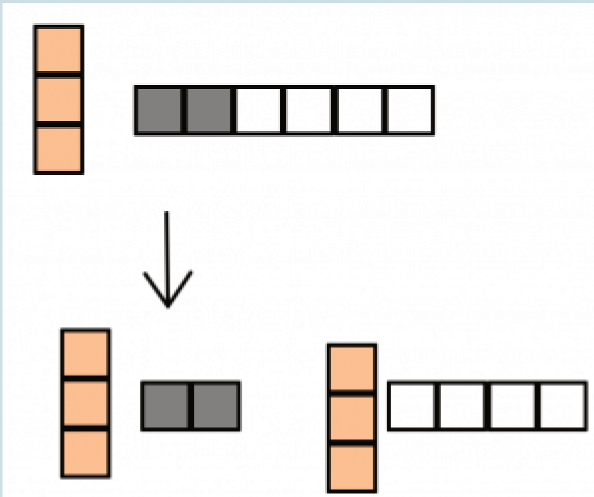


(c)



(d)



(e) 

Problem 21

Use the distributive law to easily compute each of these in your head (no calculators!). Explain your solutions.

$45 \times 11 \qquad 63 \times 101 \qquad 172 \times 1001.$

Think / Pair / Share

Use one of our models for multiplication and addition to explain why the distributive rule works every time.

PROPERTIES OF DIVISION

It's natural to wonder which, if any, of these properties also hold for division (since you know that the operations of multiplication and division are connected).

Example: Is Division Associative?

If division were associative, then for any choice of three whole numbers a, b, and c, we would have

$$a : (b : c) = (a : b) : c.$$

Remember, the parentheses tell you which two numbers to divide first.

Let's try the example $a = 9$, $b = 3$, and $c = 1$. Then we have:

$$9 : (3 : 1) = 9 : 3 = 3$$

and

$$(9 : 3) : 1 = 3 : 1 = 3.$$

So is it true? Is division associative? Well, we can't be sure.

This is just one example. But “division is associative” is a *universal statement*. If it’s true, it has to work for *every possible example*. Maybe we just stumbled on a good choice of numbers, but it won’t always work.

Let’s keep looking. Try $a = 16$, $b = 4$, and $c = 2$.

$$16 : (4 : 2) = 16 : 2 = 8$$

and

$$(16 : 4) : 2 = 4 : 2 = 2.$$

That’s all we need! A single counterexample lets us conclude:

Division is **not** associative.

What about the other properties? It’s your turn to decide!

Problem 22

1. State what it would mean for division to be commutative. You should use words and symbols.
2. Decide if division is commutative or not. Carefully explain how you made your decision and *how you know you’re right.*

Problem 23

1. State what it would mean for division to distribute over addition. You definitely want to use symbols!
2. Decide if division distributes over addition or not. Carefully explain how you made your decision and *how you know you're right.*

Problem 24

1. State what it would mean for the number 1 to be an identity for division. You should use words and symbols.
2. Decide if 1 is an identity for division or not. Carefully explain how you made your decision and *how you know you're right.*

ZERO PROPERTY FOR MULTIPLICATION AND DIVISION

Problem 25

You probably know another property of multiplication that hasn't been mentioned yet:

If I multiply any number times 0 (in either order), the product is 0. This is sometimes called the zero property of multiplication. Notice that the *zero property* is very different from the property of being an identity!

1. Write what the zero property means using both words and symbols:

For every whole number n . . .

2. Give at least three examples of the zero property for multiplication.

3. Use one of our models of multiplication to explain why the zero property holds.

Think / Pair / Share

- For each division problem below, turn it into a multiplication problem. Solve those problems if you can. If you can't, explain what is wrong.

$5 : 0 \qquad 0 : 5 \qquad 7 : 0 \qquad 0 : 7 \qquad 0 : 0$

- Use your work to explain why we say that *division by* 0 *is undefined.*
- Use one of our models of division to explain why *division by* 0 *is undefined.*

Four Fact Families

In elementary school, students are often encouraged to memorize "four fact families," for example:

$$2 + 3 = 5 \qquad 5 - 3 = 2$$
$$3 + 2 = 5 \qquad 5 - 2 = 3$$

Here's a different "four fact family":

$$2 \cdot 3 = 6 \qquad 6 : 3 = 2$$
$$3 \cdot 2 = 6 \qquad 6 : 2 = 3$$

Think / Pair / Share

- In what sense are these groups of equations "families"?
- Write down at least two more addition / subtraction four fact families.
- Use properties of addition and subtraction to explain *why* these four fact families are each really one fact.
- Write down at least two more multiplication / division four fact families.
- Use properties of multiplication and division to explain *why* these four fact families are each really one fact.

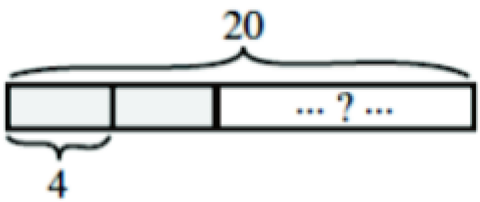
Problem 26

1. Here's a true fact in base six: $2_{\text{six}} + 3_{\text{six}} = 5_{\text{six}}$. Write the rest of this four fact family.
2. Here's a true fact in base six: $11_{\text{six}} - 5_{\text{six}} = 2_{\text{six}}$. Write the rest of this four fact family.

Going Deeper with Division

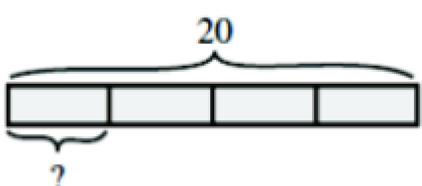
So far we've been thinking about division in what's called the *quotative model*. In the quotative model, we want to make groups of equal size. We know the *size of the group*, and we ask *how many groups*. For example, we think of 20 : 4 as:

How many groups of 4 are there in a group of 20?



Thinking about four fact families, however, we realize we can turn the question around a bit. We could think about the *partitive model* of division. In the partitive model, we want to make an equal number of groups. We know *how many groups*, and we ask the *size of the group*. In the partitive model, we think of 20 : 4 as:

20 is 4 groups of what size?



When we know the original amount and the number of parts, we use partitive division to find the size of each part.

When we know the original amount and the size of each part, we use quotative division to find the number of parts.

Here are some examples in word problems:

Partitive	Quotative
number of groups known find the number in each group	number in each group known find the number of groups
A movie theater made $6450 in one night of ticket sales. 430 people purchased a ticket. How much does one ticket cost?	A movie theater made $6450 in one night of ticket sales. Each ticket cost $12.50. How many people purchased a ticket?

Think / Pair / Share

For each word problem below:

- Draw a picture to show what the problem is asking.
- Use your picture to help you decide if it is a quotative or a partitive division problem.
- Solve the problem using any method you like.

1. David made 36 cookies for the bake sale. He packaged the cookies in boxes of 9. How many boxes did he use?
2. David made 36 cookies to share with his friends at lunch. There were 12 people at his lunch table (including David). How many cookies did each person get?
3. Liz spent one summer hiking the Appalachin trail. She completed 1,380 miles of the trail and averaged 15 miles per day. How many days was she out hiking that summer?
4. On April 1, 2012, **Chase Norton** became the first person to hike the entire Koʻolau summit in a single trip. (True story!) It took him eight days to hike all 48 miles from start to finish. If he kept a steady pace, how many miles did he hike each day?

Think / Pair / Share

Write your own word problems: Write one partitive division problem and one quotative division problem. Choose your numbers carefully so that the answer works out nicely. Be sure to solve your problems!

Why think about these two models for division? You won't be teaching the words *partitive* and *quotative* to your students. But recognizing the two kinds of division problems (and being able to come up with examples of each) will make you a better teacher.

It's important that your students are exposed to both ways of thinking about division, and to problems of both types. Otherwise, they may think about division too narrowly and not really understand what's going on. If you understand the two kinds of problems, you can more easily diagnose and remedy students' difficulties.

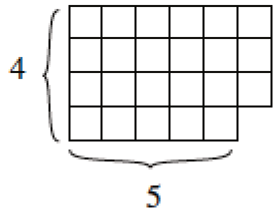
Most of the division problems we've looked at so far have come out evenly, with no remainder. But of course, that doesn't always happen! Sometimes, a whole number answer makes sense, and the context of the problem should tell you which whole number is the right one to choose.

Problem 27

What is 43 : 4?

1. Write a problem that uses the computation 43 : 4 and gives 10 as the correct answer.
2. Write a problem that uses the computation 43 : 4 and gives 11 as the correct answer.
3. Write a problem that uses the computation 43:4 and gives 10.75 as the correct answer.

We can think about division with remainder in terms of some of our models for operations. For example, we can calculate that 23 : 4 = 5 R3. We can picture it this way:



$$23 = 5 \cdot 4 + 3.$$

Think / Pair / Share

- Explain how the picture above illustrates 23 = 5 · 4 + 3. Where do you see the remainder of 3 in the picture?
- Explain the connection between these two equations.

  23 : 4 = 5 R3   and   23 = 5 · 4 + 3.

- How could you use the number line model to show the calculation 23 = 5 · 4 + 3? What does a "remainder" look like in this model?
- Draw area models for each of these division problems. Find the quotient and remainder.

  40 : 12   59 : 10   91 : 16

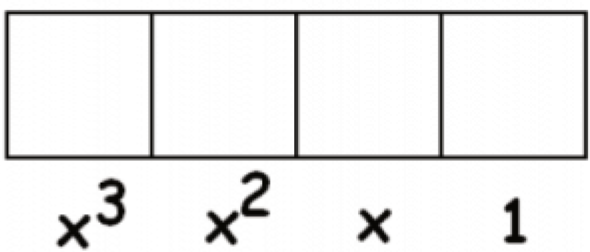
25. Division Explorations

Anu refuses to tell anyone if she is working in a 1←10 system, or a 1←5 system, or any other system. She makes everyone call it a $1 \leftarrow x$ system but won't tell anyone what *x* stands for.

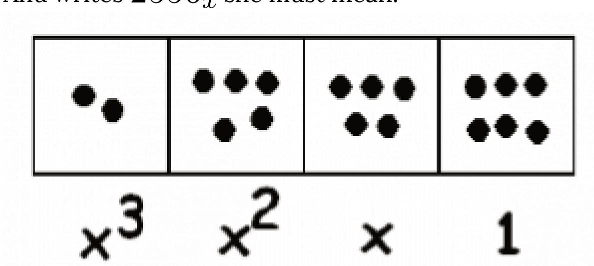
We know that boxes in a 1←10 have values that are powers of ten: 1, 10, 100, 1000, 10000…

And boxes in a 1←5 system are powers of five: 1, 5, 25, 125, 625…

So Anu's system, whatever it is, must be powers of *x*: $1, x, x^2, x^3, x^4 \ldots$

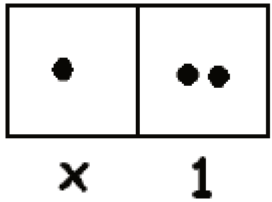


When Anu writes 2556_x she must mean:



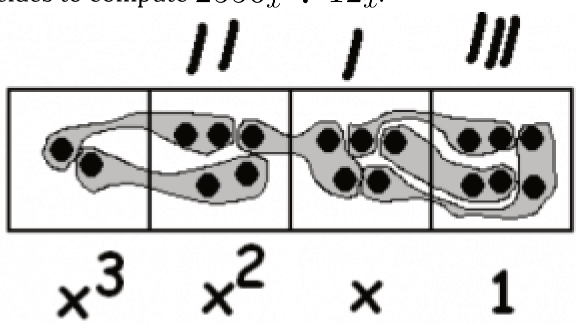
$$2x^3 + 5x^2 + 5x + 6.$$

And when she writes 12_x she means:



$x+2.$

Anu decides to compute $2556_x \div 12_x$.



She obtains:

$$(2x^3+5x^2+5x+6)\div(x+2)=2x^2+x+3.$$

Problem 28

1. Check Anu's division by computing the product

   $$(x+2)(2x^2+x+3).$$

   Did it work?

2. Use Anu's method to find

   $$(3x^2+7x+2)\div(x+2).$$

3. Use Anu's method to find

$$(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1).$$

4. Use Anu's method to find

$$(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1).$$

Anu later tells use that she really was thinking of a 1←10 system so that x does equal ten. Then her number 2556 really was two thousand, five hundred and fifty six and 12 really was twelve. Her statement:

$$(2x^3 + 5x^2 + 5x + 6) \div (x + 2) = 2x^2 + x + 3$$

is actually 2556 : 12 = 213.

Problem 29

1. Check that 2556 : 12 = 213 is correct in base 10.
2. Keeping with the 1←10 system, what division problems did you actually solve in parts (b), (c), and (d) of Problem 28? Check that your answers are correct.

Uh Oh! Anu has changed her mind. She now says she was thinking of a 1←11 system.

Now 2556_x means $2 \cdot 11^3 + 5 \cdot 11^2 + 5 \cdot 11 + 6 = 3328_{\text{ten}}$.

Similarly, 12_x means $1 \cdot 11 + 2 = 13_{\text{ten}}$, and 213_x means $2 \cdot 11^2 + 1 \cdot 11 + 3 = 256_{\text{ten}}$.

So Anu's computation $2556_x \div 12_x = 213_x$ is actually the (base 10) statement:

$$3328 : 13 = 256.$$

Problem 30

1. Check that 3328 : 13 = 256 is also correct in base ten.
2. Keeping with the 1←11 system, what division problems did you actually solve in parts (b), (c), and (d) of Problem 28? Check that they are correct.

Problem 31

1. Use Anu's method to show that

   $$(x^4 + 4x^3 + 6x^2 + 4x + 1) \div (x + 1) = (x^3 + 3x^2 + 3x + 1).$$

2. What is this saying for $x = 10$? Check that the division is correct.
3. What is this saying for $x = 2$? Check that the division is correct.
4. What is this saying for x equal to each of 3, 4, 5, 6, 7, 8, 9, and 11? Check that each division is correct.

5. What is this saying for $x = 0$?

26. Problem Bank

Problem 32

Compute the following using dots and boxes:

64212 : 3

44793 : 21

6182 : 11

99916131 : 31

637824 : 302

2125122 : 1011

Problem 33

1. Fill in the squares using the digits 4, 5, 6, 7, 8, and 9 exactly one time each to make the largest possible sum:

$$\begin{array}{r} \square\ \square\ \square \\ +\square\ \square\ \square \\ \hline \end{array}$$

2. Fill in the squares using the digits 4, 5, 6, 7, 8, and 9 exactly one time each to make the smallest possible (positive) difference:

$$\begin{array}{r} \square\ \square\ \square \\ -\square\ \square\ \square \\ \hline \end{array}$$

Problem 34

1. Make a base six addition table.

$+$	0_{six}	1_{six}	2_{six}	3_{six}	4_{six}	5_{six}
0_{six}	0_{six}	1_{six}				
1_{six}						10_{six}
2_{six}						
3_{six}						
4_{six}						
5_{six}				12_{six}		

2. Use the table to solve these subtraction problems.

$13_{\text{six}} - 5_{\text{six}}$ $\qquad$ $12_{\text{six}} - 3_{\text{six}}$ $\qquad$ $10_{\text{six}} - 4_{\text{six}}$.

Problem 35

Do these calculations in base four. Don't translate to base 10 and then calculate there – try to work in base four.

1. $33_{\text{four}} + 11_{\text{four}}$
2. $123_{\text{four}} + 22_{\text{four}}$
3. $223_{\text{four}} - 131_{\text{four}}$
4. $112_{\text{four}} - 33_{\text{four}}$

Problem 36

1. Make a base five multiplication table.

$\times$	0_{five}	1_{five}	2_{five}	3_{five}	4_{five}
0_{five}	0_{five}	0_{five}			
1_{five}					
2_{five}					
3_{five}			11_{five}		
4_{five}				22_{five}	

2. Use the table to solve these division problems.

$11_{five} \div 2_{five}$ $22_{five} \div 3_{five}$ $13_{five} \div 4_{five}$.

Problem 37

1. Here is a true fact in base five:

$$2_{five} \cdot 3_{five} = 11_{five}$$

   Write the rest of this four fact family.

2. Here is a true fact in base five:

$$13_{five} \div 2_{five} = 4_{five}$$

Write the rest of this four fact family.

Directions for AlphaMath Problems (Problems 38 – 41):

- Letters stand for digits 0–9.
- In a given problem, the same letter always represents the same digit, and different letters always represent different digits.
- There is no relation between problems (so "A" in part 1 and "A" in part 3 might be different).
- Two, three, and four digit numbers never start with a zero.
- Your job: Figure out what digit each letter stands for, so that the calculation shown is correct.

Problem 38

Notes: In part 2, "O" represents the letter "oh," not the digit zero.

1. $$\begin{array}{r} A \\ A \\ +A \\ \hline H\ A \end{array}$$

2. $$\begin{array}{r} O\ N\ E \\ +O\ N\ E \\ \hline T\ W\ O \end{array}$$

3. $$\begin{array}{r} A\ B\ C \\ +A\ C\ B \\ \hline C\ B\ A \end{array}$$

Problem 39

Here's another AlphaMath problem.

$$\begin{array}{r} T\ E\ N \\ +N\ O\ T \\ \hline N\ I\ N\ E \end{array}$$

1. Solve this AlphaMath problem in base 10.
2. Now solve it in base 6.

Problem 40

Find all solutions to this AlphaMath problem **in base 9**.

Notes: Even though this is two calculations, it is a *single problem*. All T's in both calculations represent the same digit, all B's represent the same digit, and so on.

Remember that "O" represents the letter "oh" and not the digit zero, and that two and three digit numbers never start with the digit zero

$$\begin{array}{r} T\ O \\ -B\ E \\ \hline O\ R \end{array} \qquad \begin{array}{r} N\ O\ T \\ -\quad T\ O \\ \hline B\ E \end{array}$$

Problem 41

This is a single AlphaMath problem. (So all G's represent the same digit. All A's represent the same digit. And so on.)

Solve the problem in **base 6**.

$$GALON = (GOO)^2 \qquad ALONG = (OOG)^2$$

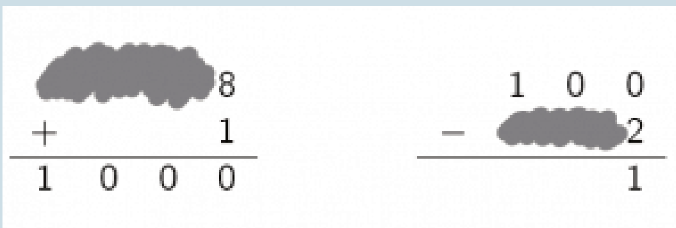
Problem 42

A *perfect square* is a number that can be written as $a \cdot a$ or a^2 (some number times itself).

1. Which of the following *base seven numbers* are perfect squares? For each number, answer **yes** (it is a perfect square) or **no** (it is not a perfect square) and give a justification of your answer.
   4_{seven} 25_{seven} 51_{seven}
2. For which choices of base b is the number $b^2 a perfect square? Justify your answer

Problem 43

Geoff spilled coffee on his homework. The answers were correct. Can you determine the missing digits and the bases?



Problem 44

1. Rewrite each subtraction problem as an addition problem:

   $x - 156 = 279 \qquad 279 - 156 = x \qquad a - x = b.$

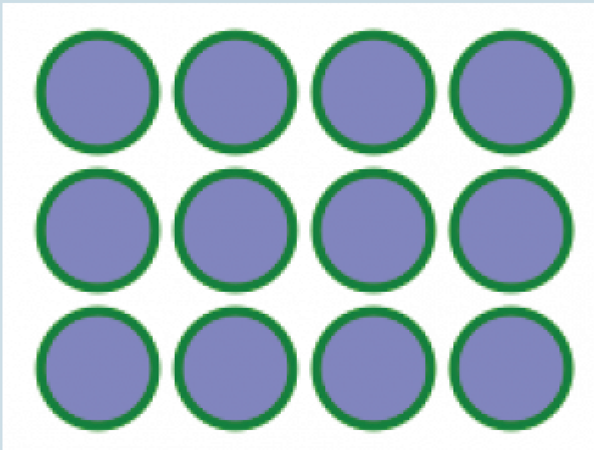
2. Rewrite each division problem as a multiplication problem:

   $24 \div x = 12 \qquad x \div 3 = 27 \qquad a \div b = x.$

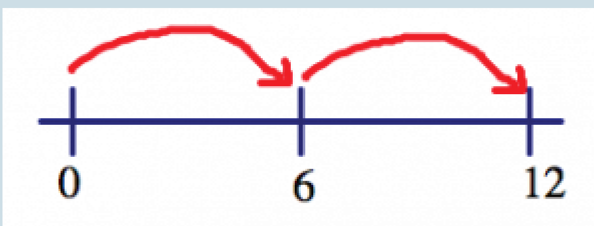
Problem 45

Which of the following models represent the same multiplication problem? Explain your answer.

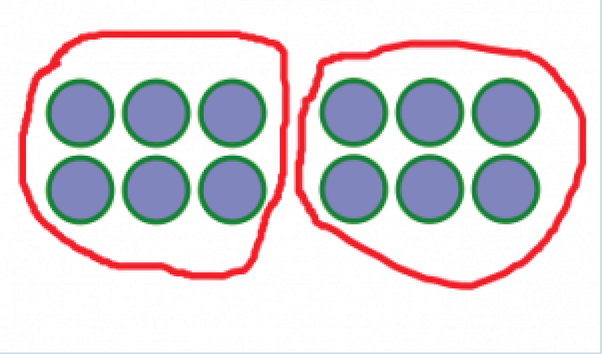
(a)



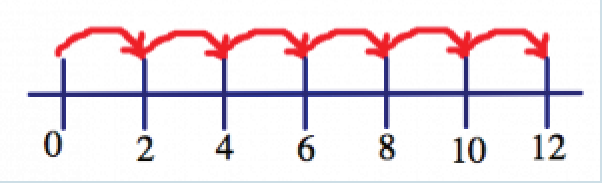
(b)



(c)



(d)



Problem 46

Show an area model for each of these multiplication problems. Write down the standard computation next to the area model and see how it compares.

20 × 33 24 × 13 17 × 11

Problem 47

Suppose the 2 key on your calculator is broken. How could you still use the calculator compute these products? Think about what properties of multiplication might be helpful. (Write out the calculation you would do on the calculator, not just the answer.)

1592×3344 $\qquad$ 2008×999 $\qquad$ 655×525

Problem 48

Today is Jennifer's birthday, and she's twice as old as her brother. When will she be twice as old as him again? Choose the best answer and justify your choice.

1. Jennifer will always be twice as old as her brother.
2. It will happen every two years.
3. It depends on Jennifer's age.
4. It will happen when Jennifer is twice as old as she is now.
5. It will never happen again.

Problem 49

1. Find the quotient and remainder for each problem.

   $7 \div 3$ $3 \div 7$ $7 \div 1$ $1 \div 7$ $15 \div 5$

2. How many possible remainders are there when dividing by these numbers? Justify what you say.

   2 12 62 23

Problem 50

Identify each problem as either partitive or quotative division and say why you made that choice. Then solve the problem.

1. Adriana bought 12 gallons of paint. If each room requires three gallons of paint, how many rooms can she paint?
2. Chris baked 15 muffins for his family of five. How many muffins does each person get?
3. Prof. Davidson gave three straws to each student for an activity. She used 51 straws. How many students are in her class?

Problem 51

Use the digits 1 through 9. Use each digit exactly once. Fill in the squares to make all of the equations true.

$$\begin{array}{ccccc} \square & - & \square & = & \square \\ & & & & \times \\ \square & \div & \square & = & \square \\ & & & & = \\ \square & + & \square & = & \square \end{array}$$

PART IV
FRACTIONS

> A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction.
>
> —Leo Tolstoy

The "Pies Per Child[1]" approach to fractions used in this part comes from James Tanton, and is used with his permission. See his development of these and other ideas at http://gdaymath.com/.

1. Pie image by Claus Ableiter (Own work) [GFDL, CC-BY-SA-3.0 or CC BY-SA 2.5-2.0-1.0], via Wikimedia Commons

27. Introduction

Fractions are one of the hardest topics to teach (and learn!) in elementary school. What is the reason for this? In this part of the book, will try to provide you with some insight about this (as well as some better ways for understanding, teaching, and learning about fractions). But for now, think about what makes this topic so hard.

Think / Pair / Share

You may have struggled learning about fractions in elementary school. Maybe you still find them confusing. Even if you were one of the lucky ones who did not struggle when learning about fractions, you probably had friends who did struggle.

With a partner, talk about why this is. What is so difficult about understanding fractions? Why is the topic harder than other ones we tackle in elementary schools?

Remember that teachers should have lots of mental models — lots of ways to explain the same concept. In this chapter, we will look at some different ways to understand the idea of fractions as well as basic operations on them.

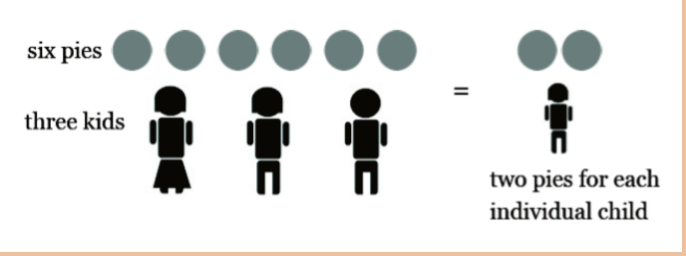
28. What is a Fraction?

One of the things that makes fractions such a difficult concept to teach and to learn is that you have to think about them in a lot of different ways, depending on the problem at hand. For now, we are going to think of a fraction as the answer to a division problem.

Example: Pies per child

Suppose 6 pies are to be shared equally among 3 children. This yields 2 pies per kid. We write

$$\frac{6}{3} = 2.$$



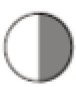
The fraction $\frac{6}{3}$ is equivalent to the division problem $6 \div 3 = 2$. It represents the number of pies one whole child receives when three kids share six pies equally.

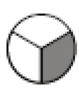
In the same way ...

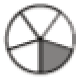
- Sharing 10 pies among 2 kids yields $\frac{10}{2} = 5$ pies per kid.
- Sharing 8 pies among 2 children yields $\frac{8}{2} = 4$ pies per child.
- Sharing 5 pies among 5 kids yields $\frac{5}{5} = 1$ pie per kid.
- Sharing 1 pie among 2 children yields $\frac{1}{2}$, which we call "one-half."

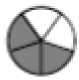
This final example is actually saying something! It also represents how fractions are usually taught to students:

If one pie is shared *equally* between two kids, then each child receives a portion of a pie which we choose to call "half."

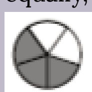
Thus students are taught to associate the number "$\frac{1}{2}$" to the picture .

In the same way, the picture  is said to represent "one-third," that is, $\frac{1}{3}$. (And this is indeed the amount of pie an individual child would receive if one pie is shared among three.)

The picture  is called “one-fifth” and is indeed $\frac{1}{5}$, the amount of pie an individual receives if three pies are shared among five children.

And the picture  is called “three-fifths” to represent $\frac{3}{5}$, the amount of pie an individual receives if three pies are shared among five children.

Think / Pair / Share

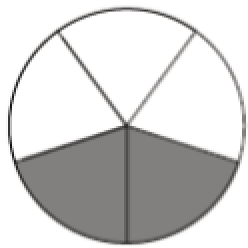
Carefully explain why this is true: If five kids share three pies equally, each child receives an amount that looks like this: .

Your explanation will probably require both words and pictures.

On Your Own

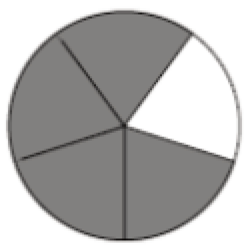
Work on the following exercises on your own or with a partner.

1. Draw a picture associated with the fraction $\frac{1}{6}$.
2. Draw a picture associated with the fraction $\frac{3}{7}$. Is your picture really the amount of pie an individual would receive if three pies are shared among seven kids? Be very clear on this!
3. Let’s work backwards! Here’s the answer to a division problem:



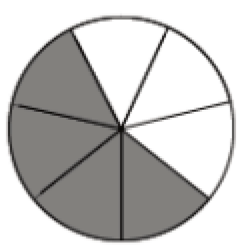
This represents the amount of pie an individual kid receives if some number of pies is shared among some number of children. How many pies? How many children? How can you justify your answers?

4. Here's another answer to a division problem:



How many pies? How many children? How can you justify your answers?

5. Here is another answer to a division problem:



How many pies? How many children? How can you justify your answers?

6. Leigh says that "$\frac{3}{5}$ is three times as big as $\frac{1}{5}$." Is this right? Explain your answer.

7. Draw a picture for the answer to the division problem $\frac{4}{8}$. Describe what you notice about the answer.

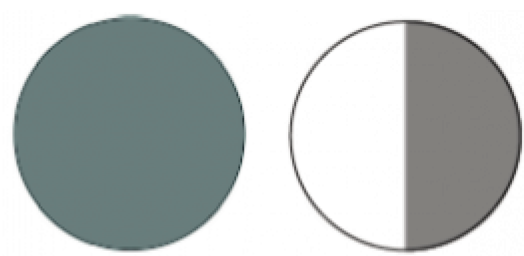
8. Draw a picture for the answer to the division problem $\frac{2}{10}$. Describe what you notice about the answer.

9. What does the division problem $\frac{1}{1}$ represent? How much pie does an individual child receive?

10. What does the division problem $\frac{5}{1}$ represent? How much pie does an individual child receive?

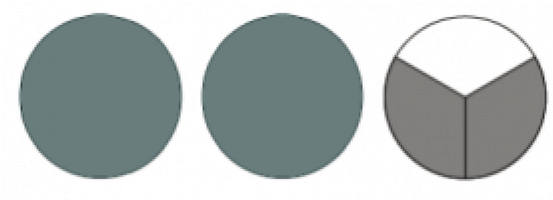
11. What does the division problem $\frac{5}{5}$ represent? How much pie does an individual child receive?

12. Here is the answer to another division problem. This is the amount of pie an individual child receives:



How many pies were in the division problem? How many kids were in the division problem? Justify your answers.

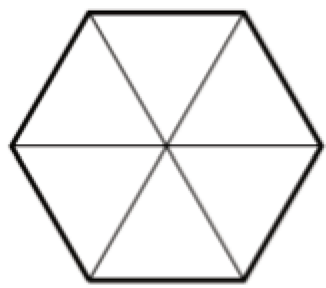
13. Here is the answer to another division problem. This is the amount of pie an individual child receives:



How many pies were in the division problem? How many kids were in the division problem? Justify your answers

14. Many teachers have young students divide differently shaped pies into fractions. For example, a hexagonal pie is good for illustrating the fractions:

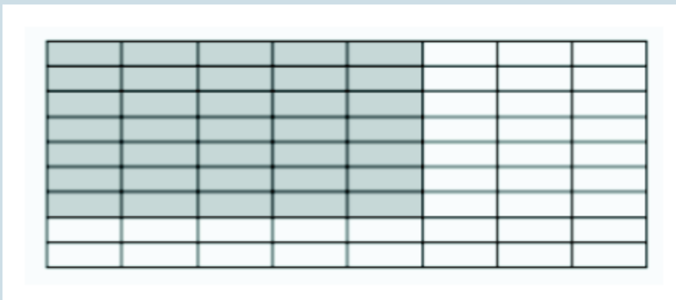
$$\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \text{ and } \frac{6}{6}.$$



- Why is this shape used? What does $\frac{1}{6}$ of a pie look like?
- What does $\frac{6}{6}$ of a pie look like?
- What shape pie would be good for illustrating the fractions $\frac{1}{8}$ up to $\frac{8}{8}$?

Problem 1

Some rectangular pies are distributed to some number of kids. This picture represents the amount of pie an individual child receives. The large rectangle represents one whole pie.



How many pies? How many kids? Carefully justify your answers!

Pies Per Child Model

In our model, a fraction $\frac{a}{b}$ represents the amount of pie an individual child receives when a pies are shared equally by b kids.

$$\begin{array}{l}\#\textbf{pies} \longrightarrow \\[1em] \#\textbf{kids} \longrightarrow\end{array} \frac{a}{b} = \textbf{amount per individual child}$$

Think / Pair / Share

- What is $\frac{2}{2}$? What is $\frac{7}{7}$? What is $\frac{100}{100}$? How can you use the "Pies Per Child Model" to make sense of $\frac{a}{a}$ for any positive whole number a?
- What is $\frac{2}{1}$? What is $\frac{7}{1}$? What is $\frac{1876}{1}$? How can you use the "Pies Per Child Model" to make sense of $\frac{b}{1}$ for any positive whole number b?
- Write the answer to this division problem: "I have no pies to share among thirteen kids." How can you generalize this division problem to make a general statement about fractions?

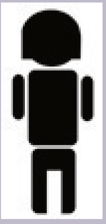
Definition

For a fraction $\frac{a}{b}$, the top number a (which, for us, is the number of pies) is called the **numerator** of the fraction, and the bottom number b (the number of kids), is called the **denominator** of the fraction.

Most people insist that the numerator and denominator each be whole numbers, but they do not have to be.

Think / Pair / Share

To understand why the numerator and denominator need not be whole numbers, we must first be a little gruesome. Instead of dividing pies, let's divide kids! Here is one child:



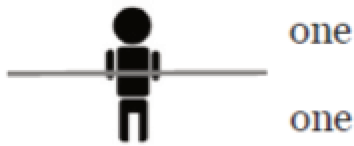
- What would half a kid look like?
- What would one-third of a kid look like?
- What would three-fifths of a child look like?

So, what would

$$\frac{1}{\left(\frac{1}{2}\right)}$$

represent?

This means assigning one pie to each "group" of half a child. So how much would a whole child receive? Well, we would have a picture like this:



The whole child gets two pies, so we have:

$$\frac{1}{\left(\frac{1}{2}\right)} = 2.$$

Think / Pair / Share

Draw pictures for these problems if it helps!

1. What does
$$\frac{1}{\left(\frac{1}{3}\right)}$$
represent? Justify your answer using the "Pies Per Child Model."
2. What is
$$\frac{1}{\left(\frac{1}{6}\right)}?$$
Justify your answer.

3. Explain why the fraction

$$\frac{5}{\left(\frac{1}{2}\right)}$$

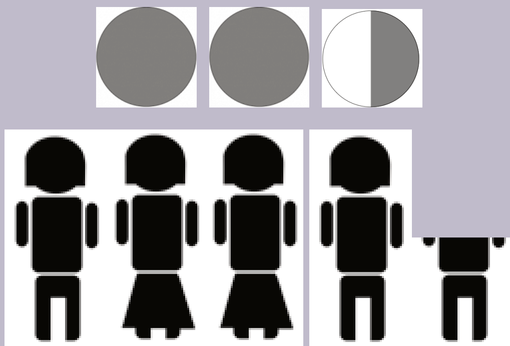
represents the number 10. (How much pie is given to half a kid? To a whole kid?)

4. What is

$$\frac{4}{\left(\frac{1}{3}\right)}?$$

Justify your answer.

5. **Challenge:** Two-and-a-half pies are to be shared equally among four-and-a-half children. How much pie does an individual (whole) child receive? Justify your answer.



Jargon: Improper fractions

A fraction with a numerator smaller than its denominator is called (in school math jargon) a *proper fraction*. For example, $\frac{45}{58}$ is "proper."

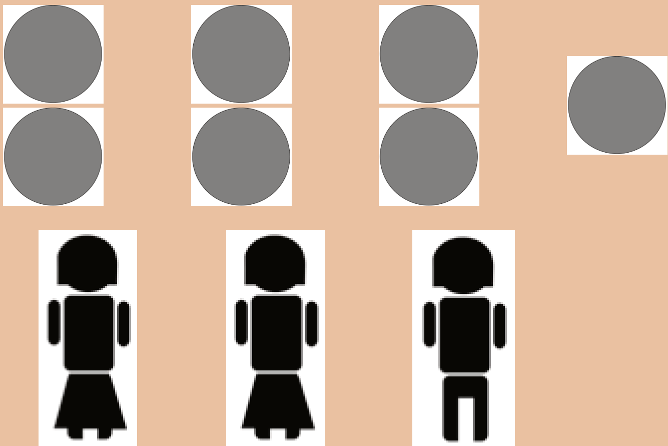
A fraction with numerator larger than its denominator is called (in school math jargon) an *improper fraction*. For example, $\frac{7}{3}$ is "improper." (In the 1800's, these fractions were called *vulgar fractions*.)

For some reason, improper fractions are considered, well, *improper* by some teachers. So students are often asked to write improper fractions as a combination of a whole number and a proper fraction (often called "mixed numbers"). Despite their name and these prejudices, improper fractions are useful nonetheless!

With a mixed number, you have a good sense of the overall size of the number: "a little more than five," or "a bit less than 17." But it is often easier to do calculations with improper fractions (why do you think that is?).

Example: white$7/3$

If seven pies are shared among three kids, then each kid will certainly receive two whole pies, leaving one pie to share among the three children.



Thus, $\frac{7}{3}$ equals 2 plus $\frac{1}{3}$. People write:

$$\frac{7}{3} \;=\; 2\tfrac{1}{3}$$

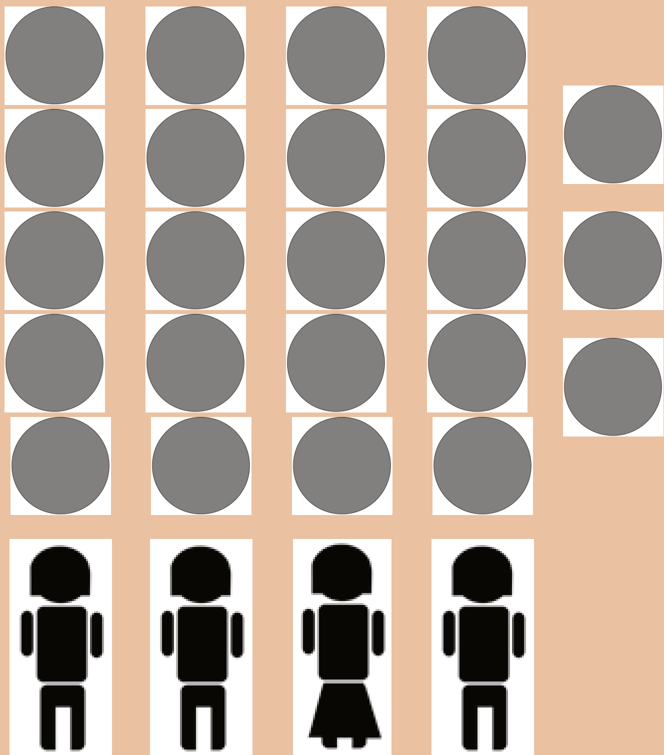
and call the result a *mixed number*. One can also write:

$$2+\frac{1}{3},$$

which is what $2\frac{1}{3}$ really means. But most people choose to omit the plus sign.

Example: white$23/4$

If 4 children share 23 pies, we can give them each 5 whole pies. That uses 20 pies, and there are 3 pies left over.



Those three pies are still to be shared equally by the four kids. We have:

$$\frac{23}{4} \quad = \quad 5\tfrac{3}{4}.$$

Example: white $2\frac{1}{5}$

For fun, let us write the number 2 as a fraction with denominator 5:

$$2 = \frac{10}{5}.$$

So:

$$2\tfrac{1}{5} \quad = \quad 2 + \frac{1}{5} \quad = \quad \frac{10}{5} + \frac{1}{5} \quad = \quad \frac{11}{5}.$$

We have written the mixed number $2\frac{1}{5}$ as the improper fraction $\frac{11}{5}$.

Think / Pair / Share

- Write each of the following as a mixed number. Explain how you got your answer.

$$\frac{17}{3}, \qquad \frac{8}{5}, \qquad \frac{100}{3}, \qquad \frac{200}{199}.$$

- Convert each of these mixed numbers into "improper" fractions. Explain how you got your answer.

$$3\frac{1}{4}, \qquad 5\frac{1}{6}, \qquad 1\frac{3}{11}, \qquad 200\frac{1}{200}.$$

Students are often asked to memorize the names "proper fractions," "improper fractions," and "mixed number" so that they can follow directions on tests and problem sets.

But, to a mathematician, these names are not at all important! There is no "correct" way to express an answer (assuming, that the answer is mathematically the right number). We often wish to express our answer in a simpler form, but sometimes the context will tell you what form is "simple" and what form is more complicated.

As you work on problems in this chapter, decide for yourself which type of fraction would be best to work with as you do your task.

29. The Key Fraction Rule

We know that $\frac{a}{b}$ is the answer to a division problem:

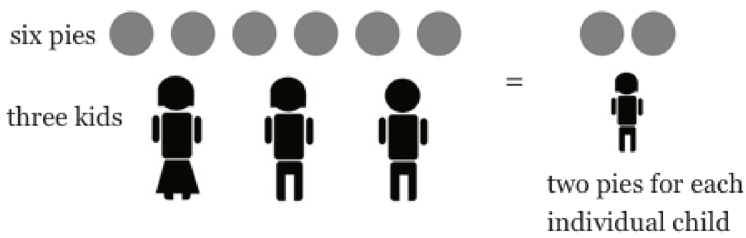
$$\frac{a}{b}$$

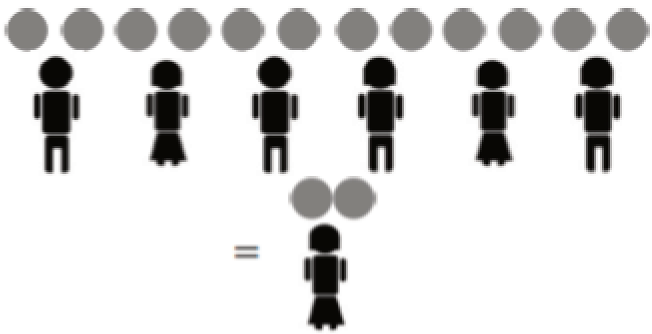
represents the amount of pie an individual child receives when a pies are shared equally by b children.

What happens if we double the number of pie and double the number of kids? Nothing! The amount of pie per child is still the same:

$$\frac{2a}{2b} = \frac{a}{b}.$$

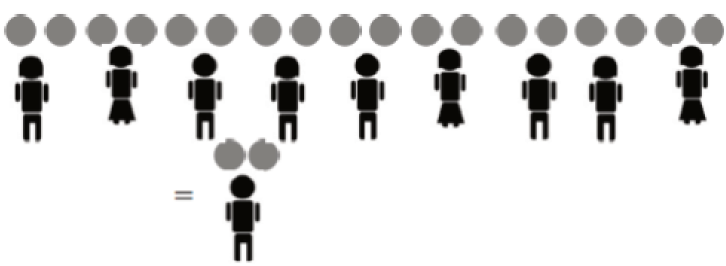
For example, as the picture shows, $\frac{6}{3}$ and $\frac{12}{6}$ both give two pies for each child.





And tripling the number of pies and the number of children also does not change the final amount of pies per child, nor does quadrupling each number, or one trillion-billion-tupling the numbers!

$$\frac{6}{3} = \frac{12}{6} = \frac{18}{9} = \ldots = \text{two pies per child.}$$



This leads us to want to believe:

Key Fraction Rule

$$\frac{xa}{xb} = \frac{a}{b},$$

(at least for positive whole numbers x).

We say that the fractions $\frac{xa}{xb}$ and $\frac{a}{b}$ are **equivalent**.

Example: Fractions equivalent to 3/5

For example,

$$\frac{3}{5} \text{ (sharing three pies among five kids)}$$

yields the same result as

$$\frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10} \text{ (sharing six pies among ten kids)}$$

and as

$$\frac{3 \cdot 100}{5 \cdot 100} = \frac{300}{500} \text{ (sharing 300 pies among 500 kids).}$$

Think / Pair / Share

Write down a lot of equivalent fractions for $\frac{1}{2}$, for $\frac{10}{3}$, and for 1.

Example: Going Backwards

$$\frac{20}{32} \text{ (sharing 20 pies among 32 kids)}$$

is the same problem as:

$$\frac{5 \cdot 4}{8 \cdot 4} = \frac{5}{8} \text{ (sharing five pies among eight kids).}$$

Most people say we have *cancelled* or taken a common factor 4 from the numerator and denominator.

Mathematicians call this process *reducing* the fraction to lowest terms. (We have made the numerator and denominator smaller, in fact as small as we can make them!)

Teachers tend to say that we are *simplifying* the fraction. (You have to admit that $\frac{5}{8}$ does look simpler than $\frac{20}{32}$.)

Example: How Low Can You Go?

As another example, $\frac{280}{350}$ can certainly be simplified by noticing that there is a common factor of 10 in both the numerator and the denominator:

$$\frac{280}{350} = \frac{28 \cdot 10}{35 \cdot 10} = \frac{28}{35}.$$

We can go further as 28 and 35 are both multiples of 7:

$$\frac{28}{35} = \frac{4 \cdot 7}{5 \cdot 7} = \frac{4}{5}.$$

Thus, sharing 280 pies among 350 children gives the same result as sharing 4 pies among 5 children!

$$\frac{280}{350} = \frac{4}{5}.$$

Since 4 and 5 share no common factors, this is as far as we can go with this example (while staying with whole numbers).

On Your Own

Mix and Match: On the top are some fractions that have not been simplified. On the bottom are the simplified answers, but in random order. Which simplified answer goes with which fraction? (Notice that there are fewer answers than questions!)

1. $\frac{10}{20}$ 2. $\frac{50}{75}$ 3. $\frac{24000}{36000}$ 4. $\frac{24}{14}$ 5. $\frac{18}{32}$ 6. $\frac{1}{40}$

$a.\ \frac{2}{3}$ $b.\ \frac{9}{16}$ $c.\ \frac{12}{7}$ $d.\ \frac{1}{40}$ $e.\ \frac{1}{2}$

Think / Pair / Share

Use the "Pies Per Child Model" to explain **why** the key fraction rule holds. That is, explain why each individual child gets the same amount of pie in these two situations:

- if you have a pies and b kids, or
- if you have xa pies and xb kids.

30. Adding and Subtracting Fractions

Here are two very similar fractions: $\frac{2}{7}$ and $\frac{3}{7}$. What might it mean to add them? It might seem reasonable to say:

$$\frac{2}{7} \text{ represents 2 pies shared by 7 kids.}$$

$$\frac{3}{7} \text{ represents 3 pies shared by 7 kids.}$$

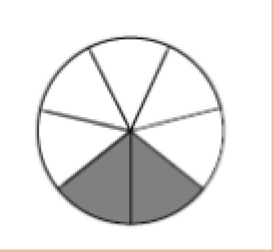
So maybe $\frac{2}{7} + \frac{3}{7}$ represents 5 pies among 14 kids, giving the answer $\frac{5}{14}$. It is very tempting to say that "adding fractions" means "adding pies and adding kids."

The trouble is that a fraction is not a pie, and a fraction is not a child. So adding pies and adding children is not actually adding fractions. A fraction is something different. It is related to pies and kids, but something more subtle. A fraction is an *amount of pie per child.*

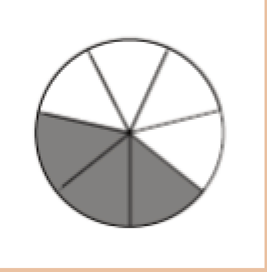
One cannot add pies, one cannot add children. One must add instead the amounts individual kids receive.

Example: 2/7 + 3/7

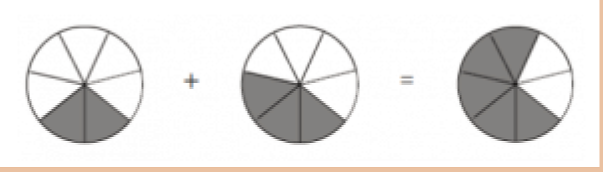
Let us take it slowly. Consider the fraction $\frac{2}{7}$. Here is a picture of the amount an individual child receives when two pies are given to seven kids:



Consider the fraction $\frac{3}{7}$. Here is the picture of the amount an individual child receives when three pies are given to seven children:



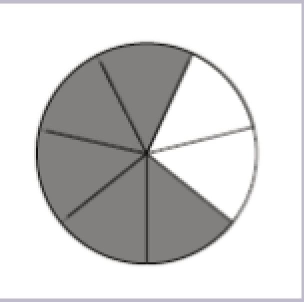
The sum $\frac{2}{7} + \frac{3}{7}$ corresponds to the sum:



The answer, from the picture, is $\frac{5}{7}$.

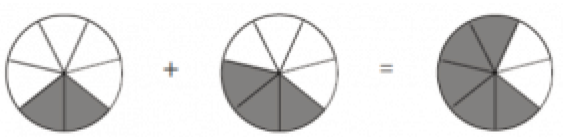
Think / Pair / Share

Remember that $\frac{5}{7}$ means "the amount of pie that one child gets when five pies are shared by seven children." Carefully explain *why* that is the same as the picture given by the sum above:



Your explanation should use both words and pictures!

Most people read this as "two sevenths plus three sevenths gives five sevenths" and think that the problem is just as easy as saying "two apples plus three apples gives five apples." And, in the end, they are right!



This is how the addition of fractions is first taught to students: Adding fractions with the same denominator seems just as easy as adding apples:

4 tenths + 3 tenths + 8 tenths = 15 tenths.

$$\frac{4}{10}+\frac{3}{10}+\frac{8}{10}=\frac{15}{10}.$$

(And, if you like, $\frac{15}{10}=\frac{5\cdot 3}{5\cdot 2}=\frac{3}{2}$.)

82 sixty-fifths + 91 sixty-fifths = 173 sixty-fifths:

$$\frac{82}{65}+\frac{91}{65}=\frac{173}{65}.$$

We are really adding **amounts per child** not amounts, but the answers match the same way.

We can use the "Pies Per Child Model" to explain *why* adding fractions with like denominators works in this way.

Example: 2/7 + 3/7

Think about the addition problem $\frac{2}{7}+\frac{3}{7}$:

$$\begin{array}{r} \text{amount of pie each kid gets when 7 kids share 2 pies} \\ + \text{ amount of pie each kid gets when 7 kids share 3 pies} \\ \hline \text{?????} \end{array}$$

Since in both cases we have 7 kids sharing the pies, we can imagine that it is the same 7 kids in both cases. First, they share 2 pies. Then they share 3 more pies. The total each child gets by the time all the pie-sharing is done is the same as if the 7 kids had just shared 5 pies to begin with. That is:

$$\begin{array}{rr} & \text{amount of pie each kid gets when 7 kids share 2 pies} \\ + & \text{amount of pie each kid gets when 7 kids share 3 pies} \\ \hline & \text{amount of pie each kid gets when 7 kids share 5 pies.} \end{array}$$

$$\frac{2}{7}+\frac{3}{7}=\frac{5}{7}.$$

Now let us think about the general case. Our claim is that

$$\frac{a}{d}+\frac{b}{d}=\frac{a+b}{d}.$$

Translating into our model, we have d kids. First, they share a pies between them, and $\frac{a}{d}$ represents the amount each child gets. Then they share b more pies, so the additional amount of pie each child gets is $\frac{b}{d}$. The total each kid gets is $\frac{a}{d}+\frac{b}{d}$.

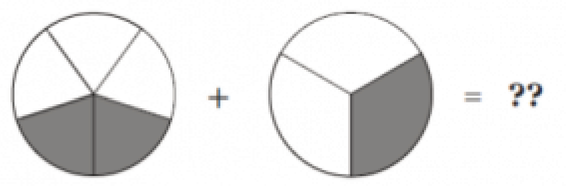
But it does not really matter that the kids first share a pies and then share b pies. The amount each child gets is the same as if they had started with all of the pies – all $a+b$ of them – and shared them equally. That amount of pie is represented by $\frac{a+b}{d}$.

Think / Pair / Share

- How can you *subtract* fractions with the same denominator? For example, what is
$$\frac{400}{903} - \frac{170}{903}?$$
- Use the "Pies Per Child" model to *carefully explain why*
$$\frac{a}{d} - \frac{b}{d} = \frac{a-b}{d}.$$
- Explain why the fact that the denominators are the same is *essential* to this addition and subtraction method. Where is that fact used in the explanations?

Fractions with Different Denominators

This approach to adding fractions suddenly becomes tricky if the denominators involved are not the same common value. For example, what is $\frac{2}{5} + \frac{1}{3}$?



Let us phrase this question in terms of pies and kids:

Suppose Poindexter is part of a team of five kids that shares two pies. Then later he is part of a team of three kids that shares one pie. How much pie does Poindexter receive in total?

Think / Pair / Share

Talk about these questions with a partner before reading on. It is actually a very difficult problem! What might a student say, if they do not already know about adding fractions? Write down any of your thoughts.

1. Do you see that this is the same problem as computing $\frac{2}{5} + \frac{1}{3}$?
2. What might be the best approach to answering the

problem?

One way to think about answering this addition question is to write $\frac{2}{5}$ in a series of alternative forms using our key fraction rule (that is, multiply the numerator and denominator each by 2, and then each by 3, and then each by 4, and so on) and to do the same for $\frac{1}{3}$:

$$\begin{array}{ll} \frac{2}{5} + & \frac{1}{3} \\ \frac{4}{10} & \frac{2}{6} \\ red\frac{6}{15} & \frac{3}{9} \\ \frac{8}{20} & \frac{4}{12} \\ \frac{10}{25} & red\frac{5}{15} \\ \vdots & \vdots \end{array}$$

We see that the problem $\frac{2}{5}+\frac{1}{3}$ is actually the same as $\frac{6}{15}+\frac{5}{15}$. So we can find the answer using the same-denominator method:

$$\frac{2}{5}+\frac{1}{3} \quad = \quad \frac{6}{15}+\frac{5}{15} \quad = \quad \frac{11}{15}.$$

Example: 3/8 + 3/10

Here is another example of adding fractions with unlike denominators: $\frac{3}{8}+\frac{3}{10}$. In this case, Valerie is part of a group of 8 kids who share 3 pies. Later she is part of a group of 10 kids who share 3 different pies. How much total pie did Valerie get?

$$\begin{array}{lllll}
\frac{3}{8} & + \frac{3}{10} & & & \\
\frac{6}{16} & \frac{6}{20} & & & \\
\frac{9}{24} & \frac{9}{30} & & & \\
\frac{12}{32} & red\frac{12}{40} & & & \\
red\frac{15}{40} & \frac{15}{50} & & & \\
\vdots & \vdots & & & \\
\frac{3}{8}+\frac{3}{10} & = & \frac{15}{40}+\frac{12}{40} & = & \frac{17}{40}.
\end{array}$$

Of course, you do not need to list all of the equivalent forms of each fraction in order to find a common denominator. If you can see a denominator right away (or think of a faster method that always works), go for it!

Think / Pair / Share

Cassie suggests the following method for the example above:

> *When the denominators are the same, we just add the numerators. So when the numerators are the same, shouldn't we just add the denominators? Like this:*
>
> $$\frac{3}{8} + \frac{3}{10} = \frac{3}{18}.$$

What do you think of Cassie's suggestion? Does it make sense? What would you say if you were Cassie's teacher?

On Your Own

Try these exercises on your own. For each addition exercise, also write down a "Pies Per Child" interpretation of the problem. You might also want to draw a picture.

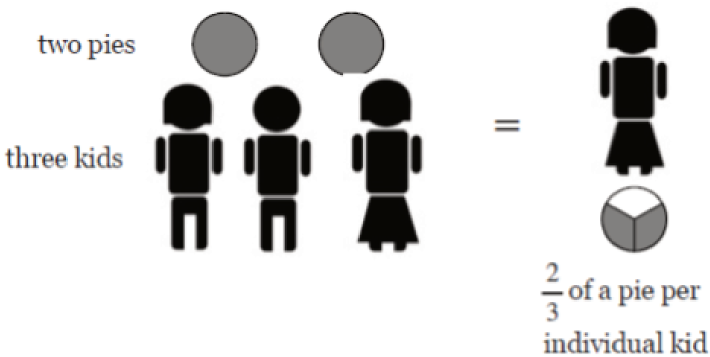
1. What is $\frac{1}{2} + \frac{1}{3}$?
2. What is $\frac{2}{5} + \frac{37}{10}$?
3. What is $\frac{1}{2} + \frac{3}{10}$?
4. What is $\frac{2}{3} + \frac{5}{7}$?
5. What is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$?
6. What is $\frac{3}{10} + \frac{4}{25} + \frac{7}{20} + \frac{3}{5} + \frac{49}{50}$?

Now try these subtraction exercises.

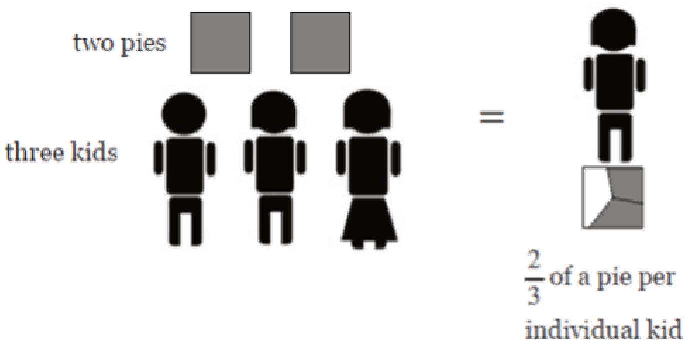
1. What is $\frac{7}{10} - \frac{3}{10}$?
2. What is $\frac{7}{10} - \frac{3}{20}$?
3. What is $\frac{1}{3} - \frac{1}{5}$?
4. What is $\frac{2}{35} - \frac{2}{7} + \frac{2}{5}$?
5. What is $\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16}$?

31. What is a Fraction? Revisited

So far, we have been thinking about a fraction as the answer to a division problem. For example, $\frac{2}{3}$ is the result of sharing two pies among three children.

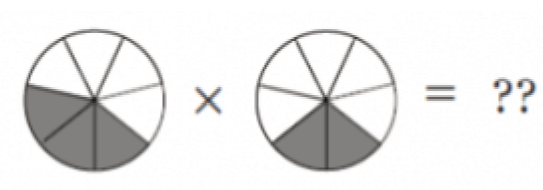


Of course, pies do not have to be round. We can have square pies, or triangular pies or squiggly pies or any shape you please.



This “Pies Per Child Model” has served us perfectly well in thinking about the meaning of fractions, equivalent fractions, and even adding and subtracting fractions.

However, there is no way to use this model to make sense of multiplying fractions! What would this mean?



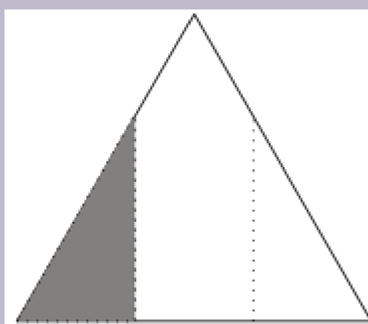
So what *are* fractions, if we are asked to multiply them? We are forced to switch models and think about fractions in a new way.

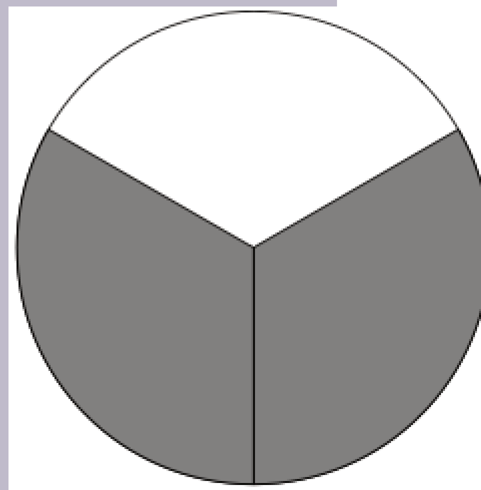
This switch is fundamentally perturbing. Think about students learning this for the first time. We keep switching concepts and models, and speak of fractions in each case as though all is naturally linked and obvious. None of this is obvious, it is all absolutely confusing. This is just one of the reasons that fractions can be such a difficult concept to teach and to learn in elementary school!

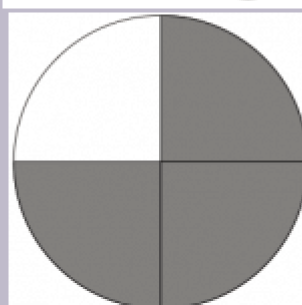
Think / Pair / Share (What's wrong here?)

For each of the following visual representations of fractions, there is a corresponding **incorrect** symbolic expression.

- Why is the symbolic representation incorrect?
- What might elementary students find confusing in these visual representations?

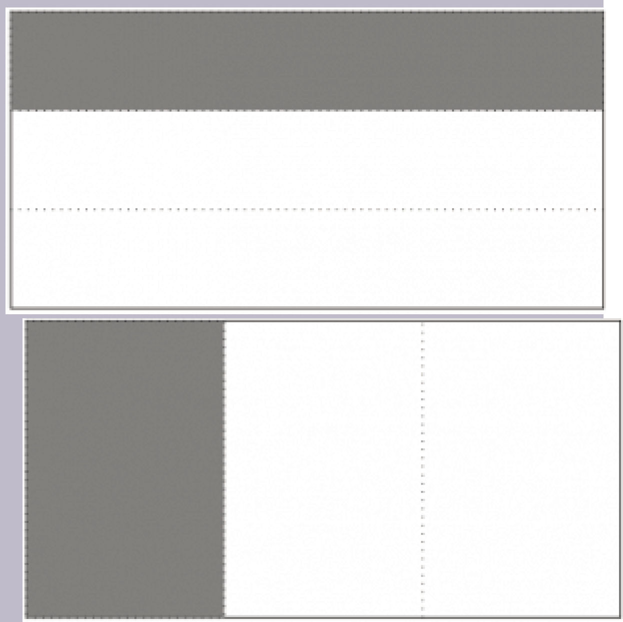






$\frac{1}{3}$

$\frac{2}{3} > \frac{3}{4}$



$$\frac{1}{3} \neq 13$$

Units and unitizing

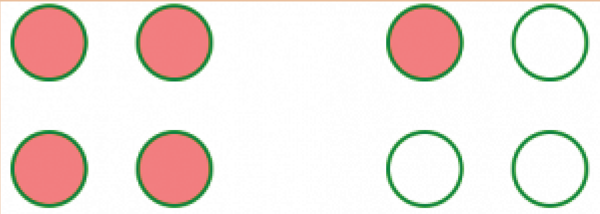
In thinking about fractions, it is important to remember that there are always *units* attached to a fraction, even if the units are hidden. If you see the number $\frac{1}{2}$ in a problem, you should ask yourself “half of what?” The answer to that question is your unit, the amount that equals 1.

So far, our units have been consistent: the "whole" (or unit) was a whole pie, and fractions were represented by pies cut into equal-sized pieces. But this is just a model, and we can take anything, cut it into equal-sized pieces, and talk about fractions *of that whole*.

One thing that can make fraction problems so difficult is that the fractions in the problem may be given in different *units* (they may be "parts" of different "wholes").

Example (Everyone is right!)

Mr. Li shows this picture to his class and asks what number is shown by the shaded region.

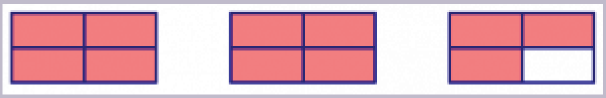


- Kendra says the shaded region represents the number 5.
- Dylan says it represents $2\frac{1}{2}$.
- Kiana says it represents $\frac{5}{8}$.
- Nate says it is $1\frac{1}{4}$.

Mr. Li exclaims, "Everyone is right!"

Think / Pair / Share

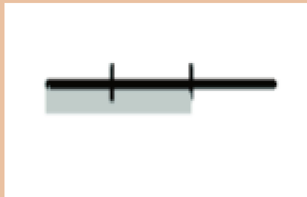
1. How can it be that everyone is right? Justify each answer by explaining what each student thought was the *unit* in Mr. Li's picture.
2. Now look at this picture:



- If the shaded region represents $3\frac{2}{3}$, what is the unit?
- Find three other numbers that could be represented by the shaded region, and explain what the unit is for each answer.

Example (Segments)

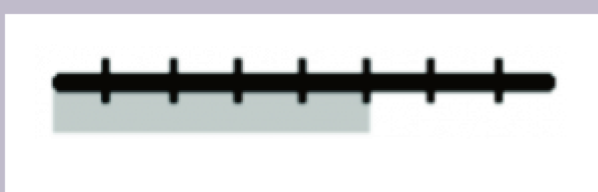
This picture

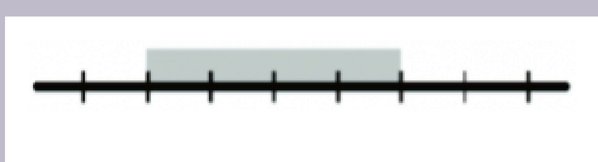


represents $\frac{2}{3}$. The whole segment (the *unit*) is split into three equal pieces by the tick marks, and two of those three equal pieces are shaded.

Think / Pair / Share

For each picture below, say what fraction it represents and how you know you are right.

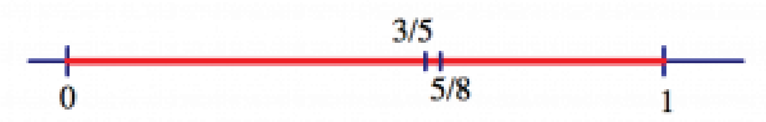




Ordering Fractions

If we think about fractions as "portions of a segment," then we can talk about their locations on a number line. We can start to treat fractions like numbers. In the back of our minds, we should remember that fractions are always relative to some unit. But on a number line, the unit is clear: it is the distance between 0 and 1.

This measurement model makes it much easier to tackle questions about the relative size of fractions based on where they appear on the number line. We can mark off different fractions as parts of the unit segment. Just as with whole numbers, fractions that appear farther to the right are larger.



3/5 and 5/8 are very close, but 5/8 is just a bit bigger.

Think / Pair / Share (Ordering fractions)

1. What quick method can you use to determine if a fraction is greater than 1?

2. What quick method can you use to determine if a fraction is greater than $\frac{1}{2}$?
3. Organize these fractions from smallest to largest using *benchmarks*: 0 to $\frac{1}{2}$, $\frac{1}{2}$ to 1, and greater than 1. Justify your choices.
$$\frac{25}{23}, \quad \frac{4}{7}, \quad \frac{17}{35}, \quad \frac{2}{9}, \quad \frac{14}{15}.$$
4. Arrange each group of fractions in *ascending order*. Keep track of your thinking and your methods.

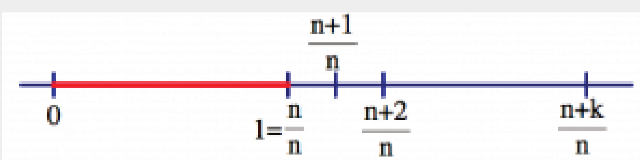
- $\frac{7}{17}, \quad \frac{4}{17}, \quad \frac{12}{17}.$
- $\frac{3}{7}, \quad \frac{3}{4}, \quad \frac{3}{8}.$
- $\frac{5}{6}, \quad \frac{7}{8}, \quad \frac{3}{4}.$
- $\frac{8}{13}, \quad \frac{12}{17}, \quad \frac{1}{6}.$

- $\frac{5}{6}$, $\frac{10}{11}$, $\frac{2}{3}$.

You probably came up with benchmarks and intuitive methods to think about the relative sizes of fractions. Here are some of these methods. (Did you come up with others?)

Fraction Intuition

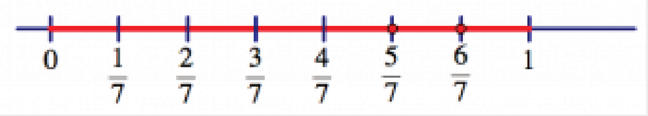
Greater than 1: A fraction is greater than 1 if its numerator is greater than its denominator. How can we see this? Well, the denominator represents how many pieces in one whole (one unit). The numerator represents how many pieces in your portion. So if the numerator is bigger, that means you have more than the number of pieces needed to make one whole.



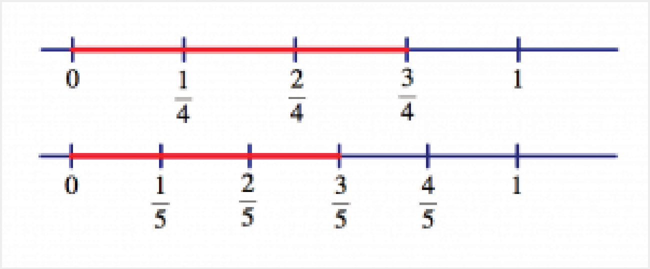
Greater than $\frac{1}{2}$: A fraction is greater than $\frac{1}{2}$ if the numerator is more than half the denominator. Another way to check (which might be an easier calculation): a fraction is greater than $\frac{1}{2}$ if twice the numerator is bigger than the denominator.

Why? Well, if we double the fraction and get something bigger than 1, then the original fraction must be bigger than $\frac{1}{2}$.

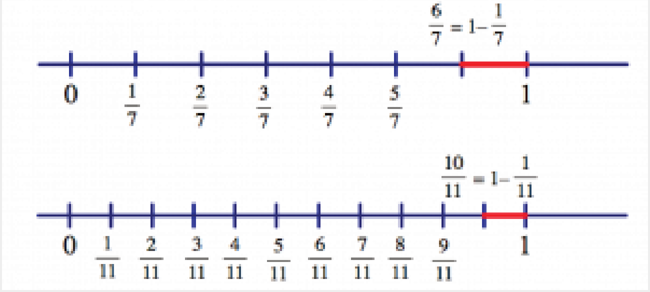
Same denominators: If two fractions have the same denominator, just compare the numerators. The fractions will be in the *same order as the numerators*. For example, $\frac{5}{7} < \frac{6}{7}$. Why? Well, the pieces are the same size since the denominators are the same. If you have more pieces of the same size, you have a bigger number.



Same numerators: If the numerators of two fractions are the same, just compare the denominators. The fractions should be in the *reverse order of the denominators*. For example, $\frac{3}{4} > \frac{3}{5}$. The justification for this one is a little trickier: The denominator tells you how many pieces make up one whole. If there are more pieces in a whole (if the denominator is bigger), then the pieces must be smaller. And if you take the same number of pieces (same numerator), then the bigger piece wins.



Numerator = denominator -1: You can easily compare two fractions whose numerators are both one less than their denominators. The fractions will be in the same order as the denominators. Think of each fraction as a pie with one piece missing. The greater the denominator, the smaller the missing piece, so the greater the amount remaining. For example, $\frac{6}{7} < \frac{10}{11}$, since $\frac{6}{7} = 1 - \frac{1}{7}$ and $\frac{10}{11} = 1 - \frac{1}{11}$.



Numerator = denominator – constant: You can extend the test above to fractions whose numerators are both the same amount less than their denominators. The fractions will again be in the same order as the denominators, for exactly the same reason. For example, $\frac{3}{7} < \frac{7}{11}$, because both are four “pieces” less than one whole, and the $\frac{1}{11}$ pieces are smaller than the $\frac{1}{7}$ pieces.

Equivalent fractions: Find equivalent fractions that lets you compare numerators or denominators, and then use one of the above rules.

Arithmetic Sequences

Consider the patterns below.

Pattern 1:

$$5, \quad 8, \quad 11, \quad 14, \quad 17, \quad 20, \quad 23, \quad 26, \ldots$$

Pattern 2:

$$2, \quad 9, \quad 16, \quad 23, \quad 30, \quad 37, \quad 44, \quad 51, \ldots$$

Pattern 3:

$$\frac{1}{5}, \quad \frac{3}{5}, \quad 1, \quad \frac{7}{5}, \quad \frac{9}{5}, \quad \frac{11}{5}, \quad \frac{13}{5}, \quad 3, \ldots$$

Think / Pair / Share

Answer these questions about each of the patterns.

- Can you predict the next five numbers?
- Can you predict the 100th number?

- What do these sequences have in common? Describe the pattern in words.

The patterns above are called **arithmetic sequences**: a sequence of numbers where the difference between consecutive terms is a constant. Here are some other examples:

Pattern A: $\underbrace{1, \quad 2,}_{+1}\underbrace{\quad}_{+1}\underbrace{\quad}_{+1}\underbrace{\quad}_{+1}$

$$\textbf{Pattern A: } \underbrace{1, \quad}_{+1}\underbrace{2, \quad}_{+1}\underbrace{3, \quad}_{+1}\underbrace{4, \quad}_{+1} 5, \ldots$$

$$\textbf{Pattern B: } \underbrace{2, \quad}_{+2}\underbrace{4, \quad}_{+2}\underbrace{6, \quad}_{+2}\underbrace{8, \quad}_{+2} 10, \ldots$$

$$\textbf{Pattern C: } \underbrace{\frac{1}{3}, \quad}_{+\frac{2}{3}}\underbrace{1, \quad}_{+\frac{2}{3}}\underbrace{\frac{5}{3}, \quad}_{+\frac{2}{3}}\underbrace{\frac{7}{3}, \quad}_{+\frac{2}{3}} 3, \ldots$$

Think / Pair / Share

If you have not done so already, find the common difference between terms for Patterns 1, 2, and 3. Are they really arithmetic sequences?

Then make up your own arithmetic sequence using whole numbers. Exchange sequences with a partner, and check if your partner's sequence is really an arithmetic sequence.

Here are several more number patterns:

Pattern 4: $1, \quad 2, \quad 4, \quad 8, \quad 16, \quad 32, \quad 64, \quad 128, \ldots$

Pattern 5: $1, \quad 3, \quad 6, \quad 10, \quad 15, \quad 21, \quad 28, \quad 36, \ldots$

Pattern 6: $\frac{2}{5}, \quad \frac{7}{10}, \quad 1, \quad \frac{13}{10}, \quad \frac{8}{5}, \quad \frac{19}{10}, \quad \frac{11}{5}, \quad \frac{5}{2}, \ldots$

Pattern 7: $\frac{3}{5}, \quad \frac{6}{5}, \quad \frac{12}{5}, \quad \frac{24}{5}, \quad \frac{48}{5}, \quad \frac{96}{5}, \ldots$

Think / Pair / Share

For each of the sequences above, decide if it is an arithmetic sequence or not. Justify your answers.

Problem 2

$$\frac{1}{4}, \quad \_\_, \quad \_\_, \quad \frac{1}{3}$$

1. Find two fractions between $\frac{1}{4}$ and $\frac{1}{3}$.
2. Are the resulting four fractions in an arithmetic sequence? Justify your answer.

Problem 3

Find two fractions between $\frac{1}{6}$ and $\frac{1}{5}$ so the resulting four numbers are in an arithmetic sequence.

$$\frac{1}{6}, \quad \underline{\quad}, \quad \underline{\quad}, \quad \frac{1}{5}$$

Problem 4

Find three fractions between $\frac{2}{5}$ and $\frac{5}{7}$ so the resulting four numbers are in an arithmetic sequence.

$$\frac{2}{5}, \quad \underline{\quad}, \quad \underline{\quad}, \quad \underline{\quad}, \quad \frac{5}{7}$$

Think / Pair / Share

Make up two fraction sequences of your own, one that *is* an arithmetic sequence and one that *is not* an arithmetic sequence.

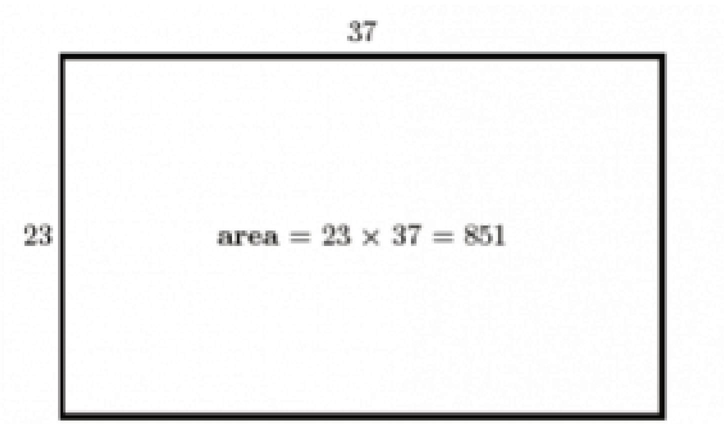
Exchange your sequences with a partner, but do not tell your partner which is which.

When you get your partner's sequences: decide which is an arithmetic sequence and which is not. Check if you and your partner agree.

32. Multiplying Fractions

Area Model

One of our models for multiplying whole numbers was an area model. For example, the product 23×37 is the area (number of 1 × 1 squares) of a 23-by-37 rectangle:

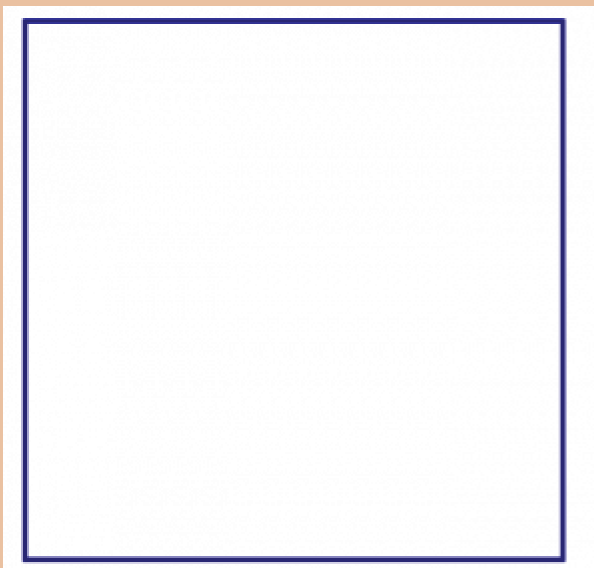


So the product of two fractions, say, $\frac{4}{7} \times \frac{2}{3}$ should also correspond to an area problem.

Example (4/7 × 2/3)

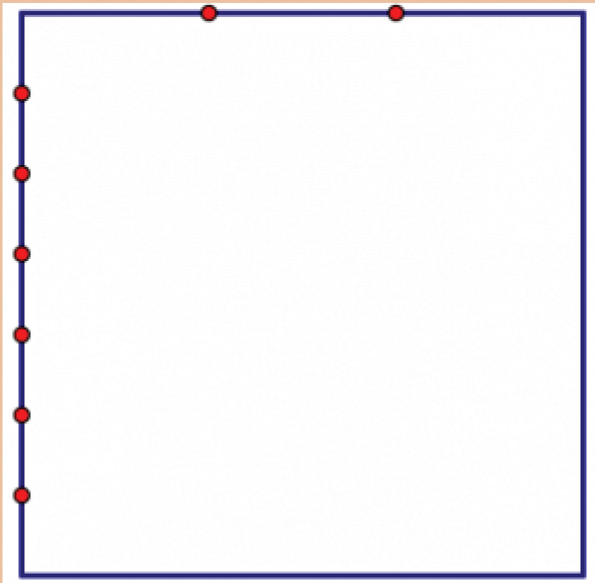
Let us start with a segment of some length that we call 1 unit:

Now, build a square that has one unit on each side:

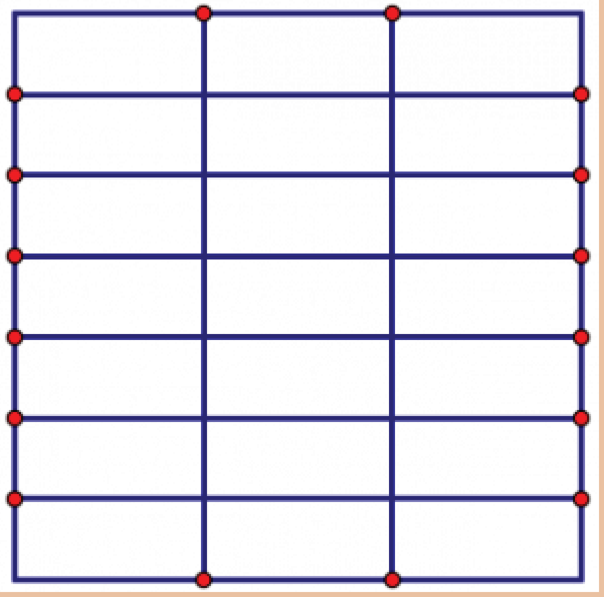


The area of the square, of course, is $1 \times 1 = 1$ square unit.

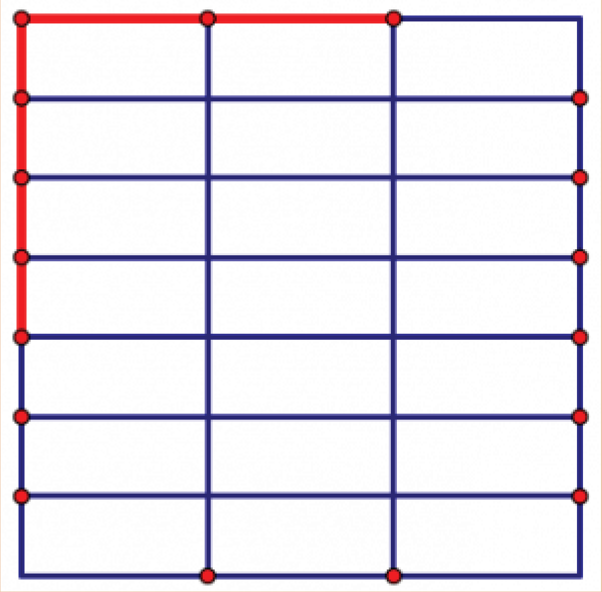
Now, let us divide the segment on top into three equal-sized pieces. (So each piece is $\frac{1}{3}$.) And we will divide the segment on the side into seven equal-sized pieces. (So each piece is $\frac{1}{7}$.)



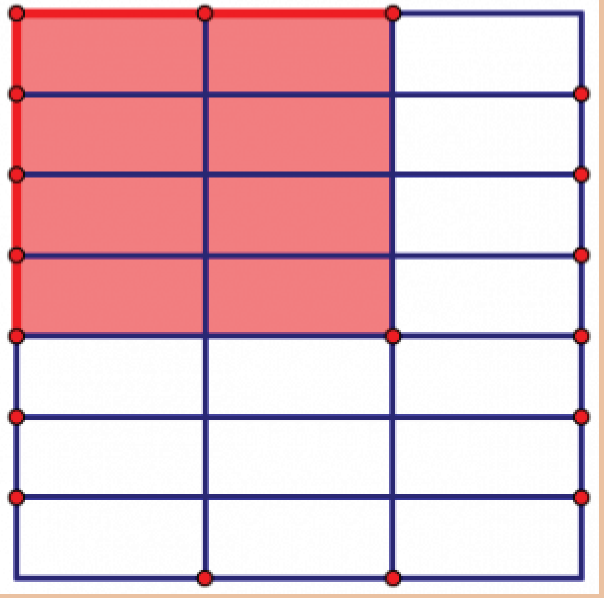
We can use those marks to divide the whole square into small, equal-sized rectangles. (Each rectangle has one side that measures $\frac{1}{3}$ and another side that measures $\frac{1}{7}$.)



We can now mark off four sevenths on one side and two thirds on the other side.



The result of the multiplication should be the area of the rectangle with $\frac{4}{7}$ on one side and $\frac{2}{3}$ on the other. What is that area?



Remember, the whole square was one unit. That one-unit square is divided into 21 equal-sized pieces, and our rectangle (the one with sides $\frac{4}{7}$ and $\frac{2}{3}$) contains eight of those rectangles. Since the shaded area is the answer to our multiplication problem we conclude that

$$\frac{4}{7} \times \frac{2}{3} \quad = \quad \frac{8}{21}.$$

Think / Pair / Share

1. Use are model to compute each of the following products. Draw the picture to see the answer clearly.

$$\frac{3}{4} \times \frac{5}{6}, \qquad \frac{3}{8} \times \frac{4}{5}, \qquad \frac{5}{8} \times \frac{3}{7}.$$

2. The area problem $\frac{4}{7} \times \frac{2}{3}$ yielded a diagram with a *total* of 21 small rectangles. Explain why 21 appears as the total number of equal-sized rectangles.
3. The area problem $\frac{4}{7} \times \frac{2}{3}$ yielded a diagram with 8 small *shaded* rectangles. Explain why 8 appears as the number of shaded rectangles.

Problem 5

How can you extend the area model for fractions greater than 1? Try to draw a picture for each of these:

$$\frac{3}{4} \cdot \frac{3}{2}, \qquad \frac{2}{5} \cdot \frac{4}{3}, \qquad \frac{3}{10} \cdot \frac{5}{4}, \qquad \frac{5}{2} \cdot \frac{7}{4}.$$

On Your Own

Work on the following exercises on your own or with a partner.

1. Compute the following products, simplifying each of the answers as much as possible. You do not need to draw pictures, but you may certainly choose to do so if it helps!

$$\frac{5}{11} \times \frac{7}{12}, \qquad \frac{4}{7} \times \frac{4}{8}, \qquad \frac{1}{2} \times \frac{1}{3}, \qquad \frac{2}{1} \times \frac{3}{1}, \qquad \frac{1}{5} \times \frac{5}{1}.$$

2. Compute the following products. (Do n0t work too hard!)

$$\frac{3}{4} \times \frac{1}{3} \times \frac{2}{5}, \qquad \frac{5}{5} \times \frac{7}{8}, \qquad \frac{88}{88} \times \frac{541}{788}, \qquad \frac{77876}{311} \times \frac{311}{77876}.$$

3. Try this one. Can you make use of the fraction rule $\frac{xa}{xb} = \frac{a}{b}$ to help you calculate? How?

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7} \times \frac{7}{8} \times \frac{8}{9} \times \frac{9}{10}.$$

Think / Pair / Share

How are these two problems different? Draw a picture of each.

1. Pam had $\frac{2}{3}$ of a cake in her refrigerator, and she ate $\frac{1}{2}$ of it. How much total cake did she eat?
2. On Monday, Pam ate $\frac{2}{3}$ of a cake. On Tuesday, Pam ate $\frac{1}{2}$ of a cake. Both cakes were the same size. How much total cake did she eat?

When a problem includes a phrase like "$\frac{2}{3}$ of …," students are taught to treat "of" as multiplication, and to use that to solve the problem. As the above problems show, in some cases this makes sense, and in some cases it does not. It is important to read carefully and understand what a problem is asking, not memorize rules about "translating" word problems.

Explaining the Rule

You probably simplified your work in the exercises above by using a multiplication rule like the following.

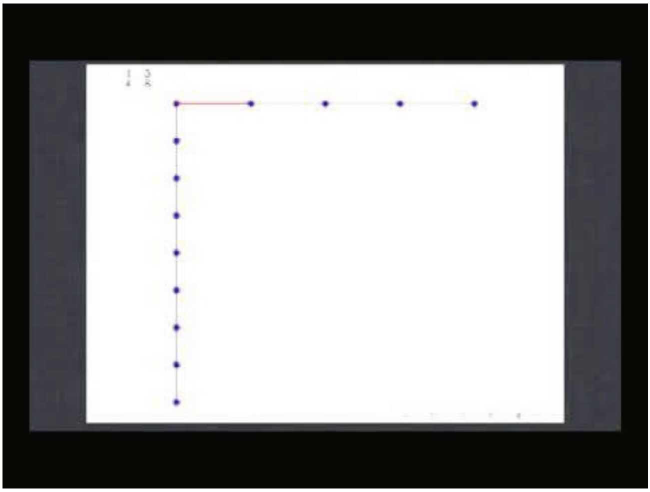
Multiplying Fractions

$$\frac{a}{b} \cdot \frac{c}{d} \quad = \quad \frac{a \cdot c}{b \cdot d}.$$

Of course, you may then choose to simplify the final answer, but the answer is always *equivalent* to this one. Why? The area model can help us explain what is going on.

First, let us clearly write out how the area model says to multiply $\frac{a}{b} \cdot \frac{c}{d}$. We want to build a rectangle where one side has length $\frac{a}{b}$ and the other side has length $\frac{c}{d}$. We start with a square, one unit on each side.

- Divide the top segment into b equal-sized pieces. Shade a of those pieces. (This will be the side of the rectangle with length $\frac{a}{b}$.)
- Divide the left segment into d equal-sized pieces. Shade c of those pieces. (This will be the side of the rectangle with length $\frac{c}{d}$.)
- Divide the whole rectangle according to the tick marks on the sides, making equal-sized rectangles.
- Shade the rectangle bounded by the shaded segments.



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If the answer is $\frac{a \cdot c}{b \cdot d}$, that means there are $b \cdot d$ total equal-sized pieces in the square, and $a \cdot c$ of them are shaded. We can see from the model why this is the case:

- The top segment was divided into b equal-sized pieces. So there are b columns in the rectangle.
- The side segment was divided into d equal-sized pieces. So there are d rows in the rectangle.
- A rectangle with b columns and d rows has $b \cdot d$ pieces. (The area model for whole-number multiplication!)

Think / Pair / Share

Stick with the general multiplication rule

$$\frac{a}{b} \cdot \frac{c}{d} \quad = \quad \frac{a \cdot c}{b \cdot d}.$$

Write a clear explanation for why $a \cdot c$ of the small rectangles will be shaded.

Multiplying Fractions by Whole Numbers

Often, elementary students are taught to multiply fractions by whole numbers using the fraction rule.

Example: Multiply Fractions

For example, to multiply $2 \cdot \frac{3}{7}$, we think of "2" as $\frac{2}{1}$, and compute this way

$$2 \cdot \frac{3}{7} = \frac{2}{1} \cdot \frac{3}{7} = \frac{2 \cdot 3}{1 \cdot 7} = \frac{6}{7}.$$

We can also think in terms of our original "Pies Per Child" model to answer questions like this.

Example: Pies per Child

We know that $\frac{3}{7}$ means the amount of pie each child gets when 7 children evenly share 3 pies.

If we compute $2 \cdot \frac{3}{7}$ that means we double the amount of pie each kid gets. We can do this by doubling the number of pies. So the answer is the same as $\frac{6}{7}$: the amount of pie each child gets when 7 children evenly share 6 pies.

Finally, we can think in terms of units and unitizing.

Example: Units

The fraction $\frac{3}{7}$ means that I have 7 equal pieces (of *something*), and I take 3 of them.

So $2 \cdot \frac{3}{7}$ means do that twice. If I take 3 pieces and then 3 pieces again, I get a total of 6 pieces. There are still 7 equal pieces in the whole, so the answer is $\frac{6}{7}$.

Think / Pair / Share

1. Use all three methods to explain how to find each product:
$$3 \cdot \frac{2}{5}, \qquad 4 \cdot \frac{3}{8}, \qquad 6 \cdot \frac{1}{5}.$$
2. Compare these different ways of thinking about fraction multiplication. Are any of them more natural to you? Does one make more sense than the others? Do the particular numbers in the problem affect your answer? Does your partner agree?

Explaining the Key Fraction Rule

Roy says that the fraction rule
$$\frac{xa}{xb} = \frac{a}{b}$$
is "obvious" if you think in terms of multiplying fractions. He reasons as follows:

We know multiplying anything by 1 does not change a number:

$$1 \cdot 4 = 4$$
$$1 \cdot 2014 = 2014$$
$$1 \cdot \frac{5}{7} = \frac{5}{7}$$

So, in general,
$$1 \cdot \frac{a}{b} \ = \ \frac{a}{b}.$$

Now, $\frac{2}{2} = 1$, *so that means that*
$$\frac{2}{2} \cdot \frac{a}{b} \ = \ 1 \cdot \frac{a}{b} \ = \ \frac{a}{b},$$

which means
$$\frac{2a}{2b} \ = \ \frac{a}{b}.$$

By the same reasoning, $\frac{3}{3} = 1$, *so that means that*
$$\frac{3}{3} \cdot \frac{a}{b} \ = \ 1 \cdot \frac{a}{b} \ = \ \frac{a}{b},$$

which means
$$\frac{3a}{3b} \ = \ \frac{a}{b}.$$

Think / Pair / Share

What do you think about Roy's reasoning? Does it make sense? How would Roy explain the general rule for positive whole numbers x:

$$\frac{xa}{xb} = \frac{a}{b}?$$

33. Dividing Fractions: Meaning

Dividing fractions is one of the hardest ideas in elementary school mathematics. By now, you are used to the rule: to divide by a fraction, multiply by its reciprocal. ("invert and multiply"). But ask yourself: Why does this rule work? Does it really make sense to you? Can you explain why it makes sense to a third grader?

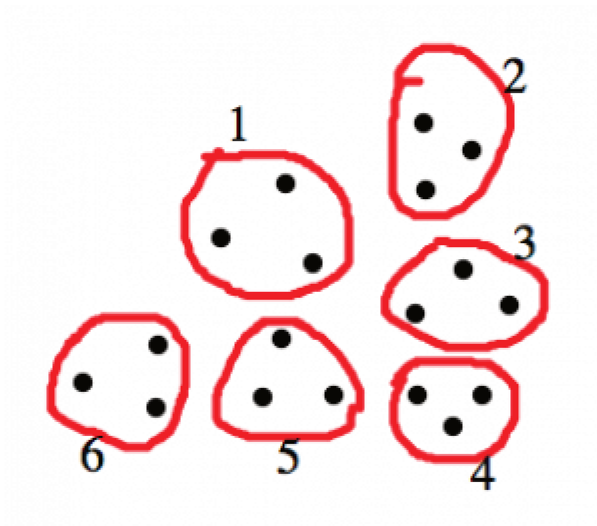
We are going to build up to the "invert and multiply" rule, but along the way, we'll find some more meaningful ways to understand division of fractions. So please play along: **pretend that you don't already know the "invert and multiply" rule,** and solve the problems in this chapter with other methods.

Groups of Equal Size

Remember the quotative model for division: $18 \div 3$ means:

> How many groups of 3 can I find in 18?

We start with 18 dots (or candy bars or molecules), and we make groups of 3 dots (or 3 whatevers). We ask: how many groups can we make?



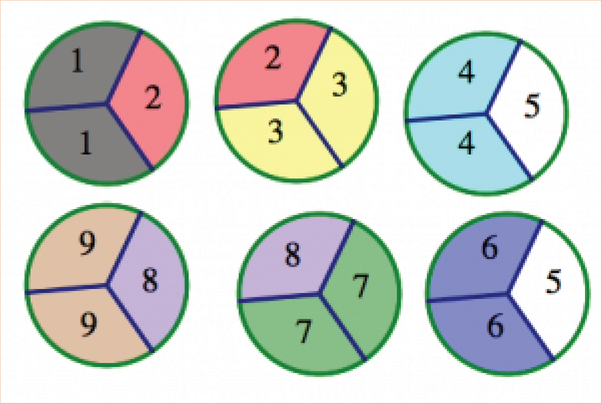
18 dots, split into groups of 3 dots. Since there are 6 groups, we have 18 : 3 = 6.

This same idea applies when we divide fractions. For example, $6 \div 23$ means:

> How many groups of $\frac{2}{3}$ can I find in 6?

Example: 6 ÷ 2/3

Let's draw a picture of 6 pies, and see how many groups of $\frac{2}{3}$ we can find:



We found nine equal groups of size $\frac{2}{3}$, so we conclude that

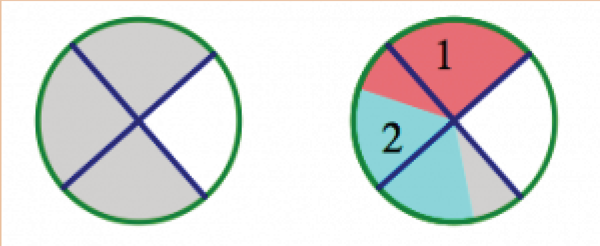
$$6 \div \frac{2}{3} = 9.$$

Unfortunately, it's not always quite so straightforward to find the equal groups. For example, $\frac{3}{4} \div \frac{1}{3}$ asks the question:

> How many groups of $\frac{1}{3}$ can I find in $\frac{3}{4}$?

Example: 3/4 ÷ 1/3

Let's draw a picture of $\frac{3}{4}$ of a pie, and see how many groups of $\frac{1}{3}$ we can find:



The first pictures shows $\frac{3}{4}$ of a pie. The second picture shows two equal groups of $\frac{1}{3}$ inside of $\frac{3}{4}$, but there's a little bit left over. We conclude

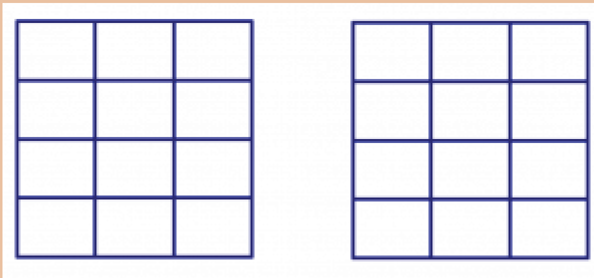
$$\frac{3}{4} \div \frac{1}{3} = 2 + \text{ a tiny bit more.}$$

But how much more? Can we figure it out exactly?

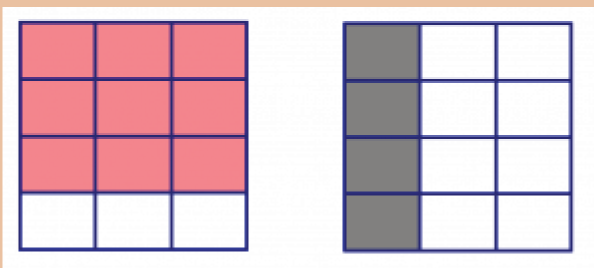
Here's a method that will let you do the computation exactly. We'll use rectangular pies, and divide them up into rows and columns based on the denominators of the numbers we're dividing.

Example: 3/4 ÷ 1/3

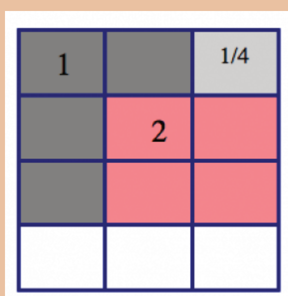
Start by drawing two identical rectangles, each with 4 rows (from the denominator of $\frac{3}{4}$ and 3 columns (from the denominator of $\frac{1}{3}$).



Shade $\frac{3}{4}$ of the first rectangle (this is exactly three rows), and shade $\frac{1}{3}$ of the second rectangle (so that's one column).



Now ask: how many copies of $\frac{1}{3}$ can I find in $\frac{3}{4}$? Well, $\frac{1}{3}$ is equal to four of the smaller squares. So we find groups equal to that:



In the picture of $\frac{3}{4}$, we can find:

- two groups of four squares (two groups of $\frac{1}{3}$), and
- one square left over, which is $\frac{1}{4}$ of the group we're looking for.

We conclude:

$$\frac{3}{4} \div \frac{1}{3} = 2\tfrac{1}{4}.$$

Think / Pair / Share

Use either method above to find the following quotients. Remember, pretend that you don't know any method to divide fractions except finding equal-sized groups.

$$\frac{3}{4} \div \frac{1}{2} \qquad \frac{1}{3} \div \frac{1}{2} \qquad \frac{4}{9} \div \frac{1}{3} \qquad \frac{4}{5} \div \frac{1}{3}$$

$$\frac{3}{5} \div \frac{3}{4} \qquad \frac{3}{2} \div \frac{1}{2} \qquad \frac{2}{3} \div \frac{1}{2}$$

Common Denominator Method

Think / Pair / Share

Solve each of the following fraction division problems using the "groups of equal size" method:

$$\frac{6}{4} \div \frac{3}{4}, \qquad \frac{6}{10} \div \frac{3}{10}, \qquad \frac{8}{9} \div \frac{4}{9}, \qquad \frac{6}{33} \div \frac{2}{33},$$

$$\frac{5}{4} \div \frac{2}{4}, \qquad \frac{5}{2} \div \frac{2}{2}, \qquad \frac{5}{10} \div \frac{2}{10}.$$

What do you notice?

This leads to our first fraction division method:

Common denominator method

If two fractions have the same denominator, then when you divide them, you can just divide the numerators. In symbols,

$$\frac{a}{d} \div \frac{b}{d} = \frac{a}{b}.$$

Think / Pair / Share

- Use the common denominator method to find these quotients:

$$\frac{1}{3} \div \frac{2}{3}, \qquad \frac{5}{8}, \div \frac{3}{8}, \qquad \frac{3}{8} \div \frac{5}{8}, \qquad \frac{15}{33} \div \frac{1}{33}, \qquad \frac{1}{2} \div \frac{1}{2}.$$

- What if the fractions do not have a common denominator? Is the method useless, or can you find a way to make it work? Can you solve these problems?

$$\frac{3}{5} \div \frac{3}{4}, \qquad \frac{3}{4} \div \frac{8}{7}, \qquad \frac{2}{3} \div \frac{1}{2}, \qquad \frac{5}{8} \div \frac{1}{4}.$$

Missing Factor Method

We know that we can always turn a division problem into a “missing factor” multiplication problem. Can that help us compute fraction division? Sometimes!

Think / Pair / Share

For each division problem, rewrite it as a missing factor multiplication question. Then find the quotient using what you know about multiplying fractions.

$$\frac{9}{10} \div \frac{3}{5}, \qquad \frac{7}{8} \div \frac{1}{4}, \qquad \frac{6}{7} \div \frac{3}{7}, \qquad \frac{10}{9} \div \frac{2}{3}, \qquad \frac{25}{12} \div \frac{5}{6}.$$

Unfortunately, the missing factor method doesn't always work out so nicely. For example,

$$\frac{3}{4} \div \frac{1}{3} = \underline{\qquad}$$

can be rewritten as

$$\frac{1}{3} \cdot \underline{\qquad} = \frac{3}{4}.$$

There isn't a nice ratio of whole numbers that obviously fills in the blank, but we'll come back to this idea and resolve it soon.

34. Dividing Fractions: Invert and Multiply

The missing factor method is a particularly nice way to understand fraction division. It builds on what we know about multiplication and division, reinforcing that these operations have the same relationship whether the numbers are whole number, fractions, or anything else. It makes sense. But we've seen that it doesn't always work out nicely. For example,

$$\frac{3}{4} \div \frac{1}{3} = \underline{\qquad}$$

can be rewritten as

$$\frac{1}{3} \cdot \underline{\qquad} = \frac{3}{4}.$$

You want to ask:

- For the numerator: $1 \cdot \underline{\quad} = 3.$ We can fill in the blank with a 3.
- For the denominator: $3 \cdot \underline{\quad} = 4.$ We can fill in the blank with $\frac{4}{3}.$ (Why does that work?)

So we have:

$$\frac{1}{3} \cdot \frac{3}{\frac{4}{3}} = \frac{3}{4}.$$

You learned about fractions like

$$\frac{3}{\frac{4}{3}}$$

back in the "What is a Fraction?" chapter. This means that each $\frac{4}{3}$ of a kid gets 3 pies. So how much does an individual kid (one whole kid) get? You could draw a picture to help you figure it out. But we can also use the key fraction rule to help us out.

$$\frac{3}{\frac{4}{3}} = \frac{3 \cdot 3}{3 \cdot \frac{4}{3}} = \frac{9}{4}.$$

This process is going to be key to understanding why the "invert and multiply" rule for fraction division actually makes sense.

Simplify an Ugly Fraction

Example

$7\frac{2}{3}$ pies are shared equally by $5\frac{1}{4}$ children. How much pie does each child get?

Technically, we could just write down the answer as

$$\frac{7\frac{2}{3}}{5\frac{1}{4}}$$

and be done! The answer is equivalent to this fraction, so why not?

Is there a way to make this look friendlier? Well, if we change those mixed numbers to "improper" fractions, it helps a little:

$$\frac{7\frac{2}{3}}{5\frac{1}{4}} = \frac{\frac{23}{3}}{\frac{21}{4}}.$$

That's a bit better, but it's still not clear how much pie each kid gets. Let's use the key fraction rule to make the fraction even friendlier. Let's multiply the numerator and denominator each by 3. (Why three?) Remember, this means we're multiplying the fraction by $\frac{3}{3}$, which is just a special form of 1, so we don't change its value.

$$\frac{3 \cdot \frac{23}{3}}{3 \cdot \frac{21}{4}} = \frac{23}{\frac{63}{4}}.$$

Now multiply numerator and denominator each by 4. (Why four?)

$$\frac{4 \cdot 23}{4 \cdot \frac{63}{4}} = \frac{92}{63}.$$

We now see that the answer is $\frac{92}{63}$. That means that sharing $7\frac{2}{3}$ pies among $5\frac{1}{4}$ children is the same as sharing 92 pies among 63 children. (In both situations, the individual child get exactly the same amount of pie.)

Example

Let's forget the context now and just focus on the calculations so that we can see what is going on more clearly. Try this one:

$$\frac{\frac{3}{5}}{\frac{2}{3}}.$$

Multiplying the numerator and denominator each by 5 (why did we choose 5?) gives

$$\frac{\frac{3}{5}}{\frac{2}{3}} = \frac{5 \cdot \frac{3}{5}}{5 \cdot \frac{2}{3}} = \frac{3}{\frac{10}{3}}.$$

Now multiply the numerator and denominator each by 3 (why did we choose 3?):

$$\frac{3 \cdot 3}{3 \cdot \frac{10}{3}} = \frac{9}{10}.$$

On Your Own

1. Each of the following is a perfectly nice fraction, but it could be written in a simpler form. So do that! Write each of them in a simpler form following the examples above.

$$\frac{\frac{2}{3}}{\frac{1}{3}}, \qquad \frac{2\frac{1}{5}}{2\frac{1}{4}}, \qquad \frac{\frac{5}{7}}{\frac{3}{5}}, \qquad \frac{\frac{3}{7}}{\frac{4}{5}}.$$

Think / Pair / Share

- Jessica calculated the second exercise above this way:

$$\frac{2\frac{1}{5}}{2\frac{1}{4}} = \frac{\cancel{2}\frac{1}{5}}{\cancel{2}\frac{1}{4}} = \frac{\frac{1}{5}}{\frac{1}{4}} = \frac{\frac{1}{5}\cdot 4}{\frac{1}{4}\cdot 4} = \frac{\frac{4}{5}}{1} = \frac{4}{5}.$$

Is her solution correct, or is she misunderstanding something? Carefully explain what is going on with her solution, and what you would do as Jessica's teacher.

- Isaac calculated the last exercise above this way:

$$\frac{\frac{3}{7}}{\frac{4}{5}} = \frac{\frac{3}{7}\cdot 7}{\frac{4}{5}\cdot 5} = \frac{3}{4}.$$

Is his solution correct, or is he misunderstanding something? Carefully explain what is going on with his solution, and what you would do as Isaac's teacher.

Perhaps without realizing it, you have just found another method to divide fractions.

Example: 3/5 ÷ 4/7

Consider $\frac{3}{5} \div \frac{4}{7}$. We know that a fraction is the answer to a division problem, meaning

$$\frac{3}{5} \div \frac{4}{7} = \frac{\frac{3}{5}}{\frac{4}{7}}.$$

And now we know how to simplify ugly fractions like this one! Multiply the numerator and denominator each by 5:

$$\frac{\left(\frac{3}{5}\right) \cdot 5}{\left(\frac{4}{7}\right) \cdot 5} = \frac{3}{\frac{20}{7}}.$$

Now multiply them each by 7:

$$\frac{(3) \cdot 7}{\left(\frac{20}{7}\right) \cdot 7} = \frac{21}{20}.$$

Done! So

$$\frac{3}{5} \div \frac{4}{7} = \frac{21}{20}.$$

Example: 5/9 ÷ 8/11

Let's do another! Consider $\frac{5}{9} \div \frac{8}{11}$:

$$\frac{5}{9} \div \frac{8}{11} = \frac{\frac{5}{9}}{\frac{8}{11}}.$$

Let's multiply numerator and denominator each by 9 and by 11 at the same time. (Why not?)

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{\left(\frac{5}{9}\right) \cdot 9 \cdot 11}{\left(\frac{8}{11}\right) \cdot 9 \cdot 11} = \frac{5 \cdot 11}{8 \cdot 9}.$$

(Do you see what happened here?)

So we have

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{5 \cdot 11}{8 \cdot 9} = \frac{55}{72}.$$

On Your Own

Compute each of the following, using the simplification technique in the examples above.

$$\frac{1}{2} \div \frac{1}{3}, \qquad \frac{4}{5} \div \frac{3}{7}, \qquad \frac{2}{3} \div \frac{1}{5}, \qquad \frac{45}{59} \div \frac{902}{902}, \qquad \frac{10}{13} \div \frac{2}{13}.$$

Invert and multiply

Consider the problem $\frac{5}{12} \div \frac{7}{11}$. Janine wrote:

$$\frac{\frac{5}{12}}{\frac{7}{11}} = \frac{\frac{5}{12} \cdot 12 \cdot 11}{\frac{7}{11} \cdot 12 \cdot 11} = \frac{5 \cdot 11}{7 \cdot 12} = \frac{5}{12} \cdot \frac{11}{7}.$$

She stopped before completing her final step and exclaimed: "Dividing one fraction by another is the same as multiplying the first fraction with the second fraction upside down!"

Think / Pair / Share

First check each step of Janine's work here and make sure that she is correct in what she did up to this point. Then answer these questions:

- Do you understand what Janine is saying? Explain it very clearly.
- Work out $\frac{\frac{3}{7}}{\frac{4}{13}}$ using the simplification method. Is the answer the same as $\frac{3}{7} \cdot \frac{13}{4}$?
- Work out $\frac{\frac{2}{5}}{\frac{3}{10}}$ using the simplification method. Is the answer the same as $\frac{2}{5} \cdot \frac{10}{3}$?
- Work out $\frac{\frac{a}{b}}{\frac{c}{d}}$ using the simplification method. Is the answer the same as $\frac{a}{b} \cdot \frac{d}{c}$?

- Is Janine right? Is dividing two fractions always the same as multiplying the two fractions with the second one turned upside down? What do you think? (Do not just think about examples. This is a question if something is *always true*.)

Summary

We now have several methods for solving problems that require dividing fractions:

Dividing fractions:

- Draw a picture using the rectangle method, and use that to solve the division problem.
- Find a common denominator and divide the numerators.
- Rewrite the division as a missing factor multiplication problem, and solve that problem.
- Simplify an ugly fraction.

- Invert the second fraction (the dividend) and then multiply.

Think / Pair / Share

Discuss your opinions about our four methods for solving fraction division problems with a partner:

- Which method for division of fractions is the easiest to *understand why it works* ?
- Which method for division of fractions is the easiest to *use in computations*?
- What are the benefits and drawbacks of each method? (Think both as a future teacher and as someone solving math problems here.)

35. Dividing Fractions: Problems

We've spent the last couple of chapters talking about dividing fractions: how to make sense of the operation, how to picture what's going on, and how to do the computations. But all of this kind of begs the question: When would you ever want to divide fractions, anyway? How does that even come up?

It's important that teachers are able to come up with situations and problems that model particular operations, which means you have to really understand what the operations mean and when they are used.

Think / Pair / Share

- Use one of our methods (draw a picture, rectangles, common denominator, missing factor) to compute $1\frac{3}{4} \div \frac{1}{2}$.
- Come up with a situation where you would want to compute $1\frac{3}{4} \div \frac{1}{2}$. (That is, write a word problem that would require you to do this computation to solve it.)

When to Multiply, When to Divide?

A common answer to

> Come up with a situation where you would want to compute $1\frac{3}{4} \div \frac{1}{2}$.

Is something like this:

> My recipe calls for $1\frac{3}{4}$ cups of flour, but I only want to make half a recipe. How much flour should I use?

But that problem doesn't ask you to divide fractions. It asks you to cut your recipe in half, which means dividing by 2 or *multiplying* by $\frac{1}{2}$.

Why is it so hard to come up with division problems that use fractions? Maybe it's because fractions are already the answer to a division problem, so you're dividing and then dividing some more. Maybe it's because they just make it look so complicated. In any case, it's worth spending some time thinking about division problems that involve fractions and how to recognize and solve them.

One handy trick: Write a problem that involves division of whole numbers, and then see if you can change the numbers to fractions in a sensible way.

Examples

Here are some division problems involving whole numbers:

- I have 10 feet of ribbon. How many 2-inch pieces can I cut from it?
- I have a fancy old clock that rings once every 15 minutes. How many times will it ring over the course of 2 hours

(120 minutes)?

- My fish tank needs 6 gallons of water, and my bucket holds 3 gallons. How many times will I need to fill my bucket in order to fill the tank?
- A recipe calls for 6 cups of flour, and my largest scoop measures exactly 2 cups. How many times should I use it?
- I ran 12 miles and went around the the same route 3 times. How long was the route?

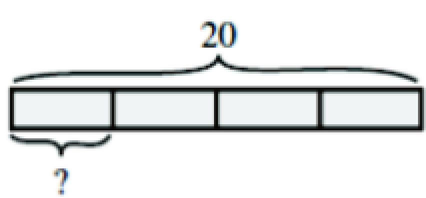
Here are some very similar problems, rewritten to use fractions instead:

- I have $1\frac{3}{4}$ feet of ribbon. How many 6-inch (that's $\frac{1}{2}$ a foot) pieces can I cut from it?
- My watch alarm goes off every half hour, and I don't know how to shut it off. How many times will it go off during the $1\frac{3}{4}$ hour movie?
- My fish tank needs $1\frac{3}{4}$ gallons of water, and my bucket holds $\frac{1}{2}$ gallon. How many times will I need to fill my bucket in order to fill the tank?
- I want to measure $1\frac{3}{4}$ cups of flour for a recipe, but I only have a $\frac{1}{2}$ cup measuring cup. How many times should I fill it?
- I ran $1\frac{3}{4}$ miles before I twisted my ankle. I only finished half the race. How long was the race course?

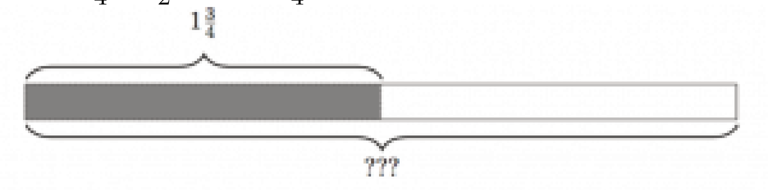
For each one of the fraction division questions, we can understand why it's a division problem:

- I have $1\frac{3}{4}$ feet of ribbon. How many 6-inch (that's $\frac{1}{2}$ a foot) pieces can I cut from it? This means making equal groups of $\frac{1}{2}$ foot each and asking how many groups. That's quotative division.
- My watch alarm goes off every half hour, and I don't know how to shut it off. How many times will it go off during the $1\frac{3}{4}$ hour movie? Again, we're making equal groups of $\frac{1}{2}$ hour each, and asking how many groups. Quotative division.
- My fish tank needs $1\frac{3}{4}$ gallons of water, and my bucket holds $\frac{1}{2}$ gallon. How many times will I need to fill my bucket in order to fill the tank? Once again: we're making equal groups of $\frac{1}{2}$ gallon each, and asking how many groups (buckets).
- I want to measure $1\frac{3}{4}$ cups of flour for a recipe, but I only have a $\frac{1}{2}$ cup measuring cup. How many times should I fill it? This is making equal groups of $\frac{1}{2}$ cup and asking how many groups.
- I ran $1\frac{3}{4}$ miles before I twisted my ankle. I only finished half the race. How long was the race course? This one is a little different. This one is a little different. It's the fraction version of partitive division.

Recall what partitive division asks: For $20 \div 4$, we ask 20 is 4 groups of what size?



So for $1\frac{3}{4} \div \frac{1}{2}$, we ask: $1\frac{3}{4}$ is half a group of what size?



Think / Pair / Share

You try it.

- First write five different division word problems that use whole numbers. (Try to write at least a couple each of partitive and quotative division problems.)
- Then change the problems so that they are fraction division problems instead. You might need to rewrite the problem a bit so that it makes sense.
- Solve your problems!

36. Fractions involving zero

Zero in the Numerator

Think / Pair Share

Does the fraction $\frac{0}{11}$ make sense?

- Write a “pies per child” story for the fraction $\frac{0}{11}$. Does it make sense? How much pie does each individual child receive in your story?
- Think of $\frac{0}{11}$ as the answer to a division problem. What is that division problem? Can you solve it?

It seems pretty clear that zero pies among eleven kids gives zero pies per child:

$$\frac{0}{11} = 0.$$

The same reasoning would lead us to say:

$$\frac{0}{b} = 0 \quad \text{for any positive number } b.$$

The “Pies Per Child Model” offers one explanation: If there are no pies for us to share, no one gets any pie. It does not matter how many children there are. No pie is no pie is no pie.

We can also justify this claim by thinking about a missing factor multiplication problem:

$$\frac{0}{b} \quad \text{is asking us to fill in the blank:} \quad \_\_ \cdot b = 0.$$

The only way to fill that in and make a true statement is with 0, so $\frac{0}{b} = 0.$

Zero in the Denominator

What happens if things are flipped the other way round?

Think / Pair / Share

Does the fraction $\frac{11}{0}$ make sense?

- Write a "pies per child" story for the fraction $\frac{11}{0}$. Does it make sense? How much pie does each individual child receive in your story?
- Think of $\frac{11}{0}$ as the answer to a division problem. What is that division problem? Can you solve it?

Students often learn in school that "dividing by 0 is undefined." But they learn this as a rule, rather than thinking about why it makes sense or how it connects to other ideas in mathematics. In this case, the most natural connection is to a multiplication fact, the zero property for multiplication:

$$\text{any number} \cdot 0 = 0.$$

That says we can never find solutions to problems like

$$\_\_ \cdot 0 = 5, \qquad \_\_ \cdot 0 = 17, \qquad \_\_ \cdot 0 = 1.$$

Using the connection between fractions and division, and the connection between division and multiplication, that means there is no number $\frac{5}{0}$. There is no number $\frac{17}{0}$. And there is no number $\frac{1}{0}$. They are all "undefined" because they are not equal to any number at all.

Think / Pair / Share

Can we give meaning to $\frac{0}{0}$ at least? After all, a zero would appear on both sides of that equation!

- Cyril says that $\frac{0}{0} = 2$ since $0 \cdot 2 = 0$.
- Ethel says that $\frac{0}{0} = 17$ since $0 \cdot 17 = 0$.
- Wonhi says that $\frac{0}{0} = 887231243$ since $0 \cdot 887231243 = 0$.

Who is right? Can they all be correct? What do you think?

Cyril says that $\frac{0}{0} = 2$, and he believes he is correct because it passes the check: $0 \cdot 2 = 0$.

But 17 also passes the check, and so does 887231243. In fact, I can choose any number for x, and $0 \cdot x = 0$ will pass the check!

The trouble with the expression $\frac{a}{0}$ (with a not zero) is that there is *no meaningful value* to assign to it. The trouble with $\frac{0}{0}$ is different: There are *too many possible values* to give it!

Dividing by zero is simply too problematic to be done! It is best to avoid doing so and never will we allow zero as the denominator of a fraction. (But all is fine with 0 as a numerator.)

37. Problem Bank

Problem 6

Harriet is part of a group of five children who share four pies. Jeff is part of a group of seven children who share four pies. Jean is part of a group of seven children who share six pies.

1. Who gets more pie, Harriet or Jeff? Justify your answer!
2. Who gets more pie, Jeff or Jean? Justify your answer!
3. Who gets more pie, Harriet or Jean? Justify your answer!

Problem 7

Yesterday was Zoe's birthday, and she had a big rectangular cake. Today, $\frac{2}{5}$ of the cake is left. The **leftover cake** is shown here.

Draw a picture of the original (whole) cake and explain your work.

Problem 8

Use benchmarks and intuitive methods to arrange the fractions below in ascending order. Explain how you decided. (The point of this problem is to think more and compute less!):

$$\frac{2}{5}, \qquad \frac{1}{3}, \qquad \frac{5}{8}, \qquad \frac{1}{4}, \qquad \frac{2}{3}, \qquad \frac{3}{4}, \qquad \frac{4}{7}.$$

Problem 9

Which of these fractions has the larger value? Justify your choice.

$$\frac{10001}{10002} \quad \text{or} \quad \frac{10000001}{10000002}.$$

Problem 10

Solve each division problem. Look for a shortcut, and explain your work.

$$\frac{251 + 251 + 251 + 251}{4}$$

$$\frac{377 + 377 + 377 + 377 + 377}{5}$$

$$\frac{123123 + 123123 + 123123 + 123123 + 123123 + 123123}{3}$$

Problem 11

Yoko says

$$\frac{16}{64} = \frac{1}{4}$$

because she cancels the sixes:

$$\frac{16}{64} = \frac{1}{4}.$$

But note:

$$\frac{16}{64} = \frac{1 \cdot 16}{4 \cdot 16} = \frac{1 \cdot 16}{4 \cdot 16} = \frac{1}{4}.$$

So is Yoko right? Does her cancelation rule always work? If it does not always work, can you find any other example where it works? Can you find every example where it works?

Problem 12

Jimmy says that a fraction does not change in value if you add the same amount to the numerator and the denominator. Is he right? If you were Jimmy's teacher, how would you respond?

Problem 13

1. Shelly says that if $ab < cd$ then $\frac{a}{b} < \frac{c}{d}$. Is Shelly's claim always true, sometimes true, or never true? If you were Shelly's teacher, what would you say to her?
2. Rob says that if $ad < bc$ then $\frac{a}{b} < \frac{c}{d}$. Is Rob's claim always true, sometimes true, or never true? If you were Rob's teacher, what would you say to him?

Problem 14

Jill, her brother, and another partner own a pizza restaurant. If Jill owns $\frac{1}{3}$ of the restaurant and her brother owns $\frac{1}{4}$ of the restaurant, what fraction does the third partner own?

Problem 15

John spent a quarter of his life as a boy growing up, one-sixth of his life in college, and one-half of his life as a teacher. He spent his last six years in retirement. How old was he when he died?

Problem 16

Nana was planning to make a red, white, and blue quilt. One-third was to be red and two-fifths was to be white. If the area of the quilt was to be 30 square feet, how many square feet would be blue?[1]

1. Image used under Creative Commons CC0 1.0 Universal Public Domain Dedication.

Ku'u Hae Aloha (My Beloved Flag), Hawaiian cotton quilt from Waimea, before 1918, Honolulu Academy of Arts.

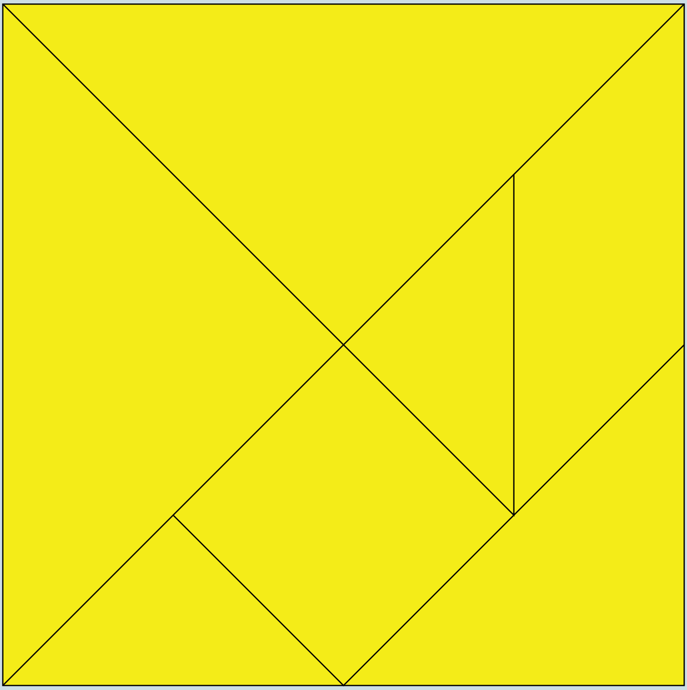
Problem 17

Rafael ate one-fourth of a pizza and Rocco ate one-third of it. What fraction of the pizza did they eat?

Problem 18

Problem 18 (Tangrams). Tangrams[2] are a seven-piece puzzle, and the seven pieces can be assembled into a big square.

2. Tangram image from Wikimedia Commons, public domain.



1. If the large square shown above is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.
2. The tangram puzzle contains a small square. If the small square (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.
3. The tangram set contains two large triangles. If a large triangle (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces.

Justify your answers.

4. The tangram set contains one medium triangle. If the medium triangle (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.
5. The tangram set contains two small triangles. If a small triangle (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers

Problem 19

Mikiko said her family made two square pizzas at home. One of the pizzas was 8 inches on each side, and the other was 12 inches on each side. Mikiko ate $\frac{1}{4}$ of the small pizza and $\frac{1}{12}$ of the large pizza. So she said that she ate

$$\frac{1}{4}+\frac{1}{12} = \frac{3}{12}+\frac{1}{12} = \frac{4}{12} = \frac{1}{3}$$

of a pizza. Do you agree with Mikiko's calculation? Did she eat $\frac{1}{3}$ of a whole pizza? Carefully justify your answer. (This question is tricky. It's probably a good idea to draw a picture!)

Problem 20

Look at the triangle of numbers. There are lots of patterns here! Find as many as you can. In particular, try to answer these questions:

1. What pattern describes the first number in each row?
2. How is each fraction related to the two fractions below it?
3. Can you write down the next two rows of the triangle?

$$\frac{1}{1}$$

$$\frac{1}{2} \quad \frac{1}{2}$$

$$\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3}$$

$$\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4}$$

$$\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5}$$

Problem 21

Marie made a sheet cake at home, but she saved some to bring to work and share with her co-workers the next day. Answer these questions about Marie's cake. (Draw a picture!)

1. Suppose Marie saved $\frac{1}{2}$ of the cake for her co-workers and the co-workers ate $\frac{3}{4}$ of this. What fraction of the entire cake did they eat?
2. What if Marie saved $\frac{1}{6}$ instead, and they ate $\frac{2}{3}$ of this?
3. What if she saved $\frac{5}{7}$ of the cake and they ate $\frac{1}{2}$ of this?

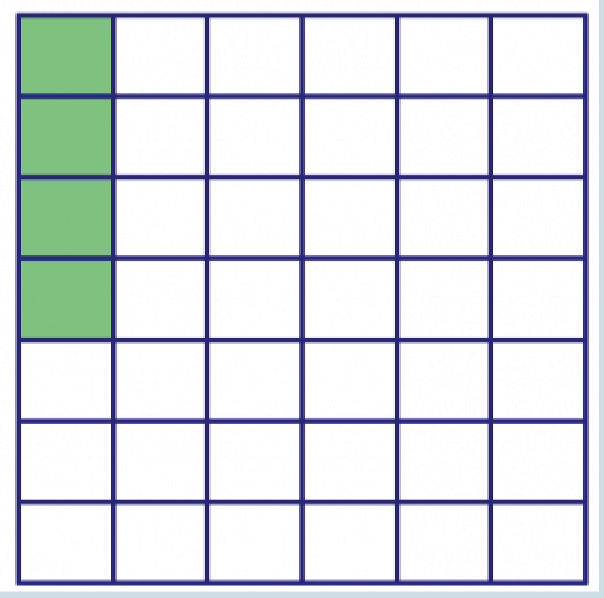
Problem 22

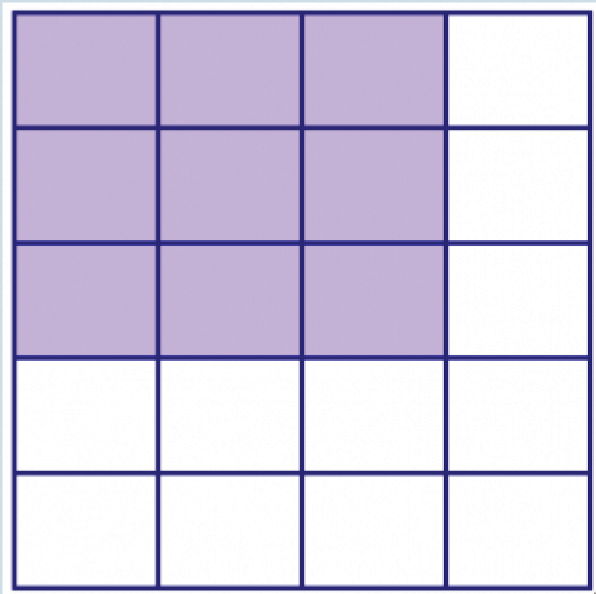
An elementary school held a "Family Math Night" event, and 405 students showed up. Two-thirds of the students who showed up won a door prize. How many students won prizes?

Problem 23

For each picture shown:

- What multiplication problem is represented?
- What is the product?





Problem 24

For each problem, use only the digits 0, 1, 2,. . . , 9 at most once each in place of the variables. Find the value closest to 1. Note that a can be a different value in each of the three problems. Justify your answer: How do you know it is the closest to 1?

1. $\frac{a}{b}$.
2. $\frac{a}{b} \cdot \frac{c}{d}$.
3. $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f}$.

Problem 25

A town plans to build a community garden that will cover $\frac{2}{3}$ of a square mile on an old farm. One side of the garden area will be along an existing fence that is $\frac{3}{4}$ of a mile long. If the garden is a rectangle, how long is the other side?

Problem 26

Nate used $90\frac{1}{2}$ pounds of seed to plant $1\frac{1}{4}$ acres of wheat. How many pounds of seed did he use per acre?

Problem 27

The family-sized box of laundry detergent contains 35 cups of detergent. Your family's machine requires $1\frac{1}{4}$ cup per load. How many loads of laundry can your family do with one box of detergent?

Problem 28

Jessica bikes to campus every day. When she is one-third of the way between her home and campus, she passes a grocery store. When she is halfway to school, she passes a Subway sandwich shop. This morning, Jessica passed the grocery store at 8:30am, and she passed Subway at 8:35am. What time did she get to campus?

Problem 29

If you place a full container of flour on a balance scale and place on the other side a $\frac{1}{3}$ pound weight plus a container of flour (the same size) that is $\frac{3}{4}$ full, then the scale balances. How much does the full container of flour weigh?

Problem 30

Geoff spent $\frac{1}{4}$ of his allowance on a movie. He spent $\frac{11}{18}$ of what was left on snacks at school. He also spent \$3 on a magazine, and that left him with $\frac{1}{24}$ of his total allowance, which he put into his savings account. How much money did Geoff save that week?

Problem 31

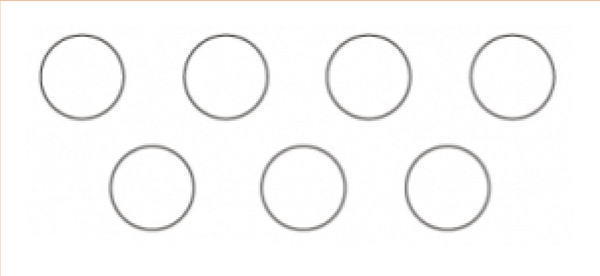
Lily was flying to San Francisco from Honolulu. Halfway there, she fell asleep. When she woke up, the distance remaining was half the distance traveled while she slept. For what fraction of the trip was Lily asleep?

38. Egyptian Fractions

Example: Egyptian fraction for 7/12

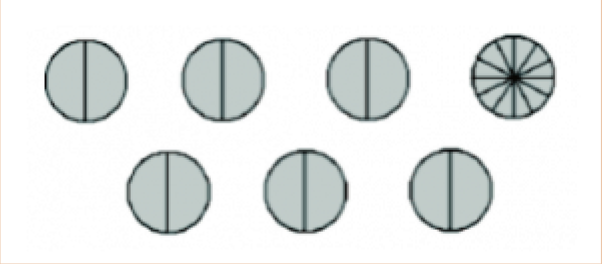
Consider the problem: Share 7 pies equally among 12 kids. Of course, given our model for fractions, each child is to receive the quantity "$\frac{7}{12}$." But this answer has little intuitive feel.

Suppose we took this task as a very practical problem. Here are the seven pies:



Is it possible to give each of the kids a whole pie? No.

How about the next best thing – can each child get half a pie? Yes! There are certainly 12 half pies to dole out. There is also one pie left over yet to be shared among the 12 kids. Divide this into twelfths and hand each kid an extra piece.



So each child gets $\frac{1}{2} + \frac{1}{12}$ of a pie, and it is indeed true that

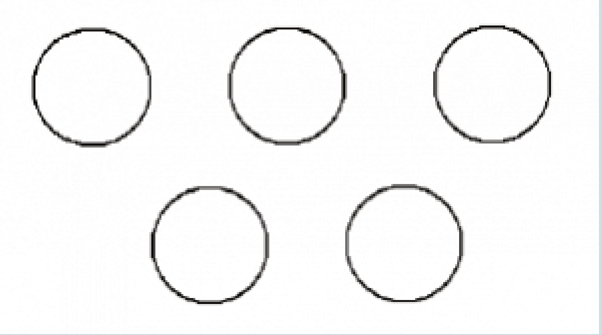
$$\frac{7}{12} = \frac{1}{2} + \frac{1}{12}.$$

(Check that calculation. . . don't just believe it!)

This seems quite reasonable. Instead of seven pieces each of size $\frac{1}{12}$, each kid gets a piece that is $\frac{1}{2}$ and a piece that is $\frac{1}{12}$. It's a lot less cutting, and a lot less messy!

Problem 32

1. Suppose you want to share five pies among six children, but you want each child to get a small number of (relatively) large pieces rather than five pieces of size $\frac{1}{6}$. Following the example above, how could you do it?



2. Using similar ideas, how could you share 4 pies among 7 kids?

History: Rhind Papyrus

The Egyptians (probably) were not particularly concerned with splitting up pies. But in fact, they did have a very strange (to us) way of expressing fractions. We know this by examining the Rhind Papyrus. This ancient document indicates that fractions were in use as many as four thousand years ago in Egypt, but the Egyptians seem to have worked primarily with **unit fractions**. They insisted on writing all of their fractions as sums of fractions with numerators equal to 1, and they insisted that the denominators of the fractions were all different.

> Accurate reckoning for inquiring into things, and the knowledge of all things, mysteries…all secrets.

The Rhind Papyrus is an ancient account of Egyptian mathematics named after Alexander Henry Rhind. Rhind was a Scotsman who acquired the ancient papyrus in 1858 in Luxor, Egypt.

The papyrus dates back to around 1650 B.C. It was copied by a scribe named Ahmes (the earliest known contributor to the field of mathematics!) from a lost text written during the reign of king Amenehat III. The opening quote is taken from Ahmes introduction to the Rhind Papyrus[1]. The papyrus covers topics relating to fractions, volume, area, pyramids, and more.

1. Image of Rhind Papyrus from Wikimedia Commons, public domain.

Rhind Papyrus

Egyptian Fractions

To write a fraction as an Egyptian fraction, you must rewrite the fraction as:

- a **sum of unit fractions** (that means the numerator is 1), and
- the denominators must all be different.

Examples: Egyptian fractions for 3/10 and 5/7

The Egyptians would not write $\frac{3}{10}$, and they would not even write $\frac{1}{10} + \frac{1}{10} + \frac{1}{10}$. Instead, they wrote

$$\frac{1}{4} + \frac{1}{20}.$$

The Egyptians would not write $\frac{5}{7}$, and they would not even write $\frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$. Instead, they wrote

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{70}.$$

(You should check that the sums above give the correct resulting fractions!)

Problem 33

Write the following as a sum of two *different* unit fractions. Be sure to check your answers.

$$\frac{2}{3}, \qquad \frac{2}{5}, \qquad \frac{2}{7}, \qquad \frac{2}{9}.$$

Can you find a general rule for how to write $\frac{2}{n}$ as an Egyptian fraction? (Assume n is an odd number.)

Problem 34

Write the following as a sum of distinct unit fractions. ("Distinct" means the fractions must have different denominators.) Note that you may need to use more than two unit fractions in some of the sums. Be sure to check your answers.

$$\frac{3}{4}, \qquad \frac{5}{6}, \qquad \frac{3}{5}, \qquad \frac{5}{9}.$$

Can you find a general process for fractions bigger than $\frac{1}{2}$?

Problem 35

Write the following fractions as Egyptian fractions.

$$\frac{17}{20}, \qquad \frac{3}{7}.$$

Can you find a general algorithm that will turn *any fraction at all* into an Egyptian fraction?

39. Algebra Connections

In an advanced algebra course students are often asked to work with complicated expressions like:

$$\frac{\frac{1}{x}+1}{\frac{3}{x}}$$

We can make it look friendlier by using the key fraction rule, exactly the same technique we used in the chapter on "Dividing Fractions: Invert and Multiply." In this example, let us multiply the numerator and denominator each by x. (Do you see why this is a good choice?) We obtain:

$$\frac{\left(\frac{1}{x}+1\right)\cdot x}{\left(\frac{1}{x}\right)\cdot x} = \frac{1+x}{3},$$

and $\frac{1+x}{3}$ is much less scary.

Notic that expressions like

$$\frac{1}{x}$$

cannot be rewritten as a decimal. Expressions like this arise in numerous applications, so it is important for math and science students to be able to work with fractions in fraction form, without always resorting to converting to decimals.

Example

As another example, given:

$$\frac{\frac{1}{a}-\frac{1}{b}}{ab},$$

one might find it helpful to multiply the numerator and the denominator each by a and then each by b:

$$\frac{\left(\frac{1}{a}-\frac{1}{b}\right)\cdot a\cdot b}{(ab)\cdot a\cdot b} = \frac{b-a}{a^2b^2}.$$

Examples

For

$$\frac{\frac{1}{(w+1)^2}-2}{\frac{1}{(w+1)^2}+5},$$

it might be good to multiply numerator and denominator each by $(w+1)^2$. (Why?)

$$\frac{\left(\frac{1}{(w+1)^2} - 2\right) \cdot (w+1)^2}{\left(\frac{1}{(w+1)^2} + 5\right) \cdot (w+1)^2} = \frac{1 - 2(w+1)^2}{1 + 5(w+1)^2}.$$

On Your Own

Can you make each of these expressions look less scary?

$$\frac{2 - \frac{1}{x}}{1 + \frac{1}{x}}, \qquad \frac{\frac{1}{x+h} + 3}{\frac{1}{x+h}}, \qquad \frac{1}{\frac{1}{a} + \frac{1}{b}}, \qquad \frac{\frac{1}{x+a} - \frac{1}{x}}{a}.$$

40. What is a Fraction? Part 3

So far, we have no single model that makes sense of fractions in all contexts. Sometimes a fraction is an action ("Cut this in half.") Sometimes it is a quantity ("We each get 2/3 of a pie!") And sometimes we want to treat fractions like numbers, like ticks on the number line in-between whole numbers.

We could say that a fraction is just a pair of numbers a and b, where we require that $b \neq 0$. We just happen to write the pair as $\frac{a}{b}$.

But again this is not quite right, since a whole infinite collection of pairs of numbers represent the same fraction! For example:

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \cdots.$$

So a single fraction is actually a whole infinite class of pairs of numbers that we consider "equivalent."

How do mathematicians think about fractions? Well, in exactly this way. They think of pairs of numbers written as $\frac{a}{b}$, where we remember two important facts:

- $b \neq 0$, and
- $\frac{a}{b}$ is really shorthand for a whole infinite class of pairs that look like $\frac{xa}{xb}$ for all $x \neq 0$.

This is a hefty shift of thinking: The notion of a "number" has changed from being a specific combination of symbols to a whole class of combinations of symbols that are deemed equivalent.

Mathematicians then *define* the addition of fractions to be given by the daunting rule:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

This is obviously motivated by something like the “Pies Per Child Model.” But if we just define things this way, we must worry about *proving* that choosing different representations for $\frac{a}{b}$ and $\frac{c}{d}$ lead to the same final answer.

For example, it is not immediately obvious that

$$\frac{2}{3}+\frac{4}{5} \quad \text{and} \quad \frac{4}{6}+\frac{40}{50}$$

give answers that are equivalent. (Check that they do!)

They also *define* the product of fractions as:

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}.$$

Again, if we start from here, we have to *prove* that you get equivalent answers for different choices of fractions equivalent to $\frac{a}{b}$ and $\frac{c}{d}$.

Then mathematicians establish that the axioms of an arithmetic system hold with these definitions and carry on from there! (That is, they check that addition and multiplication are both commutative and associative, that the distributive law holds, that all representations of 0 act like an additive identity, and so on.)

This is abstract, dry, and not at all the best first encounter to offer students on the topic of fractions. And, moreover, this approach completely avoids the question as to what a fraction really means in the “real world.” But it is the best one can do if one is to be completely honest.

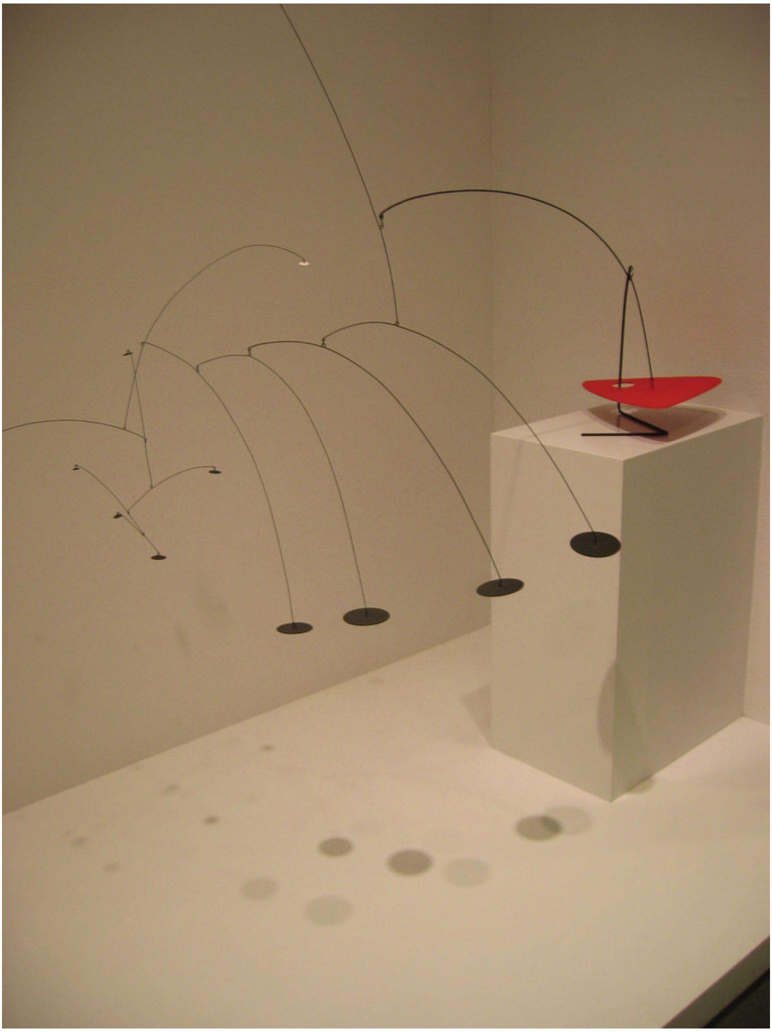
Think / Pair / Share

So… what is a fraction, really? How do you think about them?

And what is the best way to talk about them with elementary school students?

PART V

PATTERNS AND ALGEBRAIC THINKING



A mobile by the artist Alexander Calder.

[1]

> Doing mathematics should always mean finding patterns and crafting beautiful and meaningful explanations.
> -Paul Lockhart

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41. Introduction

Algebra skills are essential for your future students. Why? Here are just a few reasons:

- Mathematics, and especially algebra, is the language of science and modern technology. Thinking algebraically helps you to make sense of the world, to understand and interact with technology more productively, and to succeed in other fields.
- Algebra is a tool for solving problems. This may not be your experience so far, but it is true. If you are able to "algebratize" a problem, that often helps lead you to a solution.
- Algebra helps you to think abstractly. It is a tool for thinking about operations like addition, subtraction, multiplication, and division separate from doing calculations on particular numbers. Algebra helps you to *understand and explain why* the operations work the way they do, to *describe* their properties clearly, and to *manipulate* expressions to see the bigger picture.

You might wonder why future elementary teachers should master algebra, a topic usually studied (by that name, anyway) in 8th grade and beyond. But the Common Core Standards for School Mathematics has standards in "Operations and Algebraic Thinking" beginning in kindergarten!

Everyone who shows up to school has already learned a lot about abstraction and generalization – the fundamental ideas in algebra. They are all capable of learning to formalize these ideas. Your job as an elementary school teacher will be to provide your students with even more experiences in abstraction and generalization in a mathematical context, so that these ideas will seem quite natural when they get to a class with the name "Algebra."

Let's start with a problem:

Problem 1

I can use four 4's to make 0:

$$44 - 44 = 0.$$

I can also use four 4's to make the number 10:

$$(4 \times 4) - 4 - \sqrt{4} = 10.$$

Your challenge: Use four 4's to make all of the numbers between 0 and 20. (Try to find different solutions for 0 and 10 than the ones provided.) You can use any mathematical operations, but you can't use any digits other than the four 4's.

Think / Pair / Share

- What does "algebra" mean to you?
- What does Problem 1 have to do with "algebra"?
- What do you imagine when you think about using algebra to solve problems in school?
- Have you ever used algebra to solve problems outside of school?
- What is meant by "algebraic thinking," and what kinds of

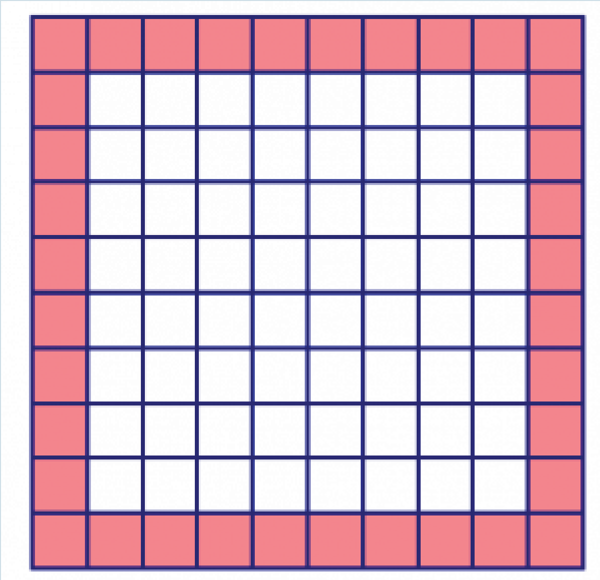
algebraic thinking can be done by elementary school students?

42. Borders on a Square

Here's another problem:

Problem 2

Here is a large square made up of 100 smaller unit squares. The unit squares along the border of the large square are colored red. Without counting one-by-one, can you figure out how many red squares there are in the picture?

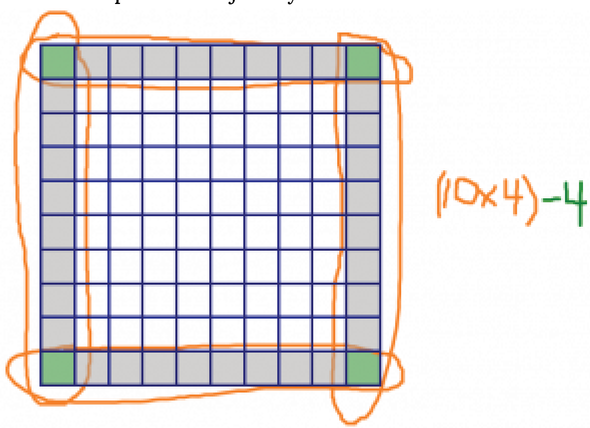


Clearly describe how you figured out the number of red squares, and how you know your answer is correct.

Justin calculated the number of squares as $(10 \times 4) - 4$. He justified his answer this way:

> Since the dimensions of the big square are 10×10, there are 10 squares along each of the four sides. So that gives me 40 red squares. But then each corner is part of two different sides. I've counted each of the corners twice. So I need to make up for that by subtracting 4 at the end.

Justin showed this picture to justify his work:



Think / Pair / Share

- What do you think about Justin's solution? Are you convinced? Could he have explained it more clearly?
- Was Justin's solution different from your solution or the same?
- Notice the color coding in Justin's picture. What do the colors represent? Why did he use the colors the way he did?

Problem 3

There are lots of different ways to calculate the number of colored squares along the border of a 10×10 square. Below are the calculations several other students did. For each calculation, write a justification and draw a picture to show why it calculates the number of squares correctly. Think about using color in your picture to make your work more clear.

1. Valerie calculated $10 + 10 + 8 + 8$.
2. Kayla calculated 4×9.

3. Linda calculated $(10 \times 10) - (8 \times 8)$.
4. Mark calculated $(4 \times 8) + 4$.
5. Allan calculated $10 + 9 + 9 + 8$.

Problem 4

Now suppose that you have a large 6×6 square with the unit squares along the border colored red. Adapt two of the techniques above to calculate the number of red unit squares.

For each technique you used, write an explanation and include a picture. Think about how to use colors or other methods to make your picture and explanation more clear.

Problem 5

Now suppose that you have a large 25×25 square with the unit squares along the border colored red. Adapt two of the techniques above to calculate the number of red unit squares.

For each technique you used, write an explanation and include a picture. Think about how to use colors or other methods to make your picture and explanation more clear.

Problem 6

1. Suppose that you have 64 red squares. Can you use all of those squares to make the border of a larger square in a picture like the one above? If yes, what are the dimensions of the larger square? If no, why not?
2. What if you have 30 red squares? Same questions.
3. What if you have 256 red squares? Same questions.

Think / Pair / Share

Describe some general rules:

- If you have a large $n \times n$ square with the border squares colored red, how many red squares will there be? Justify your answer with words and a picture.

- If you have k red squares, is there a quick test you can do to decide if you can use all of those squares to make the border of a large square? Can you tell how big the square will be?

43. Careful Use of Language in Mathematics: =

The notion of equality is fundamental in mathematics, and especially in algebra and algebraic thinking. The symbol “=”' expresses a *relationship*. It is *not* an operation in the way that + and are × operations. It should not be read left-to-right, and it definitely does not mean “… and the answer is …”.

For your work to be clear and easily understood by others, it is essential that you use the symbol = appropriately. And for your future students to understand the meaning of the = symbol and use it correctly, it is essential that you are clear and precise in your use of it.

Let’s start by working on some problems.

Problem 7

Akira went to visit his grandmother, and she gave him \$1.50 to buy a treat.

He went to the store and bought a book for \$3.20. After that, he had \$2.30 left.

How much money did Akira have before he visited his grandmother?

Problem 8

Examine the following equations. Decide: Is the statement always true, sometimes true, or never true? Justify your answers.

$(a)\ 5+3=8.$ $(b)\ \frac{2}{3}+\frac{1}{2}=\frac{3}{5}.$ $(c)\ 5+3=y.$ $(d)\ \frac{a}{5}=\frac{5}{a}.$

$(e)\ n+3=m.$ $(f)\ 3x=2x+x.$ $(g)\ 5k=5k+1.$

Problem 9

Consider the equation

$$18-7=\_\_\_\_.$$

1. Fill in the blank with something that makes the equation *always true.*
2. Fill in the blank with something that makes the equation *always false.*
3. Fill in the blank with something that makes the equation

sometimes true and sometimes false.

Problem 10

If someone asked you to *solve* the equations in Problem 8, what would you do in each case and why?

Think / Pair / Share

Kim solved Problem 7 this way this way:

Let's see:

$$2.30 + 3.20 = 5.50 - 1.50 = 4,$$

so the answer is 4.

What do you think about Kim's solution? Did she get the correct answer? Is her solution clear? How could it be better?

Although Kim found the correct numerical answer, her calculation really doesn't make any sense. It is true that

$$2.30 + 3.20 = 5.50.$$

But it is definitely *not* true that

$$2.30 + 3.20 = 5.50 - 1.40.$$

She is incorrectly using the symbol "=", and that makes her calculation hard to understand.

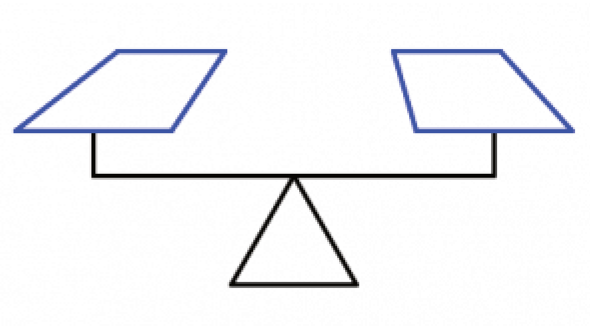
Think / Pair / Share

- Can you write a good *definition* of the symbol "="? What does it mean and what does it represent?
- Give some examples: When should the symbol "=" be used, and when should it *not* be used?
- Do these two equations express the same relationships or different relationships? Explain your answer.

$$x^2 - 1 = (x+1)(x-1).$$

$$(x+1)(x-1) = x^2 - 1.$$

This picture shows a (very simplistic) two-pan balance scale. Such a scale allows you to *compare* the weight of two objects. Place one object in each pan. If one side is lower than the other, then that side holds heavier objects. If the two sides are balanced, then the objects on each side weigh the same.

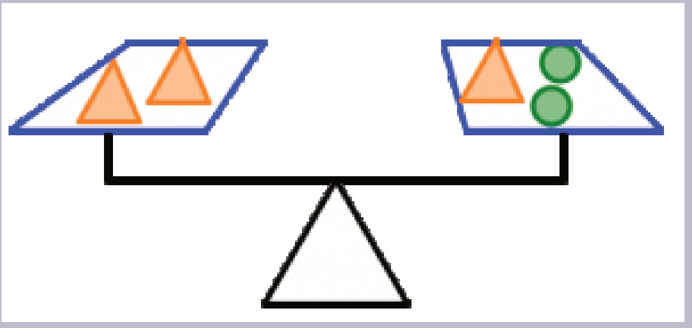


Think / Pair / Share

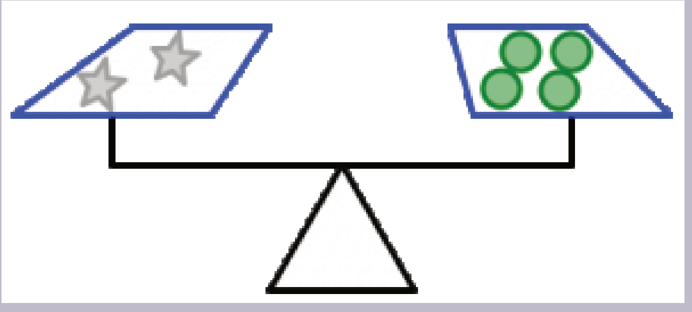
In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The silver stars all weigh the same.
- The scale is balanced.

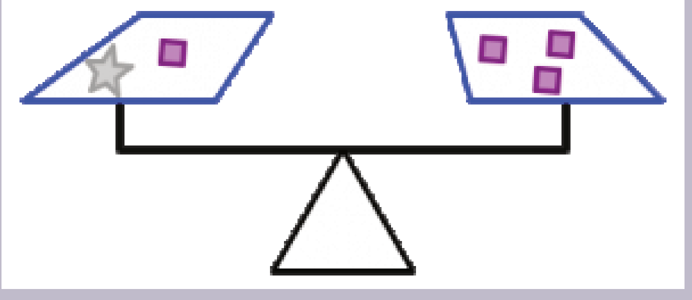
1. In the picture below, what do you know about the weights of the triangles and the circles? How do you know it?



2. In the picture below, what do you know about the weights of the circles and the stars? How do you know it?



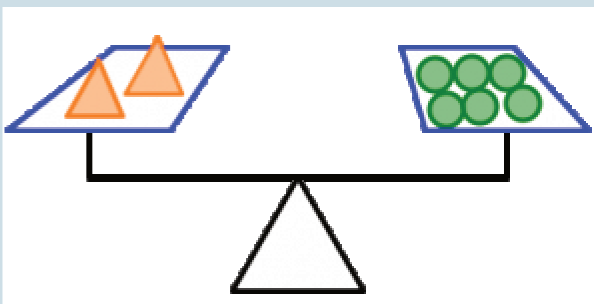
3. In the picture below, what do you know about the weights of the stars and the squares? How do you know it?

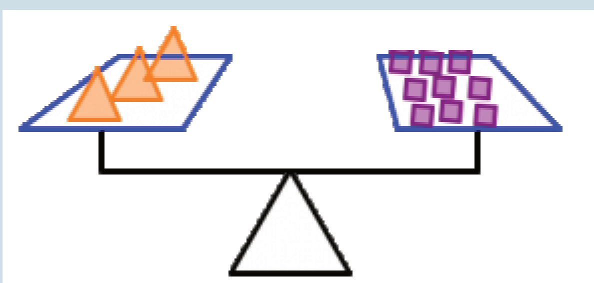


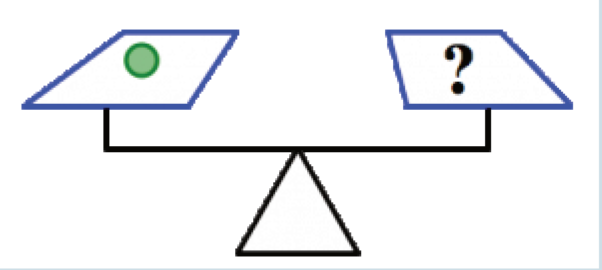
Problem 11

In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The scale is balanced.





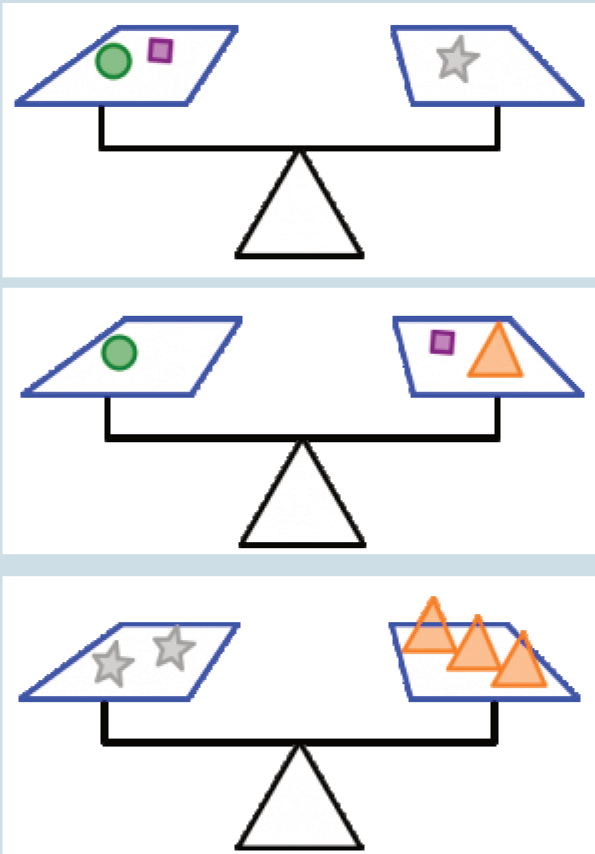


How many purple squares will balance with one circle? Justify your answer.

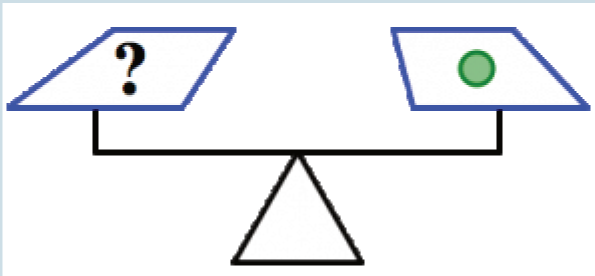
Problem 12

In the pictures below:

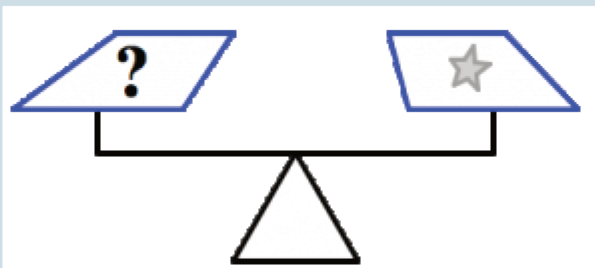
- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The silver stars all weigh the same.
- The scale is balanced.



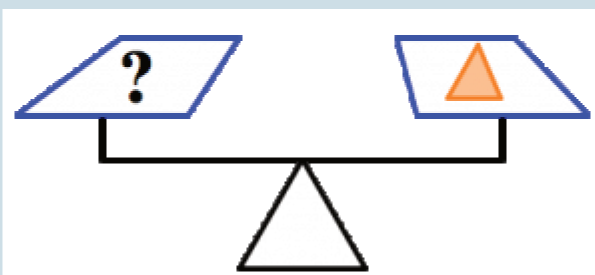
How many purple squares will balance the scale in each case? Justify your answers.



(a)



(b)

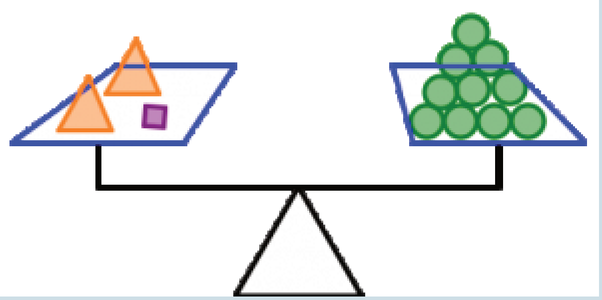


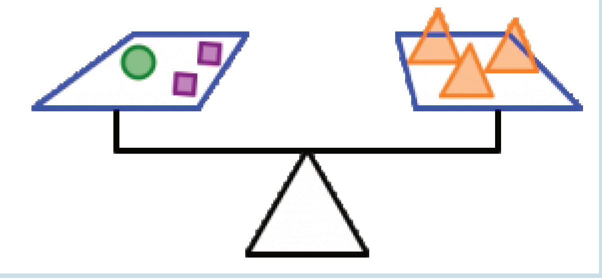
(c)

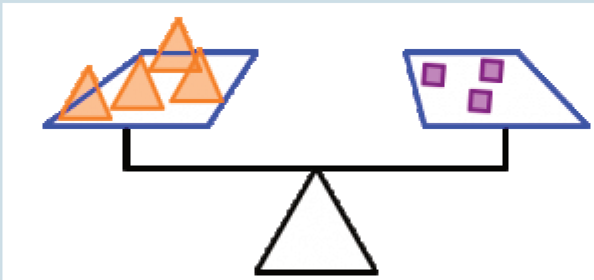
Problem 13

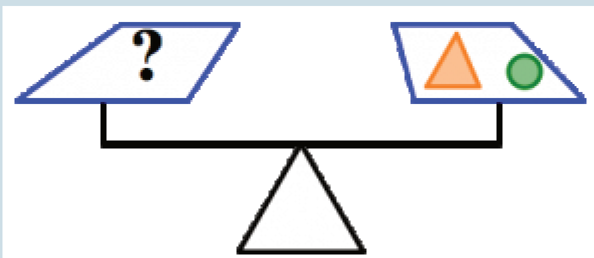
In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The scale is balanced.









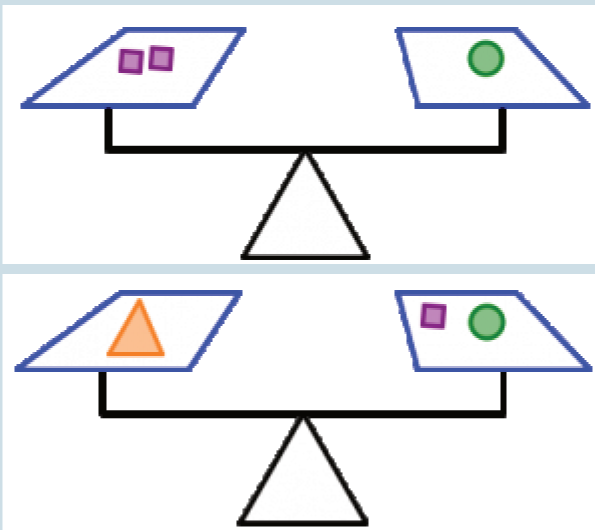
What will balance the last scale? Can you find more than one answer?

Problem 14

In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.

- The purple squares all weigh the same.
- The scale is balanced.



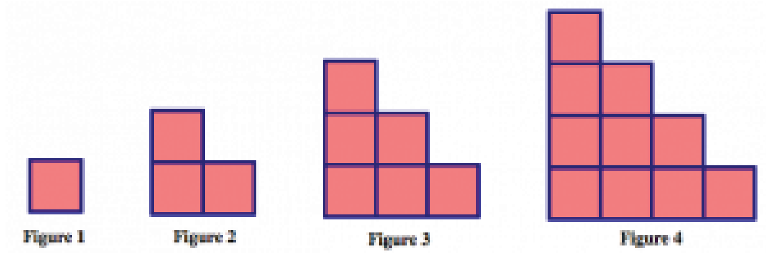
1. Which shape weighs the most: the square, the triangle, or the circle? Which shape weighs the least? Justify your answers.
2. Which of the two scales is holding the most total weight? How do you know you're right?

Think / Pair / Share

What do Problems 11–14 above have to do with the “=” symbol?

44. Growing Patterns

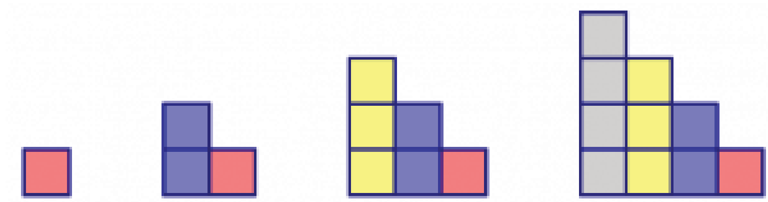
Here is a pattern made from square tiles.



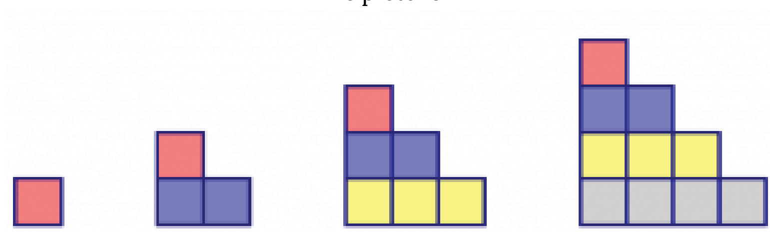
Think / Pair / Share

- Describe how you see this pattern growing. Be as specific as you can. Draw pictures and write an explanation to make your answer clear.
- Say as much as you can about this growing pattern. Can you draw pictures to extend the pattern?
- What mathematical questions can you ask about this pattern? Can you answer any of them?

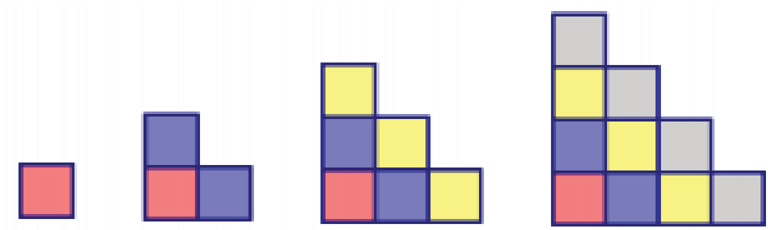
Here are some pictures that students drew to describe how the pattern was growing.



Ali’s picture



Michael’s picture



Kelli’s picture

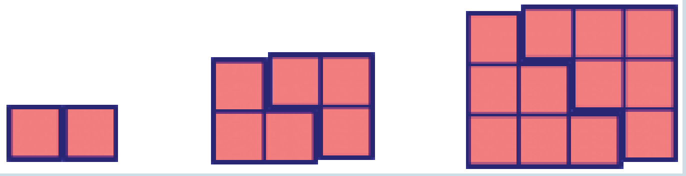
Think / Pair / Share

Describe in words how each student saw the pattern growing. Use the students’ pictures above (or your own method of seeing the growing pattern) to answer the following questions:

- How many tiles would you need to build the 5th figure in the pattern?
- How many tiles would you need to build the 10th figure in the pattern?
- How can you compute the number of tiles in any figure in the pattern?

Problem 15

Hy saw the pattern in a different way from everyone else in class. Here's what he drew:



Hy's picture.

1. Describe in words how Hy saw the pattern grow.
2. How would Hy calculate the number of tiles needed to build the 10th figure in the pattern?
3. How would Hy calculate the number of tiles needed to

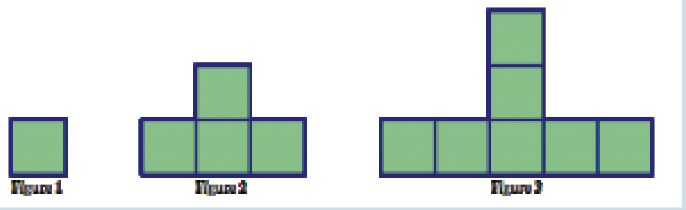
build the 100th figure in the pattern?

4. How would Hy calculate the number of tiles needed to build any figure in the pattern?

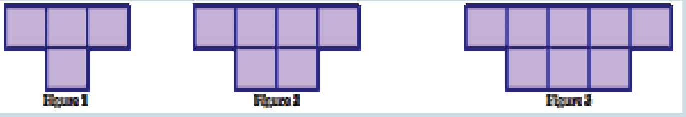
The next few problems present several growing patterns made with tiles. For each problem you work on, do the following:

1. Describe in words and pictures how you see the pattern growing.
2. Calculate the number of tiles you would need to build the 10th figure in the pattern. Justify your answer based on how the pattern grows.
3. Calculate the number of tiles you would need to build the 100th figure in the pattern.
4. Describe how you can figure out the number of tiles in any figure in the pattern. Be sure to justify your answer based on how the pattern grows.
5. Could you make one of the figures in the pattern using exactly 25 tiles? If yes, which figure? If no, why not? Justify your answer.
6. Could you make one of the figures in the pattern using exactly 100 tiles? If yes, which figure? If no, why not? Justify your answer.

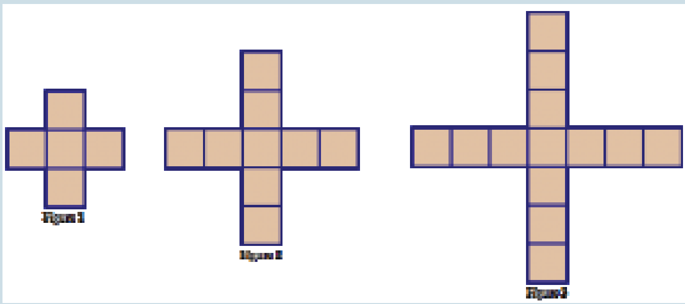
Problem 16



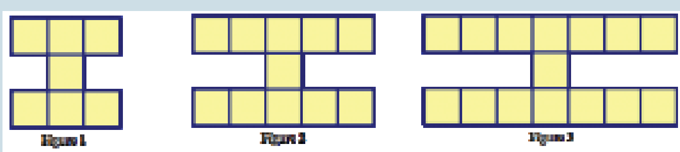
Problem 17



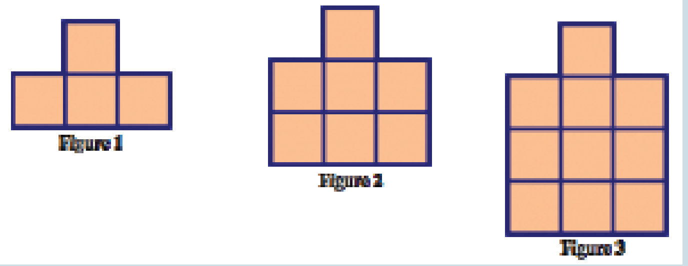
Problem 18



Problem 19



Problem 20



45. Matching Game

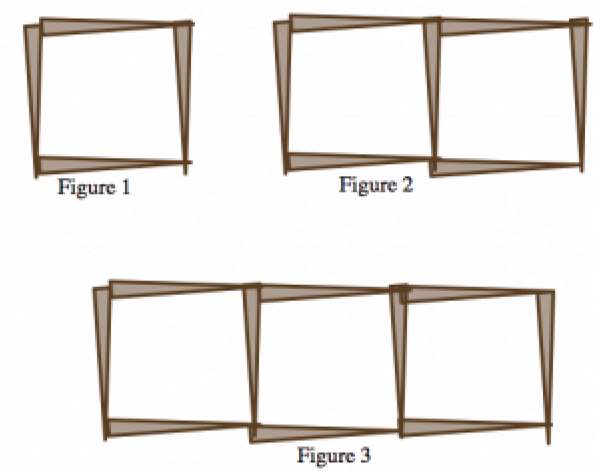
Below, you'll find patterns described in various ways: through visual representations, algebraic expressions, in tables of numbers, and in words. Your job is to match these up in a way that makes sense.

Note: there may be more than one algebraic expression to match a given pattern, or more than one pattern to match a given description. So be ready to justify your answers.

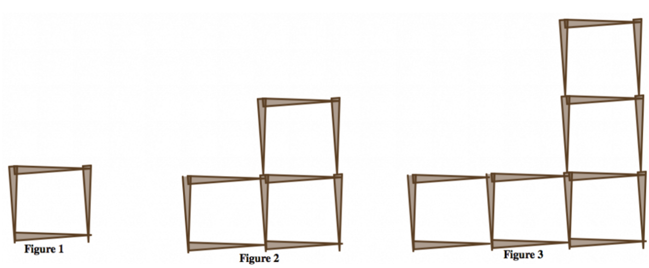
Algebraic Expressions

(a) t^2	(b) $2s+1$	(c) $2k+(k-1)+2k+(k-1)$
(d) $5n+5$	(e) $a+a$	(f) $3(\ell-1)+3(\ell-1)+4$
(g) $3b+1$	(h) $z+z+1$	(i) $m^2-(m-1)^2$
(j) $y \cdot y$	(k) $2x-1$	(l) $4e-(e-1)$
(m) $6f-2$	(n) $2c$	(o) $5(s+1)$

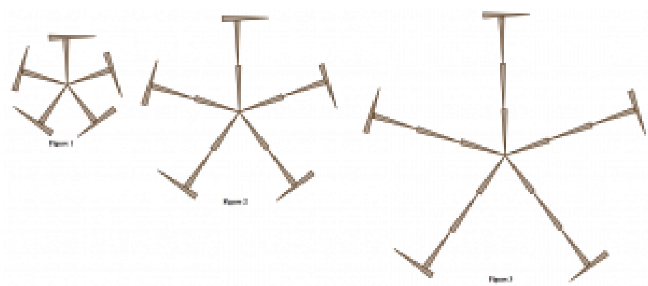
Visual Patterns



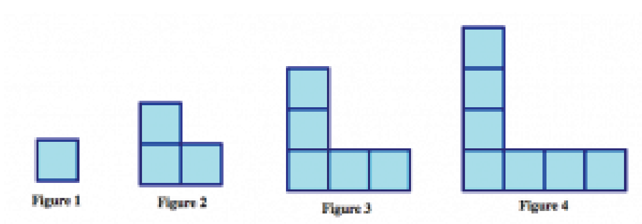
Pattern 1



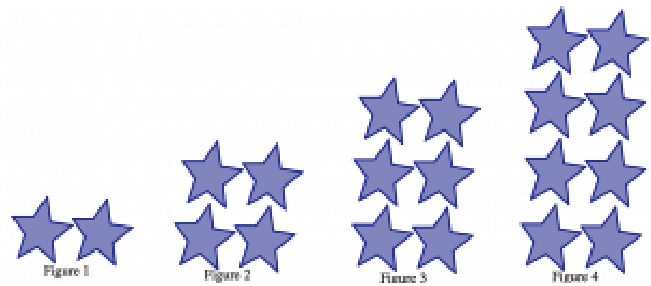
Pattern 2



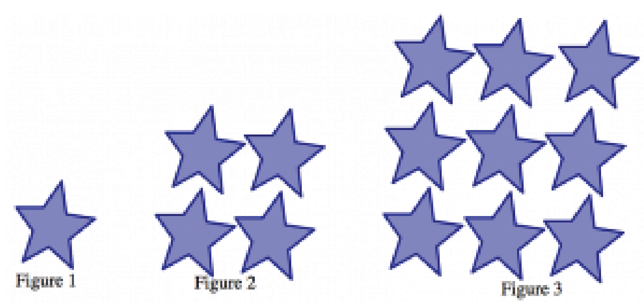
Pattern 3



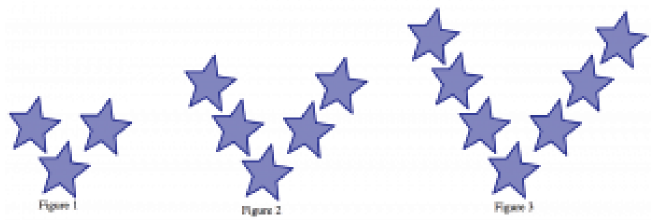
Pattern 4



Pattern 5



Pattern 6



Pattern 7

Tables of Numbers

Table A

Input	1	2	3	4
Output	1	4	9	16

Table B

Input	1	2	3	4
Output	10	15	20	25

Table C

Input	1	2	3	4
Output	1	3	5	7

Table D

Input	1	2	3	4
Output	3	5	7	9

Table E				
Input	1	2	3	4
Output	4	7	10	13

Table F				
Input	1	2	3	4
Output	4	10	16	22

Table G				
Input	1	2	3	4
Output	2	4	6	8

Descriptions in Words

1. Count horizontal and vertical toothpicks separately. Horizontal: there are two rows of n toothpicks where n is the figure number. There are n-1 more of them on the vertical arm. The vertical toothpicks are just the same. There are two columns of n along the vertical arm, and then n-1 more of them on the horizontal arm.
2. To get a figure from the previous one, you add three toothpicks in a "C"" shape on the left side of the figure. The total number of toothpicks is three times the figure number, plus one extra to close off the square on the far right.
3. There are five spikes radiating out from the center. Each spike has the same number of toothpicks as the figure number. Each spike is capped off by one additional toothpick.
4. Each arm of the "L" shape has the same number of tiles as the figure number. But then we've counted the corner of the "L" twice, so we have to subtract one to get the total number of tiles needed.
5. The stars are in two equal rows, and each row has the same

number of stars as the figure number.

6. To make the next figure, you always add five more toothpicks. Each arm has one more than the figure number of toothpicks, and there are five arms.
7. The stars are in a square, and the sides of the square have the same number of stars as the figure number.
8. Each arm of the "V" shape has the same number of stars as the figure number. Then we need to add one more star for the corner.
9. There are the same number of squares as the figure number, and each square uses four toothpicks. But then I've double-counted the toothpicks where the squares touch, so we have to subtract those out. There are one less of those than the figure number.
10. I can picture a square of tiles filled in. The side length of that square is the same as the figure number, so that's x^2. But then the square isn't really filled in. It's like I took away a square one size smaller from the top right, leaving just the border. What I took away was a square one size smaller, $(x-1)^2$.
11. Each time I go from one shape to the next, I add six new toothpicks. Three are added to the left in a "C" shape and three are added to the top in a rotated "C"" shape. So the total number will be six times the figure number plus or minus something. I can check to see that the right correction is to subtract 2.

46. Structural and Procedural Algebra

When most people think about algebra from school, they think about "solving for x." They imagine lots of equations with varying levels of complexity, but the goal is always the same: find the unknown quantity. This is a *procedural* view of algebra.

Even elementary students can be exposed to ideas in procedural algebra. This happens any time they think about unknown quantities and try to solve for them. For example, when first grade students learn to add and subtract numbers "within 10,'" they should frequently tackle problems like these:

- $3 + \_\_\_\_ = 7.$
- Find several pairs of numbers that add up to 10.

Although procedural algebra is important, it's not the most important skill, and it's certainly not the whole story.

You also need to foster thinking about *structural algebra* in your students: using symbols to express meaning in a situation. If there is an x on your page, you should be able to answer, "what does the x mean? What does it represent?"

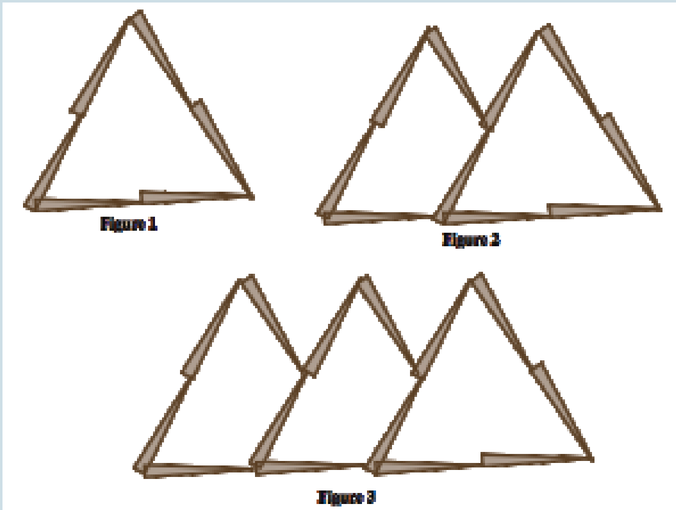
Most of what you've done so far in this chapter is *structural algebra*. You've used letters and symbols not to represent a single unknown quantity, but a *varying* quantity. For example, in Section 4 you used letters to represent the "figure number'" or "case number" in a growing pattern. The letters could take on different values, and the expressions gave you information: how many tiles or toothpicks or stars you needed to build that particular figure in that particular pattern.

Think / Pair / Share

- Consider the expression $a + 3$. Give a real world situation that could be represented by this expression. Share your answer with your partner. Together, can you come up with even more ideas?
- Suppose the expression $3c + 2$ represents the number of tiles used at any stage of a growing pattern.
  - Evaluate the expression at $c = 1, 2, \text{ and } 3$. What do the values tell you about the pattern?
  - Can you describe in words how the pattern is growing?
  - Can you design a pattern with tiles that grows according to this rule?
  - Where do you see the "3" in your pattern? Where do you see the "2""? Where do you see the "c"?

Problem 21

Krystal was looking at this pattern, which may be familiar to you from the Problem Bank:



She wrote down the equation

$$y = 4x + 2.$$

In Krystal's equation, what does x represent? What does y represent? How do you know?

Problem 22

Candice was thinking about this problem:

> Today is Jennifer's birthday, and she's twice as old as

her brother. When will she be twice as old as him again?

She wrote down the equation $2n = m$. In Candice's equation, what does n represent? What does m represent? How do you know?

Problem 23

Sarah and David collect old coins. Suppose the variable k stands for the number of coins Sarah has in her collection, and ℓ stands for the number of coins David has in his collection. What would each of these equations say about their coin collections?

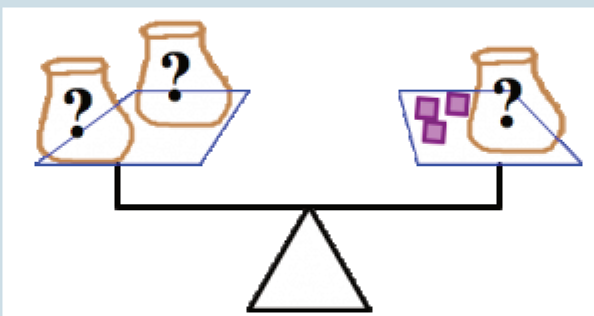
(a) $k = \ell + 1$ (b) $k = \ell$ (c) $3k = 2\ell$ (d) $k = \ell - 11$

Problem 24

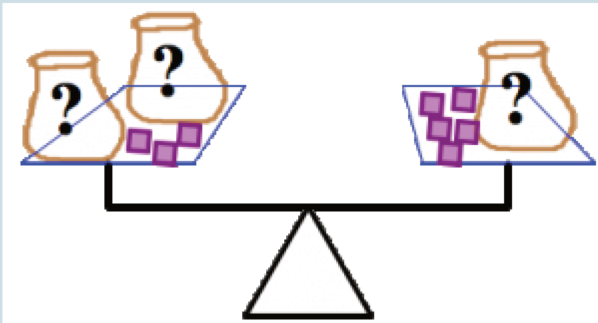
The pictures below show balance scales containing bags and blocks. The bags are marked with a "?"" because they contain some unknown number of blocks. In each picture:

- Each bag contains the same number of blocks.
- The scale is balanced.

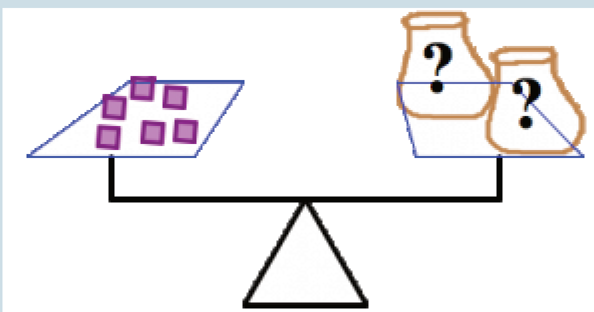
For each picture, determine how many blocks are in each bag. Justify your answers.



(a)



(b)



(c)

Problem 25

When he was working on Problem 24, Kyle wrote down these three equations.

(i) $2m = 6$.

(ii) $2x = x + 3$.

(iii) $z + 5 = 2z + 3$.

Match each equation to a picture, and justify your choices. Then solve the equations, and say (in a sentence) what the solution represents.

Problem 26

Draw a balance puzzle that represents the equation

$$2h + 3 = h + 8.$$

Now solve the balance puzzle. Where is the "h" in your puzzle? What does it represent?

Problem 27

Draw a balance puzzle that represents the equation

$$3b + 7 = 3b + 2.$$

Now solve the equation. Explain what happens.

Problem 28

Which equation below is most like the one in Problem 27 above? Justify your choice.

$(a)\ 5+3 = 8,$ $(b)\ \frac{2}{3}+\frac{1}{2} = \frac{3}{5},$ $(c)\ 5+3 = y,$ $(d)\ \frac{a}{5} = \frac{5}{a},$

$(e)\ n + 3 = m,$ $(f)\ 3x = 2x + x,$ $(g)\ 5k = 5k + 1.$

Problem 29

Draw a balance puzzle that represents the equation

$$4\ell + 7 = 4\ell + 7.$$

Now solve the equation. Explain what happens.

Problem 30

Which equation below is most like the one in Problem 29 above? Justify your choice.

$(a)\ 5+3 = 8,$ $(b)\ \frac{2}{3}+\frac{1}{2} = \frac{3}{5},$ $(c)\ 5+3 = y,$ $(d)\ \frac{a}{5} = \frac{5}{a},$

$(e)\ n + 3 = m,$ $(f)\ 3x = 2x + x,$ $(g)\ 5k = 5k + 1.$

Problem 31

Create a balance puzzle where the solution is not a whole number of blocks. Can you solve it? Explain your answer.

Problem 32

There are three piles of rocks: pile A, pile B, and pile C. Pile B has two more rocks than pile A. Pile C has four times as many rocks as pile A. The total number of rocks in all three piles is 14.

1. Use x to represent the number of rocks in pile A, and write equations that describe the rules above. Then find the number of rocks in each pile.
2. Use x to represent the number of rocks in pile B, and write equations that describe the rules above. Then find the number of rocks in each pile.
3. Use x to represent the number of rocks in pile C, and write equations that describe the rules above. Then find the number of rocks in each pile.

Think / Pair / Share

Look back at Problems 21–32. Which of them felt like *structural algebraic thinking*? Which felt like *procedural algebraic thinking*? Did any of the problems feel like they involved both kinds of thinking?

Variables and Equations

You have seen that in algebra, letters and symbols can have different meanings depending on the context.

- A symbol could stand for some *unknown quantity*.
- A symbol could stand for some quantity that *varies*. (Hence the term "variable" to describe these symbols.)

In much the same way, *equations* can represent different things.

- They can represent a problem to be solved. This is the traditional procedural algebra type of question.
- They can represent a relationship between two or more quantities. For example, $A = s^2$ represents the relationship between the area of a square and its side length.
- They can represent *identities*: mathematical truths. For example,

  $$x^2 - 1 = (x+1)(x-1)$$

  is always true, for every value of x. There is nothing to solve for, and no relationship between varying quantities. (If you do try to “solve for x,” you will get the equation $0 = 0$, much like you saw in Problem 29. Not very satisfying!)

Think / Pair / Share

Give an example of each type of equation. Be sure to say what the symbols in the equations represent.

Problem 33

Answer the following questions about the equation

$$x^2 - 1 = (x+1)(x-1).$$

1. Evaluate both sides of the above equation for $x = 1, 2, 3, 4, \text{ and } 5$. What happens?
2. Use the *distributive property* of multiplication over addition to expand the right side of the equation and simplify it.
3. Use the equation to compute 99^2 quickly, without using a calculator. Explain how you did it.

47. Problem Bank

Problems 34-36 ask you to solve problems about a strange veterinarian who created three mystifying machines.

Cat Machine: Place a cat in the input bin of this machine, press the button, and out jump two dogs and a mouse.

Dog Machine: This machine converts a dog into a cat and a mouse.

Mouse Machine: This machine can convert a mouse into a cat and three dogs.

Each machine can also operate in reverse. For example, if you have two dogs and a mouse, you can use the first machine to convert them into a cat.

Problem 34

The veterinarian hands you two cats, and asks you to convert them into exactly three dogs (no extra dogs and no other animals). Can you do it? If yes, say what process you would use. If no, say why not.

Problem 35

The veterinarian hands you one dog. He says he only wants cats, but he doesn't care how many. Can you help him? How?

Problem 36

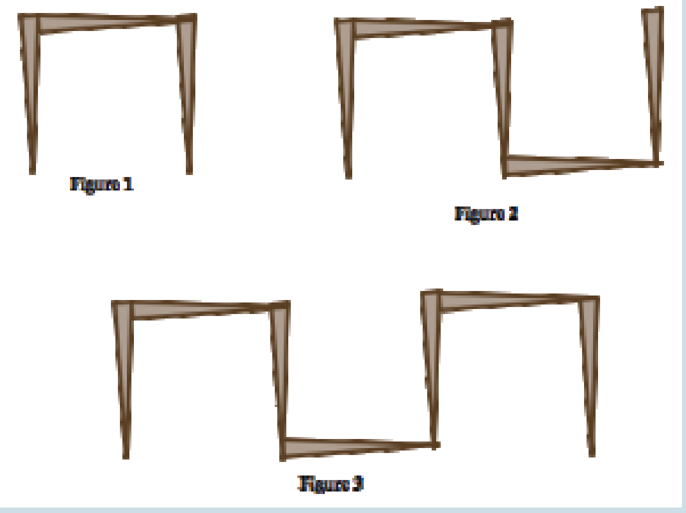
The veterinarian hands you one cat. He says he only wants dogs, but he doesn't care how many. Can you help him? How?

Problems 37-40 present several growing patterns made with toothpicks. For each problem you work on, do the following:

1. Describe in words and pictures how you see the pattern growing.
2. Calculate the number of toothpicks you would need to build the 10th figure in the pattern. Justify your answer based on how the pattern grows.
3. Calculate the number of toothpicks you would need to build the 100th figure in the pattern.
4. Describe how you can figure out the number of toothpicks in any figure in the pattern. Be sure to justify your answer based on how the pattern grows.

5. Could you make one of the figures in the pattern using exactly 25 toothpicks? If yes, which figure? If no, why not? Justify your answer.
6. Could you make one of the figures in the pattern using exactly 100 toothpicks? If yes, which figure? If no, why not? Justify your answer.

Problem 37



Problem 38

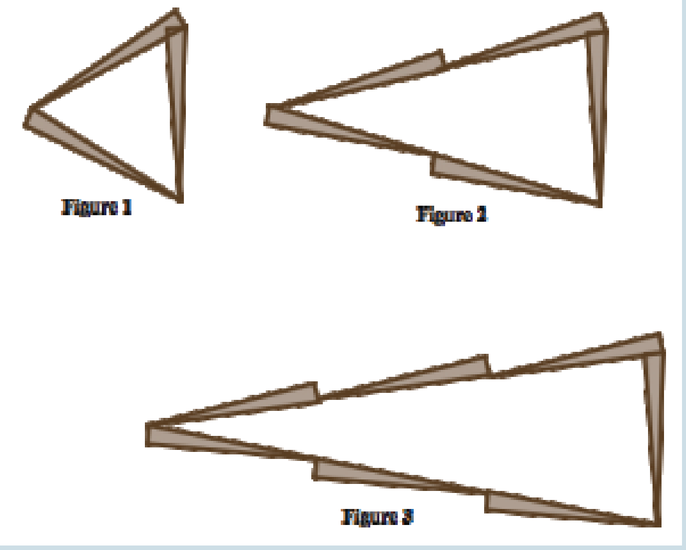
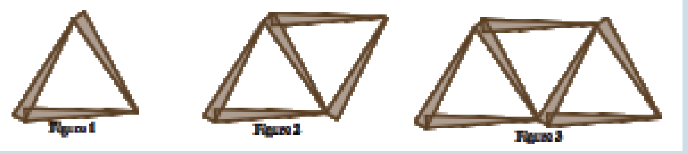
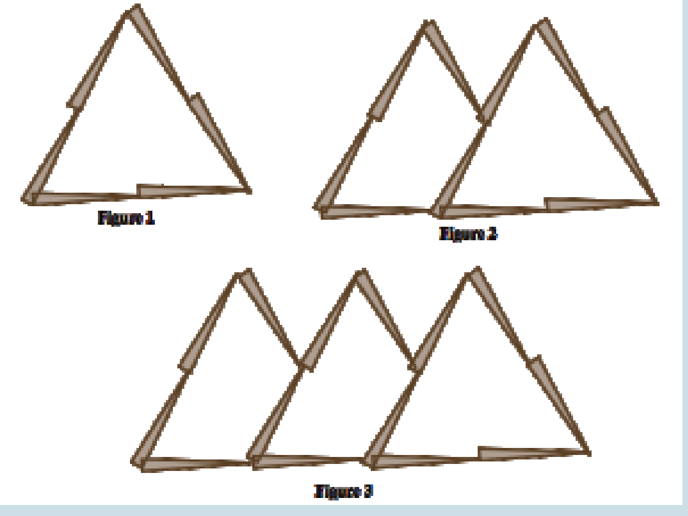


Figure 1 Figure 2 Figure 3

Problem 39



Problem 40



In a *mobile*, the arms must be perfectly balanced for it to hang properly. The artist Alexander Calder was famous for his artistic mobiles.You can view some of his amazing work here. Click "Explore Works."

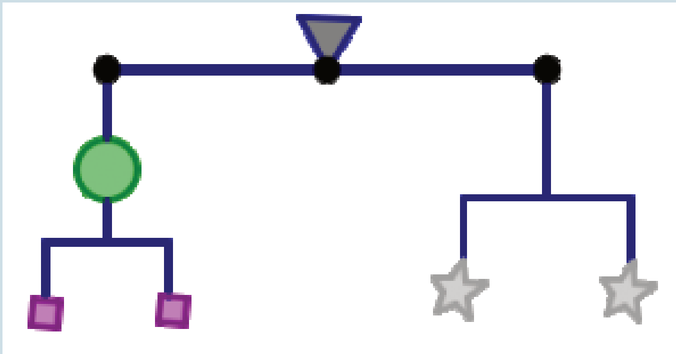
Problems 41-42 present you with mobile puzzles. In these puzzles:

- Objects that are the same shape have the same weight. (So all circles weigh the same, all squares weigh the same, etc.)
- Assume the strings and rods that hold the objects together don't factor into the total weight.
- Each arm of the mobile must have exactly the same weight.

Problem 41

In this puzzle:

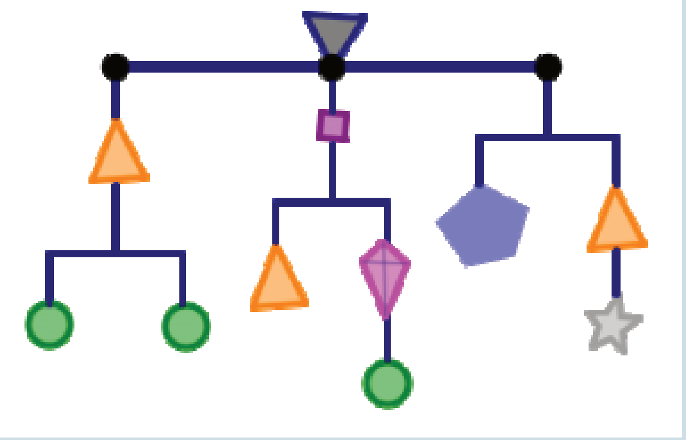
- The total weight is 36 grams.
- All shapes weigh less than 10 grams.
- All of the weights are whole numbers.
- One circle weighs more than one square.



Find the weight of each piece. Is there more than one answer? How do you know you are right?

Problem 42

In this puzzle, the total weight is 54 grams.



Find the weight of each piece. Is there more than one answer? How do you know you are right?

PART VI

PLACE VALUE AND DECIMALS

I always say when you see that old black-and-white footage of the rocket on the launch pad and it falls over and explodes, that's because people had slide rules. Not having the decimal point is a real drawback. You want the decimal point, take it from me.

-Bill Nye

The “Dots and Boxes” approach to place value used in this part (and throughout this book) and the “pies per child” approach to fractions comes from James Tanton, and are used with his permission. See his development of these and other ideas at http://gdaymath.com/.

48. Review of Dots & Boxes Model

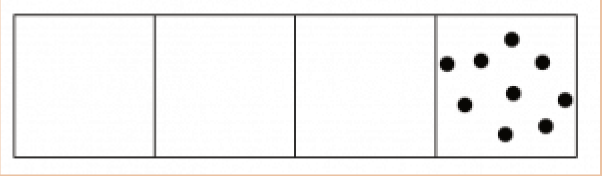
Let's start with a quick review of place value, different bases, and our "Dots & Boxes" model for thinking about these ideas.

The 1←2 Rule

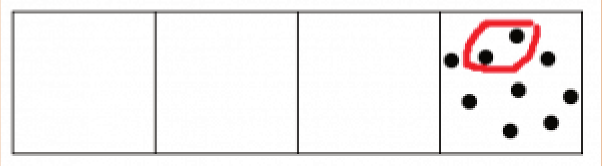
Whenever there are two dots in single box, they "explode," disappear, and become one dot in the box to the left.

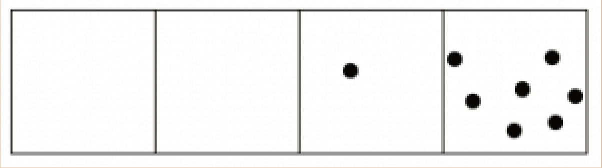
Example: Nine dots 1←2 in the system

We start by placing nine dots in the rightmost box.

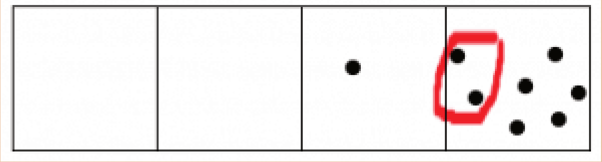


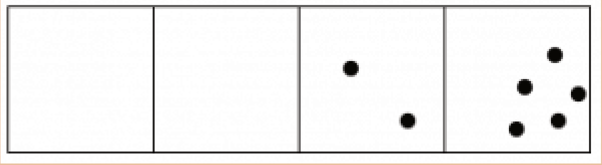
Two dots in that box explode and become one dot in the box to the left.



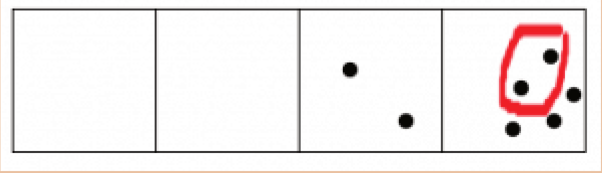


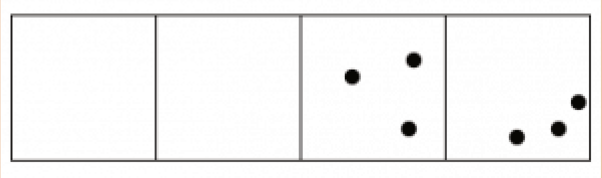
Once again, two dots in that box explode and become one dot in the box to the left.



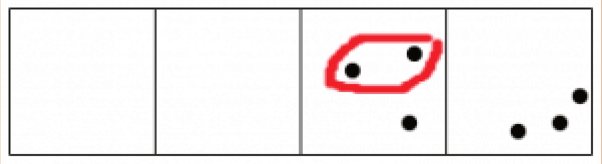


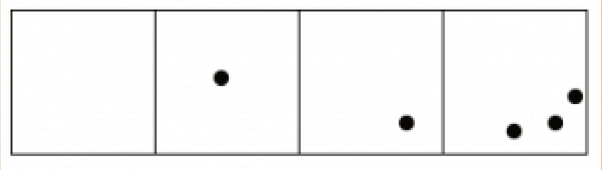
We do it again!



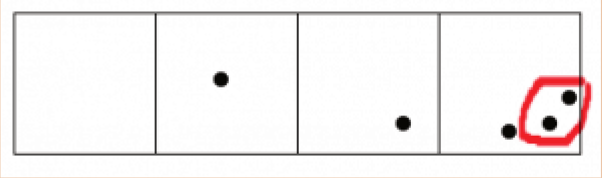


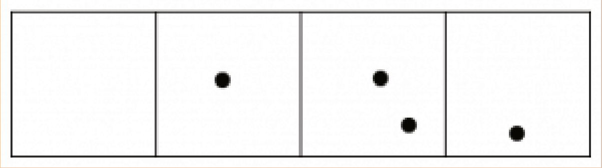
Hey, now we have more than two dots in the second box, so those can explode and move!



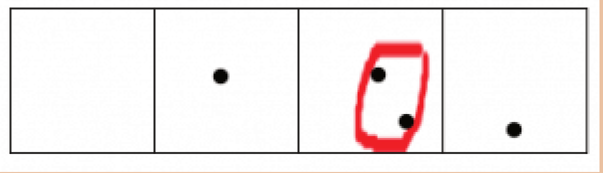


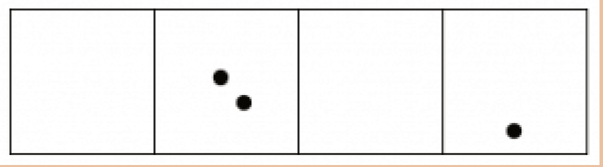
And the rightmost box still has more than two dots.

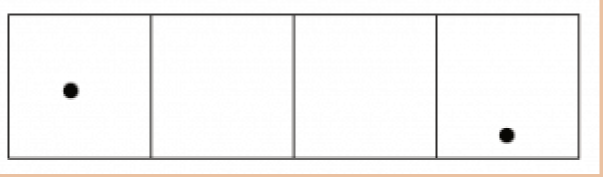




Keep going, until no box has two dots.







After all this, reading from left to right we are left with one dot, followed by zero dots, zero dots, and one final dot.

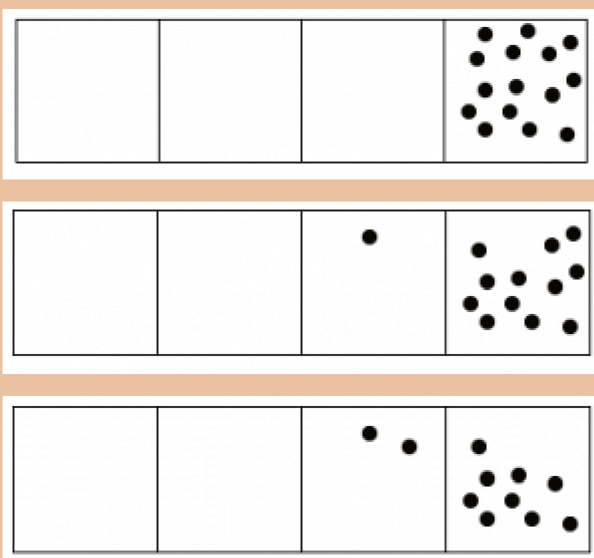
Solution: The 2←1 code for nine dots is: 1001.

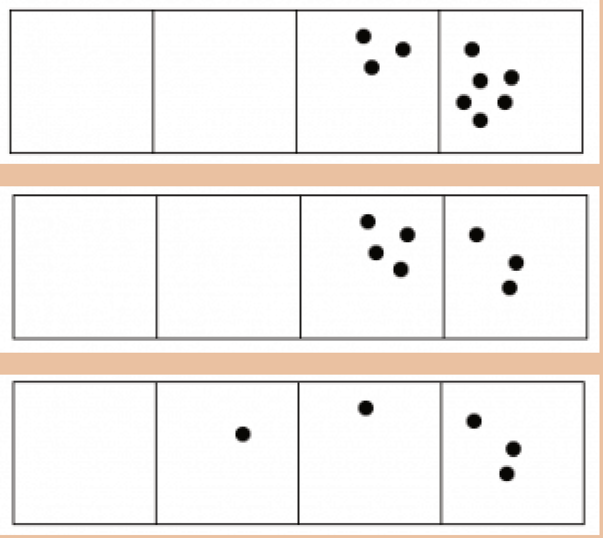
The 1←3 Rule

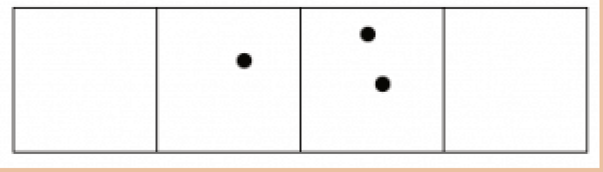
Whenever there are three dots in single box, they “explode,” disappear, and become one dot in the box to the left.

Example: Fifteen dots in the 1←3 system

Here’s what happens with fifteen dots:







Solution: The 1←3 code for fifteen dots is: 120.

Definition

Recall that numbers written in the 1←2 system are called **binary** or **base two** numbers.

Numbers written in the 1←3 system are called **base three** numbers.

Numbers written in the 1←4 system are called **base four** numbers.

Numbers written in the 1←10 system are called **base ten** numbers.

In general, numbers written in the 1←b system are called **base *b*** numbers.

In a base b number system, each place represents a power of b, which means b^n for some whole number n. Remember this means b multiplied by itself n times:

- The right-most place is the units or ones place. (Why is this a power of b?)
- The second spot is the "b" place. (In base ten, it's the tens place.)
- The third spot is the "b^2" place. (In base ten, that's the hundreds place. Note that $10^2 = 100$.)
- The fourth spot is the "b^3" place. (In base ten, that's the thousands place, since $10^3 = 1000$.)
- And so on.

Notation

Whenever we're dealing with numbers written in different bases, we use a subscript to indicate the base so that there can be no confusion. So:

- 102_{three} is a base three number (read it as "one-zero-two base three"). This is the base three code for the number eleven.
- 222_{four} is a base four number (read it as "two-two-two base four"). This is the base four code for the number forty-two.
- 54321_{ten} is a base ten number. (It's ok to say "fifty-four thousand three hundred and twenty-one." Why?)

If the base is not written, we assume it's base ten.

Remember: when you see the subscript, you are seeing the **code** for some number of dots.

Think / Pair / Share

Work through the two examples above carefully to be sure you remember and understand how the "Dots & Boxes" model works. Then answer these questions:

- When we write 9 in base 2, why do we write 1001_{two} instead of just 11_{two}?
- When we write 15 in base 3, why do we write 120_{three} instead of just 12_{three}?
- How many different digits do you need in a base 7 system? In a base 12 system? In a base b system? How do you know?

On Your Own

Work on the following exercises on your own or with a partner.

1. In base 4, four dots in one box are worth one dot in the box one place to the left.
    1. What is the value of each box?
    2. How do you write 29_{ten} in base 4?
    3. How do you write 132_{four} in base 10?
2. In our familiar base ten system, ten dots in one box are worth one dot in the box one place to the left.
    1. What is the value of each box?
    2. When we write the base ten number 7842:
        - What quantity does the "7" represent?

- The "4" is four groups of what value?
- The "8" is eight groups of what value?
- The "2" is two groups of what value?

3. Write the following numbers of dots in base two, base three, base five, and base eight. Draw the "Dots & Boxes" model if it helps you remember how to do this! (Note: these numbers are all written in base ten. When we don't say otherwise, you should assume base ten.)

$$(a)\ 2 \qquad (b)\ 17 \qquad (c)\ 27 \qquad (d)\ 63.$$

4. Convert these numbers to our more familiar base ten system. Draw out dots and boxes and "unexplode" the dots if it helps you remember.

$$(a)\ 1101_{\text{two}} \qquad (b)\ 102_{\text{three}} \qquad (c)\ 24_{\text{five}} \qquad (d)\ 24_{\text{nine}}.$$

Think / Pair / Share

Quickly compute each of the following. Write your answer in the same base as the problem.

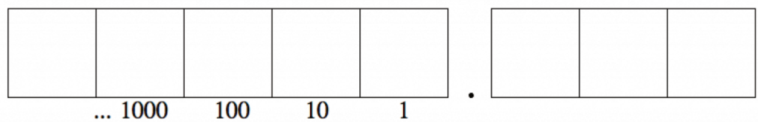
- 131_{ten} times ten.
- 263207_{eight} times eight.
- 563872_{nine} times nine.

- Use the 1←10 system to explain why multiplying a whole number in base ten by ten results in simply appending a zero to the right end of the number.
- Suppose you have a whole number written in base b. What is the effect of multiplying that number by b? Justify what you say.

49. Decimals

Up to now our "Dots & Boxes" model has consisted of a row of boxes extending infinitely far to the left. Why not have boxes extending to the right as well?

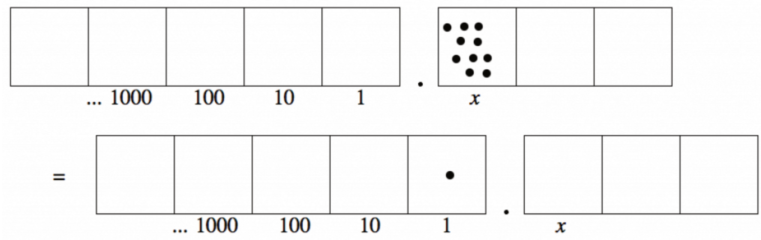
Let's work specifically with a 1←10 rule and see what boxes to the right could mean.



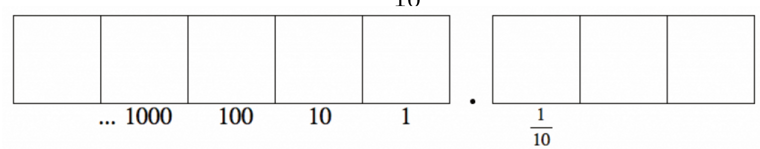
Notation

It has become convention to separate boxes to the right of the ones place with a decimal point. (At least, this is what the point is called in the base ten world… "dec" means "ten" after all!)

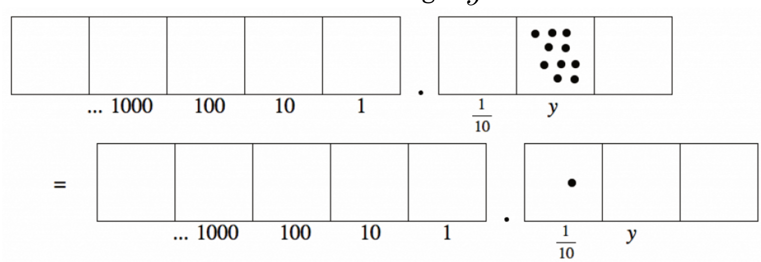
What is the value of the first box to the right of the decimal point? If we denote its value as x, we have that ten x's is equivalent to 1. (Remember, we are using a $1 \leftarrow 10$ rule.)



From $10x = 1$ we get that $x = \frac{1}{10}$.

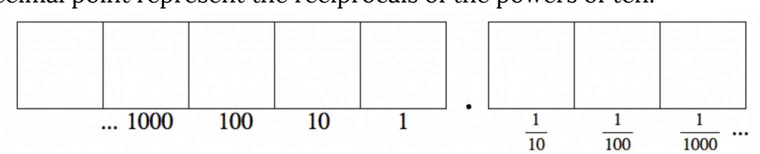


Call the value of the next box to the right y.



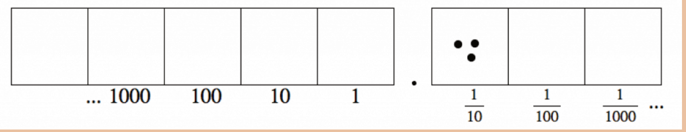
From $10y = \frac{1}{10}$ we get $y = \frac{1}{100}$.

If we keep doing this, we see that the boxes to the right of the decimal point represent the reciprocals of the powers of ten.



Example: 0.3

The decimal 0.3 is represented by the picture:

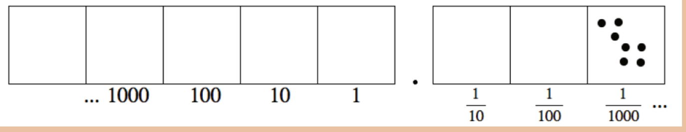


It represents three groups of $\frac{1}{10}$, that is:

$$0.3 = \frac{3}{10}.$$

Example: 0.007

The decimal 0.007 is represented by the picture:



It represents seven groups of $\frac{1}{1000}$.

Of course, some decimals represent fractions that can simplify further. For example:

$$0.5 = \frac{5}{10} = \frac{1}{2}.$$

Similarly, if a fraction can be rewritten to have a denominator that is a power of ten, then it is easy to convert it to a decimal. For example, $\frac{3}{5}$ is equivalent to $\frac{6}{10}$, and so we have:

$$\frac{3}{5} = \frac{6}{10} = 0.6.$$

Example: 12 3/4

Can you write $12\frac{3}{4}$ as a decimal? Well,

$$12\tfrac{3}{4} = 12 + \tfrac{3}{4}.$$

We can write the denominator as a power of ten using the key fraction rule:

$$\frac{3}{4} \cdot \frac{25}{25} = \frac{75}{100}.$$

So we see that:

$$12 + \tfrac{3}{4} = 12 + \tfrac{75}{100} = 12.75.$$

Think / Pair / Share

- Draw a "Dots & Boxes" picture for each of the following decimals. Then say what fraction each decimal represents:

  $0.09, \qquad 0.003, \qquad 0.7, \qquad 0.0000003.$

- Draw a "Dots & Boxes" picture for each of the following fractions. Then write the fraction as a decimal:

  $$\frac{1}{1000}, \qquad \frac{7}{100}, \qquad \frac{9}{10}.$$

- What fractions (in simplest terms) do the following decimals represent?

  $0.05, \qquad 0.2, \qquad 0.8, \qquad 0.004.$

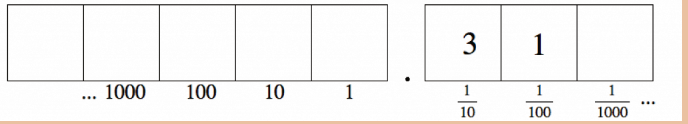
- Use the key fraction rule to write the following fractions as decimals. *Do not use a calculator!*

  $\frac{2}{5}, \qquad \frac{1}{25}, \qquad \frac{1}{20}, \qquad \frac{1}{200}, \qquad \frac{1}{1250}.$

- Some people read 0.6 out loud as "point six." Others read it out loud as "six tenths." Which is more helpful for understanding what the number really is? Why do you think so?

Example: 0.31

Here is a more interesting question: What fraction is represented by the decimal 0.31?



There are two ways to think about this.

Approach 1:

From the picture of the "Dots & Boxes" model we see:

$$0.31 = \frac{3}{10} + \frac{1}{100}.$$

We can add these fractions by finding a common denominator:

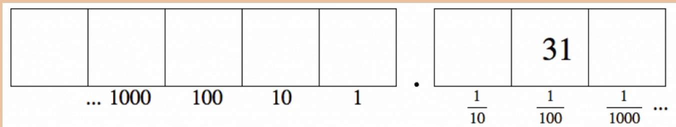
$$\frac{3}{10} + \frac{1}{100} = \frac{30}{100} + \frac{1}{100} = \frac{31}{100}.$$

So

$$0.31 = \frac{31}{100}.$$

Approach 2:

Let's unexplode the three dots in the $\frac{1}{10}$ position to produce an additional 30 dots in the $\frac{1}{100}$ position.



So we can see right away that

$$0.31 = \frac{31}{100}.$$

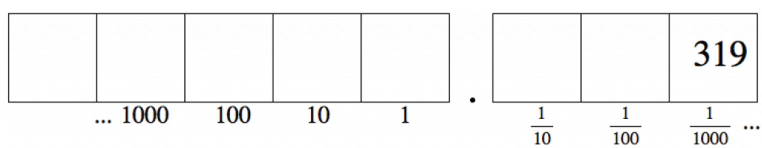
On Your Own

Work on the following exercises on your own or with a partner.

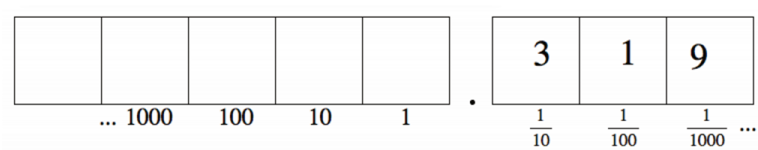
1. Brian is having difficulty seeing that 0.47 represents the fraction $\frac{47}{100}$. Describe the two approaches you could use to explain this to him.

2. A teacher asked his students to each draw a "Dots & Boxes" picture of the fraction $\frac{319}{1000}$.

Jin drew this:



Sonia drew this:



The teacher marked both students as correct.

- Are each of these solutions correct? Explain your thinking.

- Jin said he could get Sonia's solution by performing some explosions. What did he mean by this? Is he right?

3. Choose the best answer and justify your choice. The decimal 0.23 equals:

$(a)\ \frac{23}{10}$ $(b)\ \frac{23}{100}$

$(c)\ \frac{23}{1,000}$ $(d)\ \frac{23}{10,000}$.

4. Choose the best answer and justify your choice. The decimal 0.0409 equals:

$(a)\ \frac{409}{100}$ $(b)\ \frac{409}{1,000}$

$(c)\ \frac{409}{10,000}$ $(d)\ \frac{409}{100,000}$.

5. Choose the best answer and justify your choice. The decimal 0.050 equals:

$(a)\ \frac{50}{100}$ $(b)\ \frac{1}{20}$

$(c)\ \frac{1}{200}$ (d) None of these.

6. Choose the best answer and justify your choice. The decimal 0.000204 equals:

$(a)\ \frac{51}{250}$ $(b)\ \frac{51}{2500}$

$(c)\ \frac{51}{25000}$ $(d)\ \frac{51}{250000}$.

7. What fraction is represented by each of the following decimals?

(a) 0.567 (b) 0.031

(c) 0.4077 (d) $0.101.$

8. Write each of the following fractions as decimals. Don't use a calculator!

(a) $\dfrac{73}{100}$ (b) $\dfrac{519}{1000}$

(c) $\dfrac{71}{1000}$ (d) $\dfrac{7001}{10000}.$

9. Write each of the following fractions as decimals. Don't use a calculator!

(a) $\dfrac{7}{20}$ (b) $\dfrac{16}{25}$

(c) $\dfrac{301}{500}$ (d) $\dfrac{17}{50}$ (e) $\dfrac{3}{4}.$

10. Write each of the following as a fraction (or mixed number).

(a) 2.3 (b) 17.04 (c) 1003.1003

11. Write each of the following numbers in decimal notation.

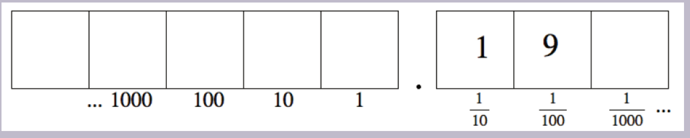
(a) $5\frac{3}{10}$ (b) $7\frac{1}{5}$ (c) $13\frac{1}{2}$ (d) $106\frac{3}{20}$

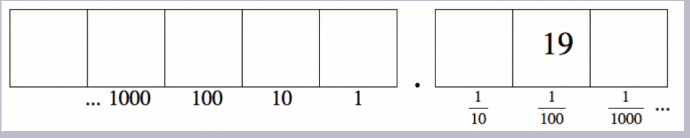
(e) $\dfrac{78}{25}$ (f) $\dfrac{9}{4}$ (g) $\dfrac{131}{40}$

Think Pair Share

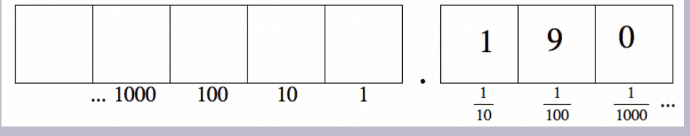
Do 0.19 and 0.190 represent the same number or different numbers?

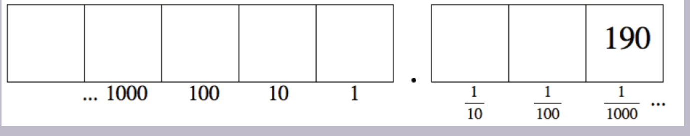
Here are two dots and boxes pictures for the decimal 0.19.





And here are two dots and boxes picture for the decimal 0.190.



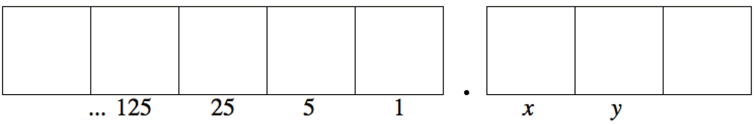


- Explain how one “unexplosion” establishes that the first picture of 0.19 is equivalent to the second picture of 0.19.

- Explain how several unexplosions establishes that the first picture of 0.190 is equivalent to the second picture of 0.190.
- Use explosions and unexplosions to show that all four pictures are equivalent to each other.
- So … does 0.19 represent the same number as 0.190?

50. x-mals

Just like in base 10, we can add boxes to the right of the decimal point in other bases, like base 5.



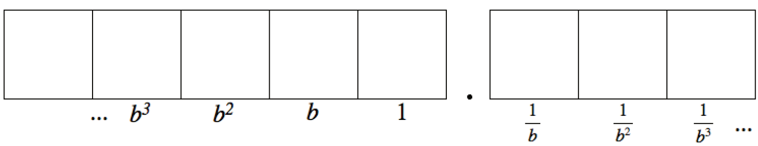
However, the prefix "dec" in "decimal point" means ten. So we really shouldn't call it a decimal point anymore. Maybe a "pentimal point"? (In fact, the general term is **radix point**.)

Think / Pair / Share

- Use reasoning like you saw on page 6 for the base ten system to think about other number systems:
- Figure out the values of x and y in the picture of the base-5 system above. Be sure you can explain your reasoning.
- Draw a base-4 "Dots & Boxes" model, including a radix point and some boxes to the right. Label at least three boxes to the left of the ones place and three boxes to the right of the ones place.
- Draw a base-6 "Dots & Boxes" model, including a radix point and some boxes to the right. Label at least three

boxes to the left of the ones place and three boxes to the right of the ones place

In general, in a base-b system, the boxes to the left of the ones place represent positive powers of the base b. Boxes to the right of the ones place represent reciprocals of those powers.



On Your Own

Work on the following exercises on your own or with a partner.

1. Draw a "Dots & Boxes" picture of each number.

$$(a)\ 0.03_{\text{five}} \qquad (b)\ 0.22_{\text{six}}$$
$$(c)\ 0.103_{\text{four}} \qquad (d)\ 0.002_{\text{three}}.$$

2. Find a familiar (base-10) fraction value for each number.

$$(a)\ 0.04_{\text{five}} \qquad (b)\ 0.3_{\text{six}}$$
$$(c)\ 0.02_{\text{four}} \qquad (d)\ 0.03_{\text{nine}}.$$

3. Find a familiar (base-10) fraction value for each number. (You might want to re-read the example of 0.31 in base ten from the previous chapter.)

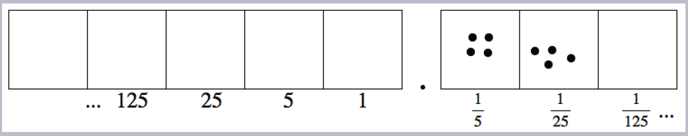
$(a)\ 0.13_{\text{five}}$ $(b)\ 0.25_{\text{six}}$ $(c)\ 0.101_{\text{two}}$

$(d)\ 0.24_{\text{seven}}$ $(e)\ 0.55_{\text{eight}}$

Think / Pair / Share

Tami and Courtney were working on converting 0.44_{five} to a familiar base-10 fraction. Courtney said this:

> The places in base five to the right of the point are like $\frac{1}{5}$ and then $\frac{1}{25}$. Since this has two places, the answer should be $\frac{44}{25}$.



Tami thought about what Courtney said and replied:

> I don't know what the right answer is, but I know that can't be right. The number 0.44_{five} is less than one, since there are no numbers in the ones place and no explosions that we can do. But the fraction $\frac{44}{25}$ is more than one. It's almost two. So they can't be the same number.

- Who makes the most sense, Courtney or Tami? Why do you think so?
- Find the right answer to the problem Courtney and Tami were working on.

Problem 1

Find the "decimal" representation of $\frac{1}{4}$ in each of the following bases. Be sure that you can justify your answer. (You might want to review the example of $12\frac{3}{4}$ in the previous chapter.)

base 2	base 4	base 6
base 8	base 10	base 12

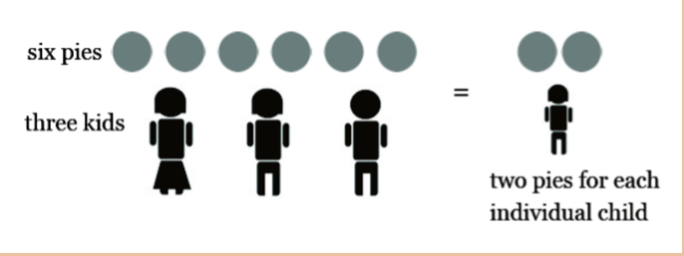
51. Division and Decimals

When you studied fractions, you had lots of different ways to think about them. But the first way, and the one we keep coming back to, is to think of a fraction as the answer to a division problem.

Example

Suppose 6 pies are to be shared equally among 3 children. This yields 2 pies per child. We write:

$$\frac{6}{3} = 2.$$



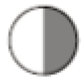
The fraction $\frac{6}{3}$ is equivalent to the answer to the division problem $6 \div 3 = 2$. It represents the number of pies one whole child receives.

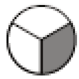
In the same way…

sharing 10 pies among 2 kids yields $\frac{10}{2} = 5$ pies per kid,
sharing 8 pies among 2 kids yields $\frac{8}{2} = 4$ pies per kid,

sharing 5 pies among 5 kids yields $\frac{5}{5} = 1$ pies per kid, and

the answer to sharing 1 pies among 2 children is $\frac{1}{2}$, which we call "one-half."

We associate the number "$\frac{1}{2}$" to the picture .

In the same way, the picture  represents "one third," that is, $\frac{1}{3}$.

(This is the amount of pie an individual child would receive if one pie is shared among three children.)

The picture  is called "one fifth" and is indeed $\frac{1}{5}$, the amount of pie an individual child receives when one pie is shared by five kids.

And the picture  is called "three fifths" to represent $\frac{3}{5}$, the amount of pie an individual receives if three pies are shared by five kids.

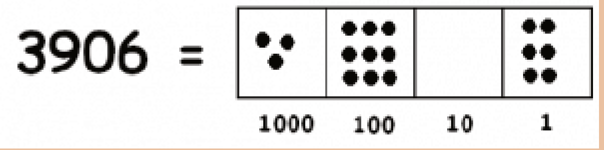
We know how to do division in our "Dots & Boxes" model.

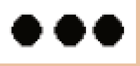
Example: 3906 ÷ 3

Suppose you are asked to compute $3906 \div 3$. One way to interpret this question (there are others) is:

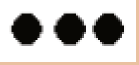
"How many groups of 3 fit into 3906?"

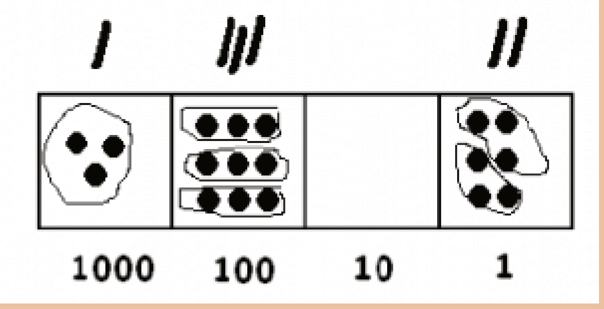
In our "Dots & Boxes" model, the dividend 3906 looks like this:



and three dots looks like this: 

So we are really asking:

"How many groups of  fit into the picture of 3906?"



Notice what we have in the picture:

- One group of 3 in the thousands box.
- Three groups of 3 in the hundreds box.
- Zero groups of 3 in the tens box.
- Two groups of 3 in the ones box.

This shows that 3 goes into 3906 one thousand, three hundreds and two ones times. That is,

$$3906 \div 3 = 1302.$$

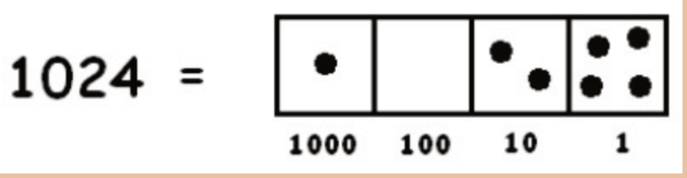
Of course, not every division problem works out evenly! Here's a different example.

Example: 1024 ÷ 3

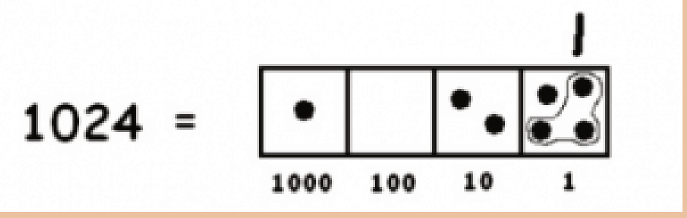
Suppose you are asked to compute $1024 \div 3$. One way to interpret this question is:

"How many groups of 3 fit into 1024?"

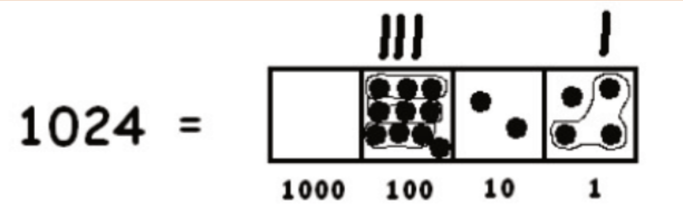
So we're looking for groups of three dots in this picture:



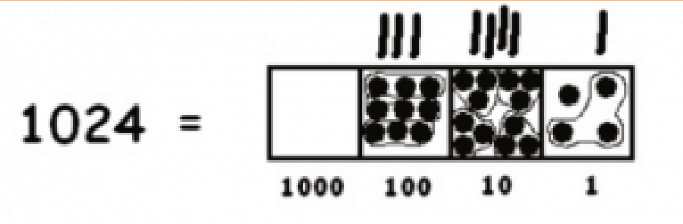
One group of three is easy to spot:



To find more groups of three dots, we must "unexplode" a dot:



We need to unexplode again:



This leaves one stubborn dot remaining in the ones box and no more group of three. So we conclude:

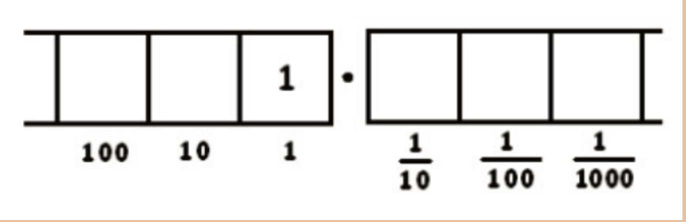
$$1024 \div 3 \;=\; 341 \;\; R1, \;\; \text{meaning } 1024 = 341 \cdot 3 + 1.$$

In words: 1024 gives 341 groups of 3, plus one extra dot.

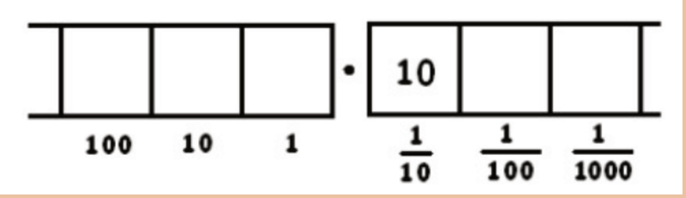
We can put these two ideas together – fractions as the answer to a division problem and what we know about division in the “Dots & Boxes” model – to help us think more about the connection between fractions and decimals.

Example: 1/8

The fraction $\frac{1}{8}$ is the result of dividing 1 by 8. Let's actually compute $1 \div 8$ in a "Dots & Boxes" model, making use of decimals. We want to find groups of eight in the following picture:

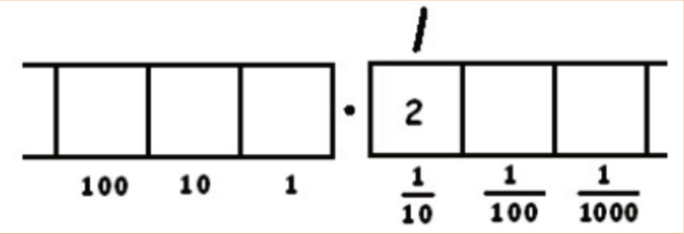


Clearly none are to be found, so let's unexplode:

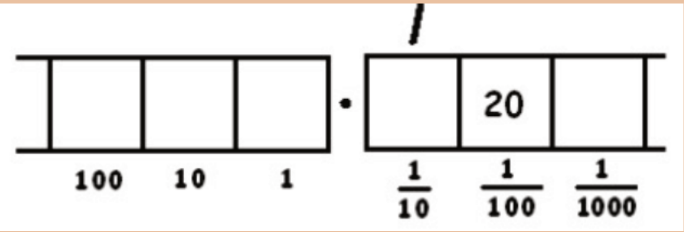


(We're being lazy and not drawing all the dots. As you follow along, you might want to draw the dots rather than the number of dots, if it helps you keep track.)

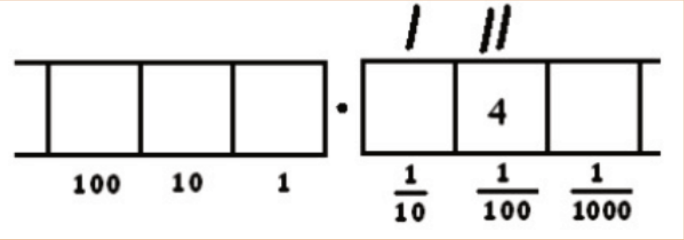
Now there is one group of 8, leaving two behind. We write a tick-mark on top, to keep track of the number of groups of 8, and leave two dots behind in the box.



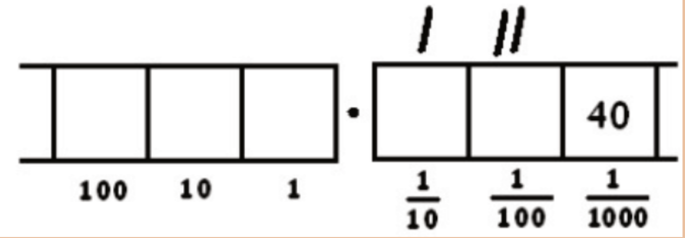
We can unexplode the two dots in the $\frac{1}{10}$ box:



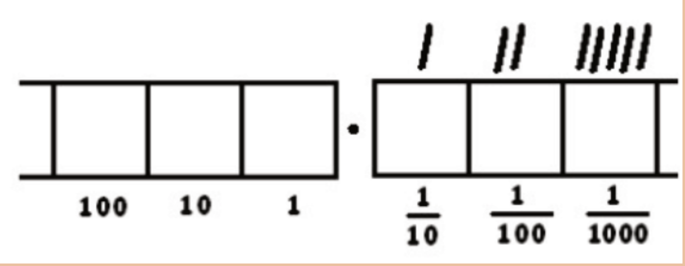
This gives two groups of 8 leaving four behind. Remember: the two tick marks represent two groups of 8. And there are four dots left in the $\frac{1}{100}$ box.



Unexploding those four remaining dots:



Now we have five groups of 8 and no remainder.



Remember: the tick marks kept track of how many groups of eight there were in each box. We have

- One group of 8 dots in the $\frac{1}{10}$ box
- Two groups of 8 dots in the $\frac{1}{100}$ box.
- Five groups of 8 dots in the $\frac{1}{1000}$ box.

So we conclude that:

$$\frac{1}{8} = 1 \div 8 = 0.125.$$

Of course, it's a good habit to check our answer:

$$0.125 = \frac{125}{1000} = \frac{5 \cdot 25}{5 \cdot 200} = \frac{5 \cdot 5}{5 \cdot 40} = \frac{5 \cdot 1}{5 \cdot 8} = \frac{1}{8}.$$

On Your Own

Work on the following exercises on your own or with a partner. Be sure to show your work.

1. Perform the division in a "Dots & Boxes" model to show that $\frac{1}{4}$, as a decimal, is 0.25.
2. Perform the division in a "Dots & Boxes" model to show that $\frac{1}{2}$, as a decimal, is 0.5.
3. Perform the division in a "Dots & Boxes" model to show that $\frac{3}{5}$, as a decimal, is 0.6.
4. Perform the division in a "Dots & Boxes" model to show that $\frac{3}{16}$, as a decimal, is 0.1875.
5. In simplest terms, what fraction is represented by each of these decimals?

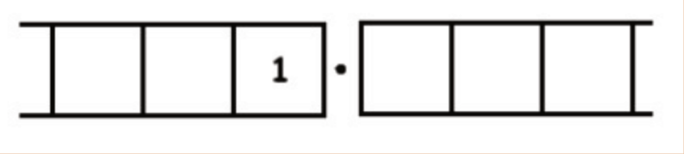
$$0.75, \qquad 0.625, \qquad 0.16, \qquad 0.85, \qquad 0.0625.$$

Repeating Decimals

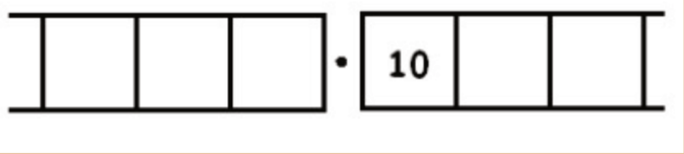
Not all fractions lead to simple decimal representations.

Example: 1/3

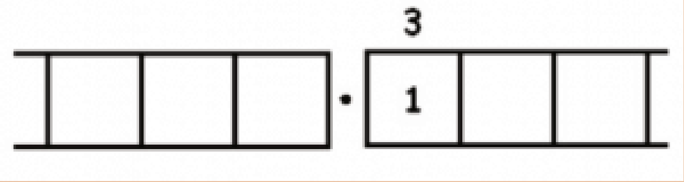
Consider the fraction $\frac{1}{3}$. We seek groups of three in the following picture:



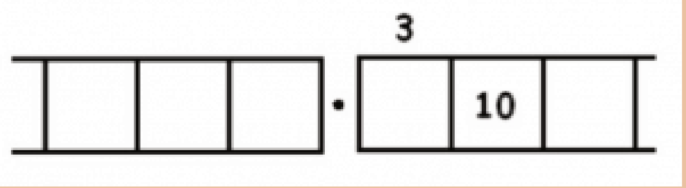
Unexploding requires us to look for groups of 3 in:



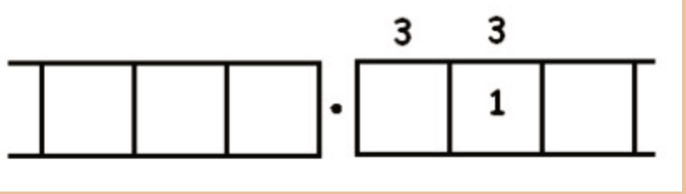
Here there are three groups of 3 leaving one behind:



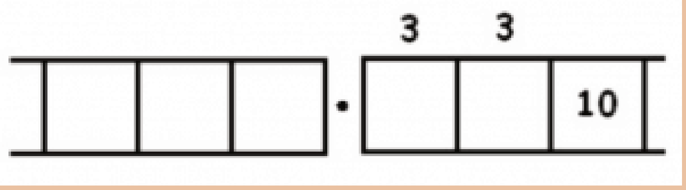
Unexploding gives:

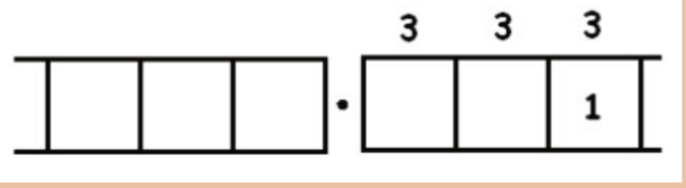


We find another three groups of 3 leaving one behind:



Unexploding gives:





And we seem to be caught in an infinitely repeating cycle.

We are now in a philosophically interesting position. As human beings, we cannot conduct this, or any, activity an infinite number of times. But it seems very tempting to write:

$$\frac{1}{3} = 0.33333\ldots,$$

with the ellipsis "..." meaning "keep going forever with this pattern." We can *imagine* what this means, but we cannot actually *write down* those infinitely many 3's represented by the . . .

Notation

Many people make use of a **vinculum** (horizontal bar) to represent infinitely long repeating decimals. For example, $0.\overline{3}$ means "repeat the 3 forever":

$$0.\bar{3} = 0.33333\ldots,$$

and $0.296\overline{412}$ means "repeat the 412 forever":

$$0.296\overline{412} = 0.296412412412412\ldots$$

Now we're in a position to give a perhaps more satisfying answer to the question $1024 \div 3$. In the example above, we found the answer to be

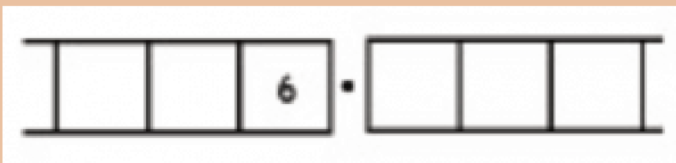
$$1024 \div 3 = 341 \;\; R1.$$

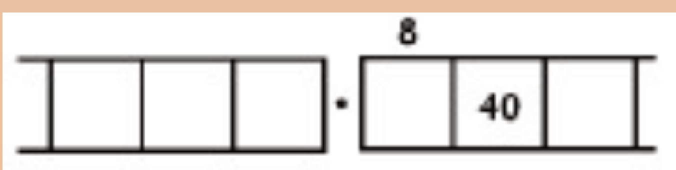
But now we know we can keep dividing that last stubborn dot by 3. Remember, that represents a single dot in the ones place, so if we keep dividing by three it really represents $\frac{1}{3}$. So we have:

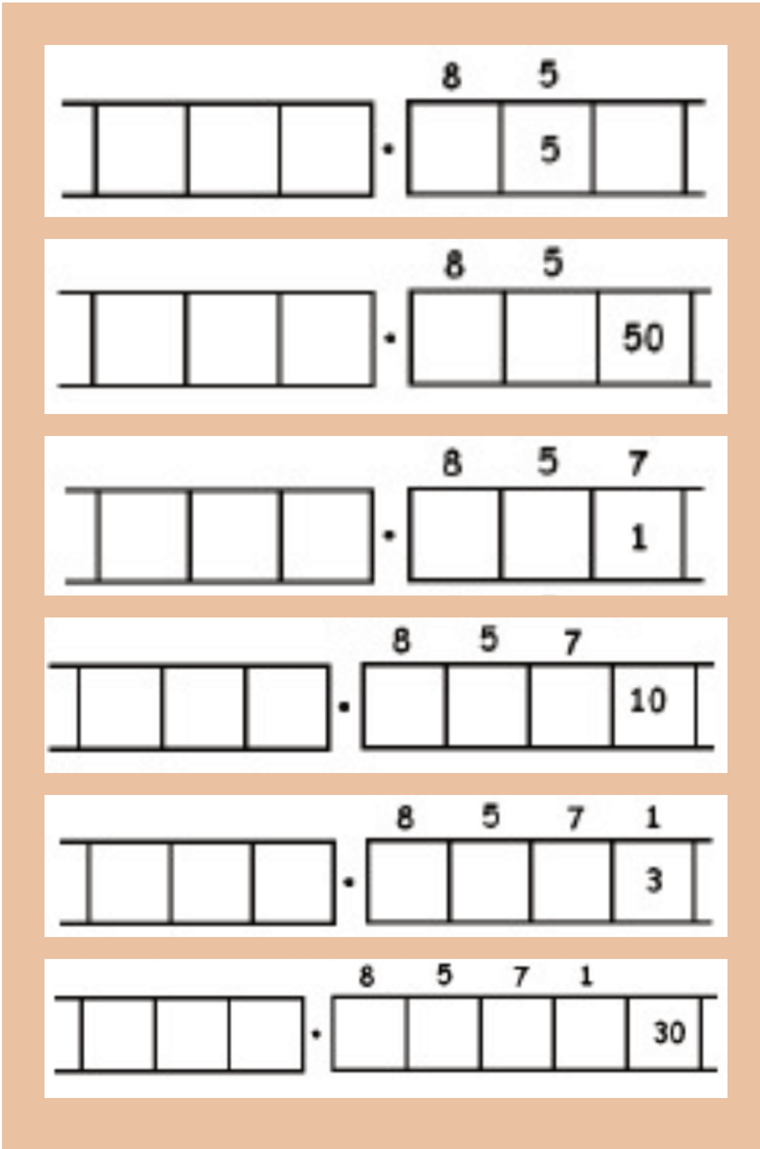
$$1024 \div 3 = 341 \;\; R1 = 341\tfrac{1}{3} = 341.333333\ldots = 341.\overline{3}.$$

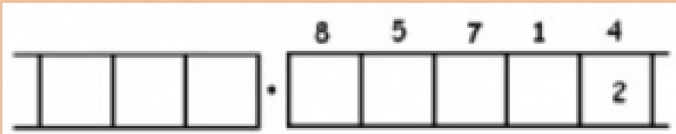
Example: 6/7

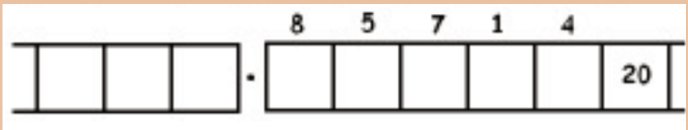
As another (more complicated) example, here is the work that converts the fraction $\frac{6}{7}$ to an infinitely long repeating decimal. Make sure to understand the steps one line to the next.

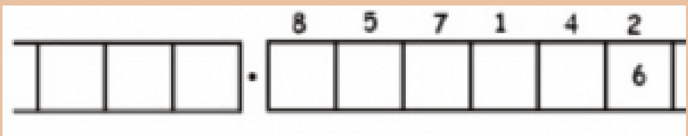





8 5
5
8 5
50
8 5 7
1
8 5 7
10
8 5 7 1
3
8 5 7 1
30







With this 6 in the final right-most box, we have returned to the very beginning of the problem. (Do you see why? Remember, we started with a six in the ones box!)

This means that we will simply repeat the work we have done and obtain the same sequence of answers: 857142. And then again, and then again, and then again. We have:

$$\begin{aligned}\frac{6}{7} &= 0.857142857142857142857142\ldots \\ &= 0.\overline{857142}.\end{aligned}$$

On Your Own

Work on the following exercises on your own or with a partner. Be sure to show your work.

1. Compute $\frac{4}{7}$ as an infinitely long repeating decimal.

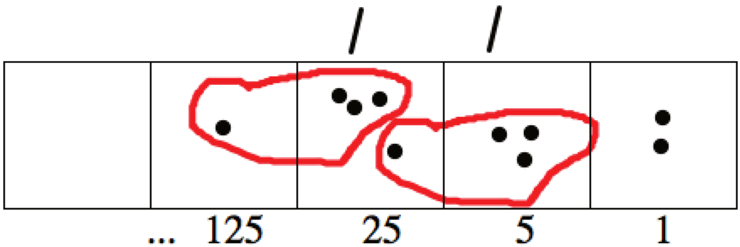
2. Compute $\frac{1}{9}$ as an infinitely long repeating decimal.
3. Use a “Dots & Boxes” model to compute $133 \div 6$. Write the answer as a decimal.
4. Use a “Dots & Boxes” model to compute $255 \div 11$. Write the answer as a decimal.

52. More x -mals

It should come as no surprise that we can use this reasoning about division in the "Dots & Boxes" model in other bases as well.

The following picture shows that working in base 5,

$$1432_{\text{five}} \div 13_{\text{five}} = 110_{\text{five}}\ \text{R}2_{\text{five}},\ \text{meaning}\ 1432_{\text{five}} = 110_{\text{five}} \cdot 13_{\text{five}} + 2_{\text{five}}.$$



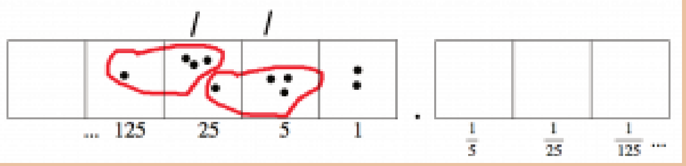
Think / Pair / Share

Carefully explain the connection between the picture and the equation shown above.

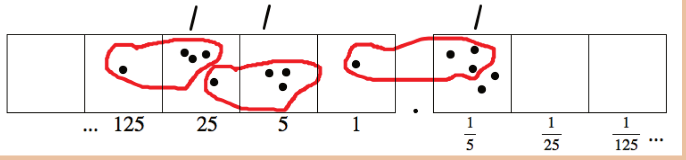
- Show in the picture where you see 1432_{five} from the equation.
- Where do you see 13_{five}?
- Where do you see 110_{five} and 2_{five}?

Example: 1432five ÷ 13five

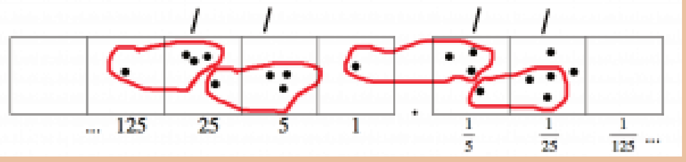
Here's where we left off the division, with a remainder of 2:



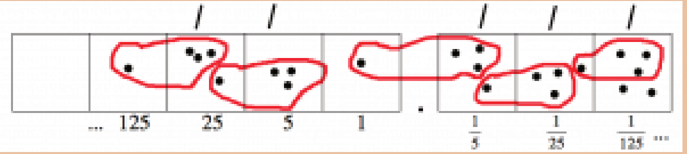
Now we can unexplode one of those two remaining dots. Then we're able to make another group of 13_{five}.



Once again, there are two dots left over, not in any group. So let's unexplode one of them.



And we still have two dots left over. Why not do it again?



It seems like we're going to be doing the same thing forever:

- Start with two dots in some box.
- Unexplode one one of the dots, so you have one dot in your original box and five in the box to the right.
- Form a group of 13_{five}. That uses the one dot in your original box and three dots in the box to the right.
- So you have two dots left in a box.
- Unexplode one of the dots, so you have one dot in your original box and five in the box to the right.
- This feels familiar…

We conclude:

$$1432_{\text{five}} \div 13_{\text{five}} = 110.111\ldots_{\text{five}} = 110.\overline{1}_{\text{five}}.$$

Think / Pair / Share

The equation

$$1432_{\text{five}} \div 13_{\text{five}} = 110.\overline{1}_{\text{five}}.$$

is a statement in base five. What is it saying in base ten?

"1432_{five}" is the number

$$1 \cdot 125 + 4 \cdot 25 + 3 \cdot 5 + 2 \cdot 1 = 242_{\text{ten}}.$$

- What is 13_{five} in base 10? Be sure to explain your answer.
- What is $110.\overline{1}_{\text{five}}$ in base 10? Explain how you got your answer.
- Translate the equation above to a statement in base ten and check that it is correct.

Problem 2

1. Draw pictures to compute $8 \div 3$ in a base ten system, and show the answer is $2.\overline{6}$.
2. Draw the pictures to compute $8_{\text{nine}} \div 3_{\text{nine}}$ in a base 9 system, and write the answer as a decimal. (Or is it a "nonimal"?)

Problem 3

1. Draw the pictures to compute $1 \div 11$ in a base ten system, and show the answer is $0.\overline{09}$.
2. Draw the base 3 pictures to compute $1_{\text{three}} \div 11_{\text{three}}$, and write the answer as a decimal ("trimal"?) number.
3. Draw the base four pictures to compute $1_{\text{four}} \div 11_{\text{four}}$, and write the answer as a decimal ("quadimal"?) number.
4. Draw the base six pictures to compute $1_{\text{six}} \div 11_{\text{six}}$, and write the answer as a decimal ("heximal"?) number.
5. Describe any patterns you notice in the computations above. Do you have a conjecture of a general rule? Can you prove your general rule is true?

Problem 4

Remember that the fraction $\frac{2}{5}$ represents the division problem $2 \div 5$. (This is all written in base ten.)

1. What is the decimal expansion (in base ten) of the

fraction $\frac{2}{5}$?

2. Rewrite the base-ten fraction $\frac{2}{5}$ as a base four division problem. Then find the decimal expansion for that fraction in base four.
3. Rewrite the base-ten fraction $\frac{2}{5}$ as a base five division problem. Then find the decimal expansion for that fraction in base five.
4. Rewrite the base-ten fraction $\frac{2}{5}$ as a base seven division problem. Then find the decimal expansion for that fraction in base seven.
5. Barry said that in base fifteen, the division problem looks like

   $$2_{\text{fifteen}} \div 5_{\text{fifteen}},$$

   and the decimal representation would be 0.6_{fifteen}. Check Barry's answer. Is he right?

Problem 5

Expand each of the following as a "decimal" number in the base given. (The fraction is given in base ten.)

(a) $\dfrac{1}{9}$ in base 10

(b) $\dfrac{1}{2}$ in base 3

(c) $\dfrac{1}{3}$ in base 4

(d) $\dfrac{1}{4}$ in base 5

(e) $\dfrac{1}{5}$ in base 6

(f) $\dfrac{1}{6}$ in base 7

(g) $\dfrac{1}{7}$ in base 8

(h) $\dfrac{1}{8}$ in base 9.

Do you notice any patterns? Any conjectures?

Problem 6 (Challenge)

What fraction has decimal expansion $0.\overline{3}_{\text{seven}}$? How do you know you are right?

53. Terminating or Repeating?

You've seen that when you write a fraction as a decimal, sometimes the decimal *terminates*, like:

$$\frac{1}{2} = 0.5 \quad \text{and} \quad \frac{33}{1000} = 0.033.$$

However, some fractions have decimal representations that go on forever in a repeating pattern, like:

$$\frac{1}{3} = 0.33333\ldots \quad \text{and} \quad \frac{6}{7} = 0.857142857142857142857142\ldots.$$

It's not totally obvious, but it is true: Those are the only two things that can happen when you write a fraction as a decimal.

Of course, you can *imagine* (but never write down) a decimal that goes on forever but doesn't repeat itself, for example:

$$0.1010010001000010000001\ldots \quad \text{and} \quad \pi = 3.14159265358979\ldots.$$

But these numbers can never be written as a nice fraction $\frac{a}{b}$ where a and b are whole numbers. They are called *irrational numbers*. The reason for this name: Fractions like $\frac{a}{b}$ are also called *ratios*. Irrational numbers cannot be expressed as a *ratio* of two whole numbers.

For now, we'll think about the question: Which fractions have decimal representations that terminate, and which fractions have decimal representations that repeat forever? We'll focus just on *unit fractions*.

Definition

A **unit fraction** is a fraction that has 1 in the numerator. It looks like $\frac{1}{n}$ for some whole number n.

Think / Pair / Share

- Which of the following fractions have infinitely long decimal representations and which do not?

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10}.$$

- Try some more examples on your own. Do you have a conjecture?

A fraction $\frac{1}{b}$ *has an infinitely long decimal expansion if:*

________________________________.

Problem 7

Complete the table below which shows the decimal expansion of unit fractions where the denominator is a power of 2. (You may want to use a calculator to compute the decimal representations. The point is to look for and then explain a pattern, rather than to compute by hand.)

Try even more examples until you can make a conjecture: What is the decimal representation of the unit fraction $\frac{1}{2^n}$?

Fraction	Denominator	Decimal
$\frac{1}{2}$	2	0.5
$\frac{1}{4}$	2^2	0.25
$\frac{1}{8}$	2^3	0.125
$\frac{1}{16}$		
$\frac{1}{32}$		
$\frac{1}{64}$		
$\frac{1}{128}$		
$\frac{1}{256}$		

Problem 8

Complete the table below which shows the decimal expansion of unit fractions where the denominator is a power of 5. (You may want to use a calculator to compute the decimal representations. The point is to look for and then explain a pattern, rather than to compute by hand.)

Try even more examples until you can make a conjecture: What is the decimal representation of the unit fraction $\frac{1}{5^n}$?

Fraction	**Denominator**	**Decimal**
$\frac{1}{5}$	5	0.2
$\frac{1}{25}$	5^2	0.04
$\frac{1}{125}$	5^3	
$\frac{1}{625}$		
$\frac{1}{3125}$		
$\frac{1}{15625}$		

Marcus noticed a pattern in the table from Problem 7, but was having trouble explaining exactly what he noticed. Here's what he said to his group:

> *I remembered that when we wrote fractions as decimals before, we tried to make the denominator into a power of ten. So we can do this:*
>
> $$\frac{1}{2} = \frac{1}{2} \cdot \frac{5}{5} = \frac{5}{10} = 0.5.$$
>
> $$\frac{1}{4} = \frac{1}{4} \cdot \frac{25}{25} = \frac{25}{100} = 0.25.$$
>
> $$\frac{1}{8} = \frac{1}{8} \cdot \frac{125}{125} = \frac{125}{1000} = 0.125.$$
>
> *When we only have 2's, we can always turn them into 10's by adding enough 5's.*

Think / Pair / Share

- Write out several more examples of what Marcus discovered.
- If Marcus had the unit fraction $\frac{1}{2^n}$, what would be his first step to turn it into a decimal? What would the decimal expansion look like and why?
- Now think about unit fractions with powers of 5 in the denominator. If Marcus had the unit fraction $\frac{1}{5^n}$, what would be his first step to turn it into a decimal? What

would the decimal expansion look like and why?

Marcus had a really good insight, but he didn't explain it very well. He doesn't really mean that we "turn 2's into 10's." And he's not doing any addition, so talking about "adding enough 5's" is pretty confusing.

Problem 9

1. Complete the statement below by filling in the numerator of the fraction.

   The unit fraction $\frac{1}{2^n}$ has a decimal representation that terminates. The representation will have n decimal digits, and will be equivalent to the fraction

   $$\frac{?}{10^n}.$$

2. Write a better version of Marcus's explanation to justify why this fact is true.

Problem 10

Write a statement about the decimal representations of unit fractions $\frac{1}{5^n}$ and justify that your statement is correct. (Use the statement in Problem 9 as a model.)

Problem 11

Each of the fractions listed below has a terminating decimal representation. Explain how you could know this for sure, without actually calculating the decimal representation.

$$\frac{1}{10} \qquad \frac{1}{20} \qquad \frac{1}{50} \qquad \frac{1}{200} \qquad \frac{1}{500} \qquad \frac{1}{4000}.$$

The Period of a Repeating Decimal

If the denominator of a fraction can be factored into just 2's and 5's, you can always form an equivalent fraction where the denominator is a power of ten.

For example, if we start with the fraction

$$\frac{1}{2^a 5^b},$$

we can form an equivalent fraction

$$\frac{1}{2^a 5^b} = \frac{1}{2^a 5^b} \cdot \frac{2^b 5^a}{2^b 5^a} = \frac{2^b 5^a}{2^{a+b} 5^{a+b}} = \frac{2^b 5^a}{10^{a+b}}.$$

The denominator of this fraction is a power of ten, so the decimal expansion is finite with (at most) $a + b$ places.

What about fractions where the denominator has other prime factors besides 2's and 5's? Certainly we *can't* turn the denominator into a power of 10, because powers of 10 have just 2's and 5's as their prime factors. So in this case the decimal expansion will go on forever. But why will it have a *repeating pattern*? And is there anything else interesting we can say in this case?

Definition

The **period** of a repeating decimal is the smallest number of digits that repeat.

For example, we saw that

$$\frac{1}{3} = 0.33333 \cdots = 0.\overline{3}.$$

The repeating part is just the single digit 3, so the period of this repeating decimal is one.

Similarly, we know that

$$\frac{6}{7} = 0.857142857142857142857142\ldots = 0.\overline{857142}.$$

The smallest repeating part is the digits 857142, so the period of this repeating decimal is 6.

You can think of it this way: the *period* is the length of the string of digits under the vinculum (the horizontal bar that indicates the repeating digits).

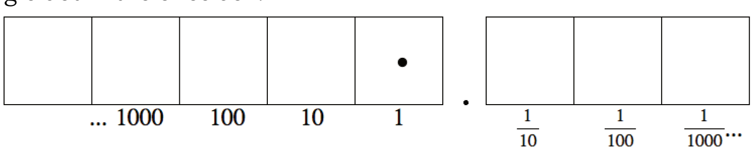
Problem 12

Complete the table below which shows the decimal expansion of unit fractions where the denominator has prime factors besides 2 and 5. (You may want to use a calculator to compute the decimal representations. The point is to look for and then explain a pattern, rather than to compute by hand.)

Try even more examples until you can make a conjecture: What can you say about the period of the fraction $\frac{1}{n}$ when n has prime factors besides 2 and 5?

Fraction	Decimal	Period
$\frac{1}{3}$	$0.\overline{3}$	1
$\frac{1}{6}$	$0.1\overline{6}$	1
$\frac{1}{7}$	$0.\overline{142857}$	6
$\frac{1}{9}$		
$\frac{1}{11}$		
$\frac{1}{12}$		
$\frac{1}{13}$		
$\frac{1}{14}$		

Imagine you are doing the "Dots & Boxes" division to compute the decimal representation of a unit fraction like $\frac{1}{6}$. You start with a single dot in the ones box:



To find the decimal expansion, you "unexplode" dots, form groups of six, see how many dots are left, and repeat.

Draw your own pictures to follow along this explanation:

Picture 1: When you unexplode the first dot, you get 10 dots in the $\frac{1}{10}$ box, which gives one group of six with remainder of 4.

Picture 2: When you unexplode those four dots, you get 40 dots in the $\frac{1}{100}$ box, which gives six group of six with remainder of 4.

Picture 3: Unexplode those 4 dots to get 40 in the next box to the right.

Picture 4: Make six groups of 6 dots with remainder 4.

Since the remainder repeated (we got a remainder of 4 again), we can see that the process will now repeat forever:

- unexplode 4 dots to get 40 in the next box to the right,
- make six groups of 6 dots with remainder 4,
- unexplode 4 dots to get 40 in the next box to the right,
- make six groups of 6 dots with remainder 4,
- and so on forever…

On Your Own

Work on the following exercises on your own or with a partner.

1. Use "Dots & Boxes" division to compute the decimal representation of $\frac{1}{11}$. Explain how you know for sure the process will repeat forever.
2. Use "Dots & Boxes" division to compute the decimal representation of $\frac{1}{12}$. Explain how you know for sure the process will repeat forever.
3. What are the possible *remainders* you can get when you use division to compute the fraction $\frac{1}{7}$? How can you be sure the process will eventually repeat?
4. What are the possible *remainders* you can get when you use division to compute the fraction $\frac{1}{9}$? How can you be sure the process will eventually repeat?

Problem 13

Suppose that n is a whole number, and it has some prime factors besides 2's and 5's. Write a convincing argument that:

1. The decimal representation of $\frac{1}{n}$ will go on forever (it will not terminate).
2. The decimal representation of $\frac{1}{n}$ will be an infinite *repeating* decimal.
3. The period of the decimal representation of $\frac{1}{n}$ will be less than n.

Problem 14

1. Find the "decimal" expansion for $\frac{1}{2}$ in the following bases. Be sure to show your work:

   $$\text{two}, \quad \text{three}, \quad \text{four}, \quad \text{five}, \quad \text{six}, \quad \text{seven}.$$

2. Make a conjecture: If I write the decimal expansion of $\frac{1}{2}$ in base b, when will that expansion be finite and when will it be an infinite repeating decimal expansion?

3. Can you prove your conjecture is true?

54. Matching Game

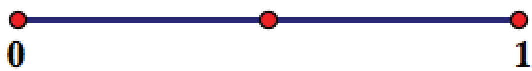
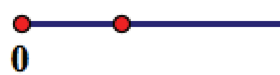
In this section, you'll find numbers described in various ways: as fractions, as points on a number line, as decimals, and in a picture. Your job is to match these up in a way that makes sense.

Note: there may be more than one fraction to match a given decimal, or more than one picture to match a given point on the number line. So be ready to justify your answers.

Fractions

(a) $\frac{1}{5}$	(b) $\frac{1}{3}$	(c) $\frac{2}{3}$
(d) $\frac{9}{8}$	(e) $\frac{15}{16}$	(f) $\frac{25}{100}$
(g) $\frac{3}{4}$	(h) $\frac{33}{100}$	(i) $\frac{3}{25}$
(j) $\frac{1}{4}$	(k) $\frac{6}{5}$	(ℓ) $\frac{2}{5}$
(m) $\frac{4}{100}$	(n) $\frac{2}{10}$	(o) $\frac{1}{2}$

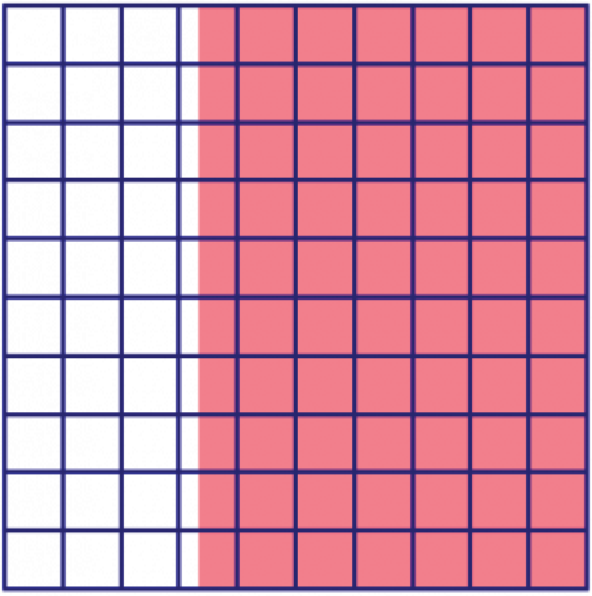
Points on a number line

 Point 1	 Point
 Point 3	 Point
 Point 5	 Point
 Point 7	 Point
 Point 9	 Point
 Point 11	 Point
 Point 13	

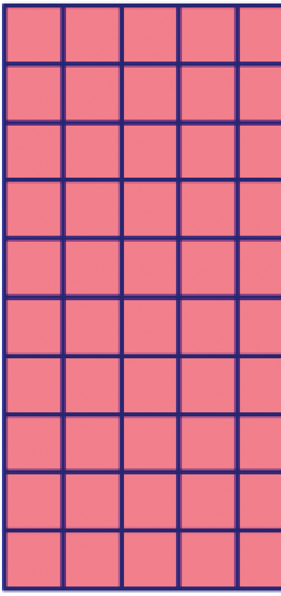
Decimals

(i) 1.20	(ii) $0.\overline{6}$	(iii) 0.33
(iv) 0.25	(v) 0.5	(vi) 0.200
(vii) 0.75	(viii) $0.\overline{3}$	(ix) 0.2
(x) 1.125	(xi) 0.12	(xii) 0.04
(xiii) 0.40	(xiv) 0.50	(xv) 0.9375

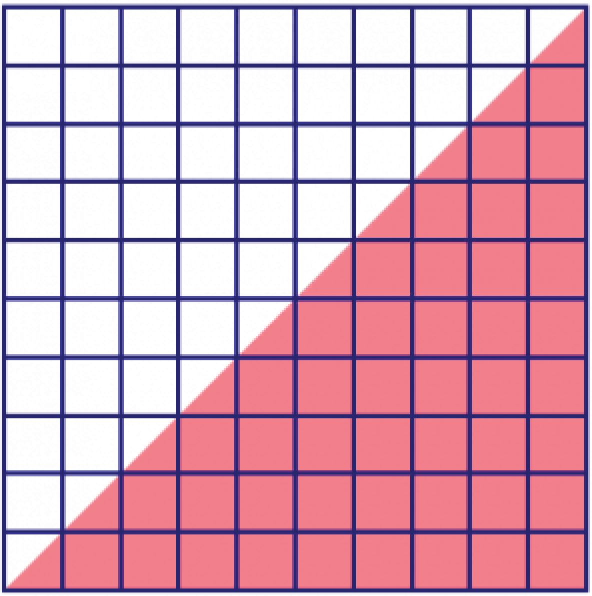
Pictures



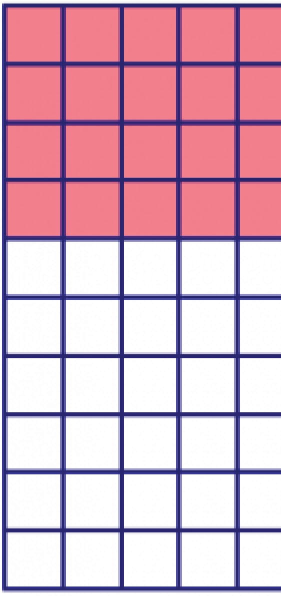
Picture A



Pic



Picture C



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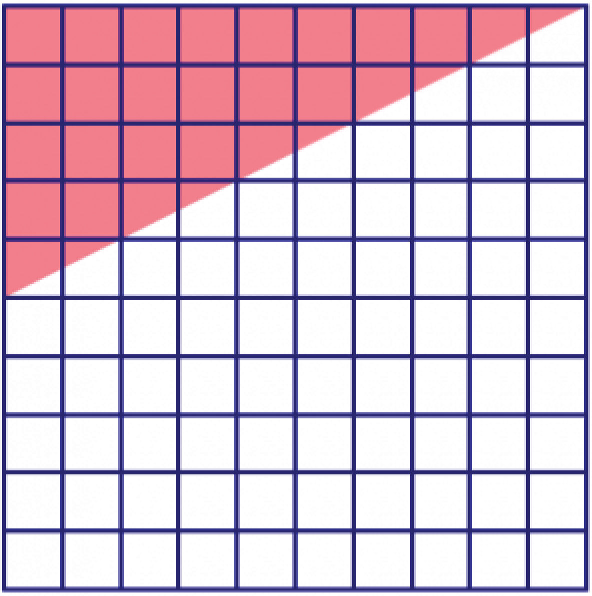

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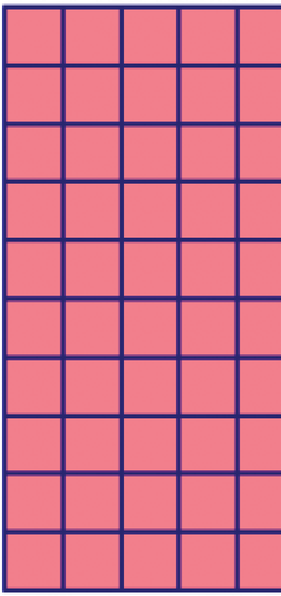

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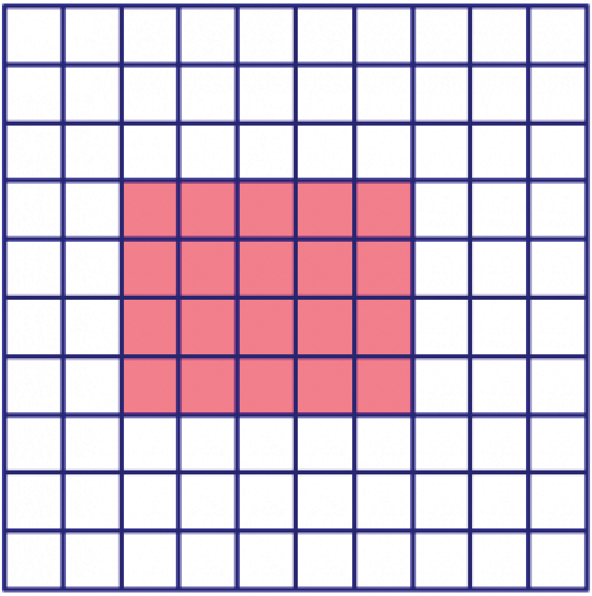

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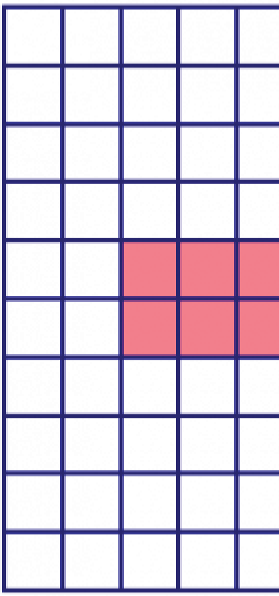

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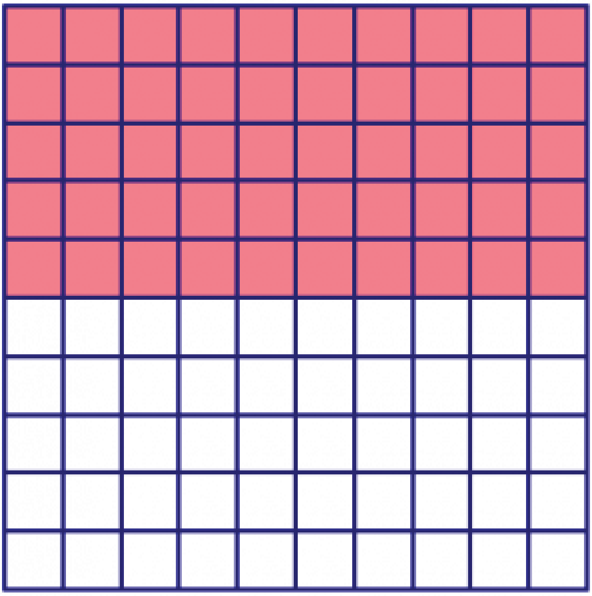

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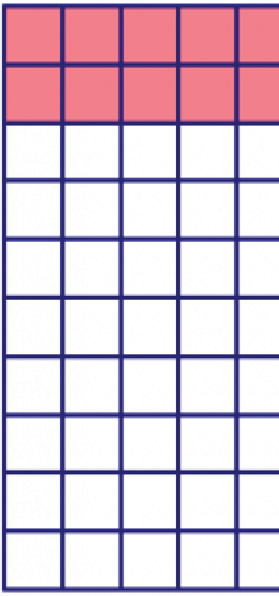

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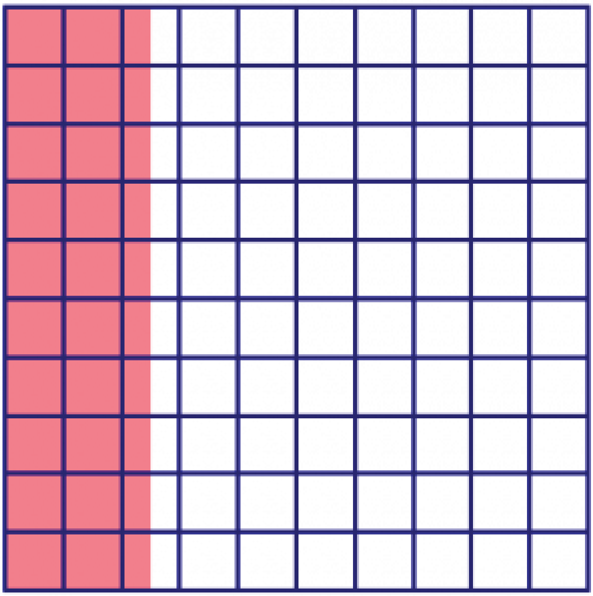

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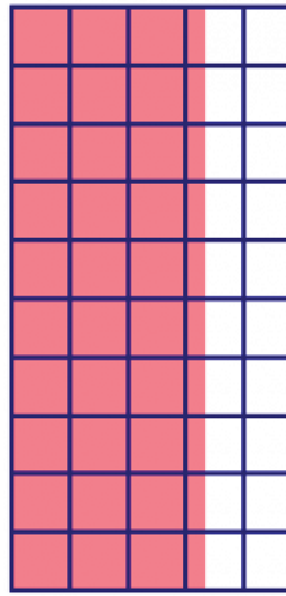

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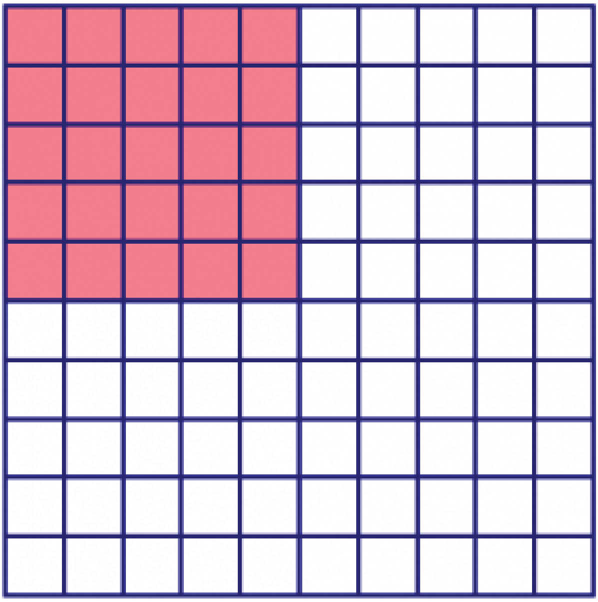

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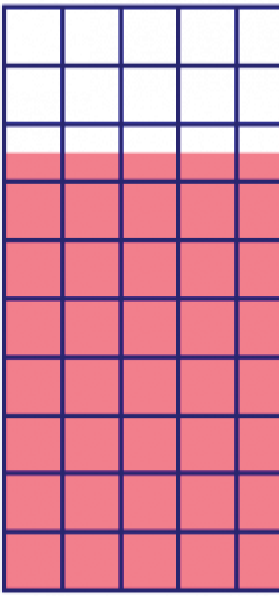

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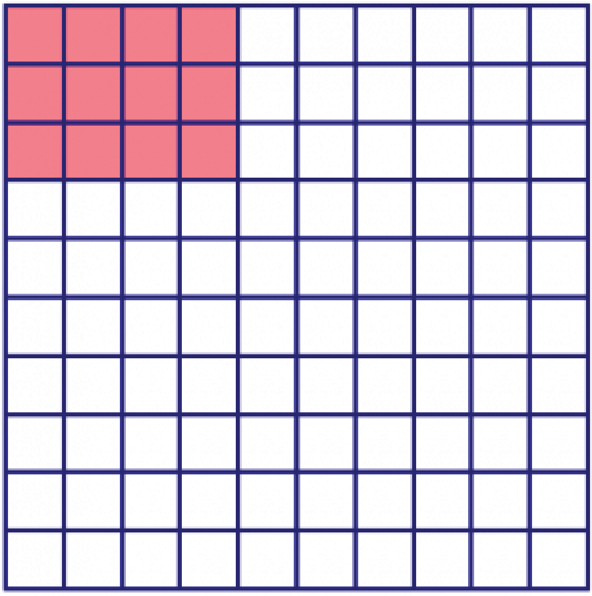
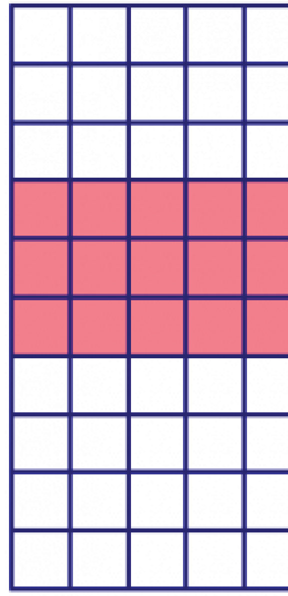


Picture O



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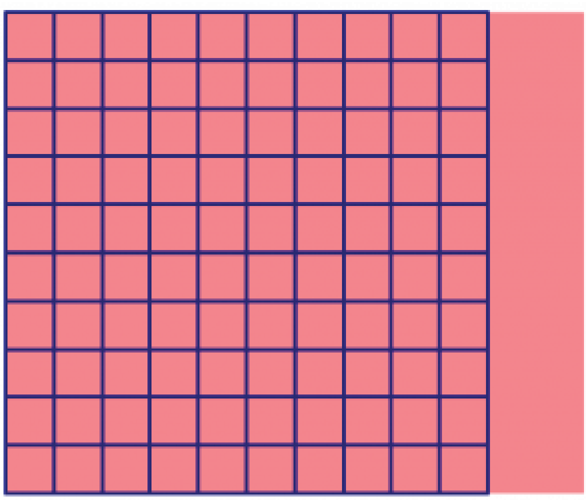
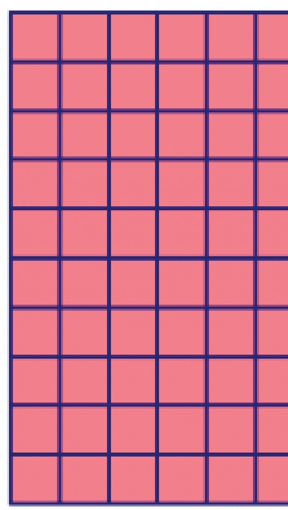


Picture Q



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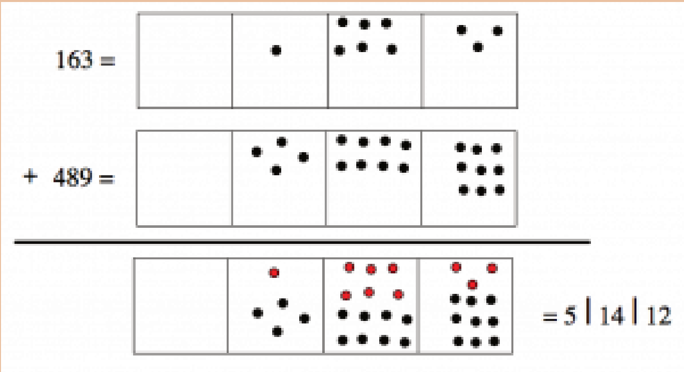
55. Operations on Decimals

Of course we can add, subtract, multiply, and divide decimal numbers by rewriting them as fractions and using the algorithms we know there. Of course, sometimes it is a lot more work to convert to fractions than it is to just work directly with the decimals (as long as you know what you're doing). So let's think about place value and computing with decimals.

Adding and Subtracting Decimals

Remember that when we used the "Dots & Boxes" model to add, it looked like this.

Example: 163 + 489



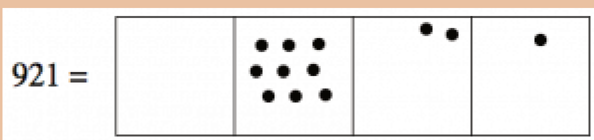
We then perform explosions until there are fewer than ten dots in each box, and we find that:

$$163 + 489 = 652.$$

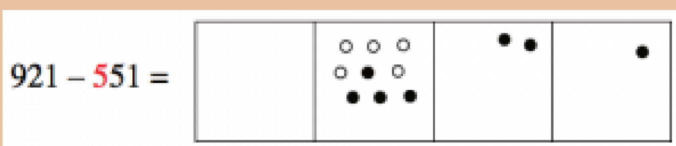
Subtraction was a little more complicated.

Example: 921 – 551

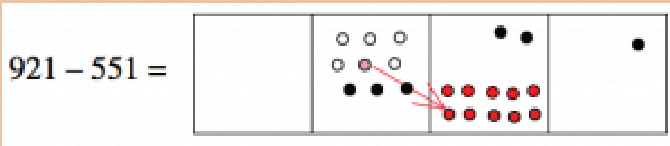
We start with the representation of 921:

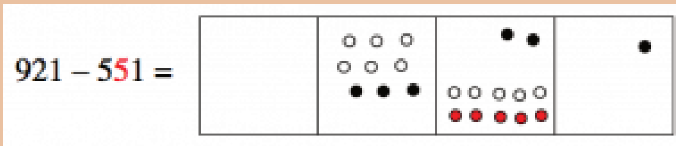


Since we want to "take away" 551, that means we take away five dots from the hundreds box, leaving four dots.



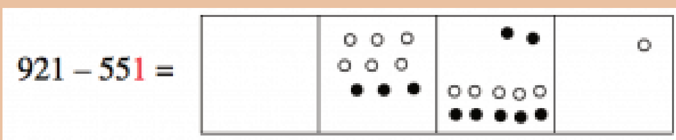
Now we want to take away five dots from the tens box, but we can't do it! There are only two dots there. What can we do? Well, we still have some hundreds, so we can "unexplode" a hundreds dot, and put ten dots in the tens box instead. Then we'll be able to take five of them away, leaving seven.





(Notice that we also have one less dot in the hundreds box; there's only three dots there now.)

Now we want to take one dot from the ones box, and that leaves no dots there.



We conclude that: $921 - 551 = 370.$

On Your Own

Work on the following exercises on your own or with a partner.

1. For each calculation, draw a “Dots & Boxes” model and use it to find the result of the calculation.

$$3.56 + 7.95 \qquad 1.452 + 32.27 \qquad 3.0205 + 409.2019$$

$$15.225 - 7.209 \qquad 14.793 - 8.95 \qquad 12.5 - 3.0002.$$

2. For each calculation below, add the decimals quickly using the

same reasoning as in the previous problems.

$$0.0066 + 0.9 \qquad 0.25 + 0.0088 \qquad 0.\overline{20} + 0.\overline{01}.$$

Think / Pair / Share

- Chloe added 0.2 and 0.02 and got an answer of 0.4. What was Chloe's likely mistake? As her teacher, how could you help Chloe understand the operation of addition better?
- In elementary school, students are taught to add and subtract decimals by "lining up the decimal points." Use the "Dots & Boxes" model to explain why this shorthand makes sense.

Multiplying and Dividing: Powers of 10

Let's quickly review the "Dots & Boxes" model for multiplication of whole numbers before we get back to talking about decimals.

Example: 243192 × 4

If we want to compute 243192×4, it helps to remember what multiplication *means*. One interpretation is: I want to add 243192 to itself a total of four times. So there will be:

- 2×4 dots in the ones place,
- 9×4 dots in the tens place,
- 1×4 dots in the hundreds place,
- and so on.

Here's the start of the computation:

$$243192 \times 4 = 8 \mid 16 \mid 12 \mid 4 \mid 36 \mid 8.$$

To finish the computation, we need to do some explosions to write the result as a familiar base 10 number:

$$243192 \times 4 = 972768.$$

On Your Own

Work on the following exercises on your own or with a partner.

1. Do each computation, using reasoning like in the multiplication example above.

   $(a)\ 2.3 \times 10 \qquad (b)\ 3.56 \times 10 \qquad (c)\ 1.452 \times 100.$
2. Do each computation, using reasoning like in the "Division and Decimals" examples.

$$7.1 \div 10 \qquad 98.55 \div 10 \qquad 145.2 \div 100.$$

You know that multiplying a base-ten whole number by 10 results in appending a zero to the right end of the number. Your work above should convince you that this does not work for decimals!

Think / Pair / Share

- Write a new rule that works for both whole numbers and decimals:

  If I multiply a whole number or a decimal by 10, a simple way to find the result is

  ____.

- Justify the claim you made above.
- One can go much further with this thinking. What is the effect of *dividing* a number written in decimal notation by ten? By one-hundred? Justify what you say.

Multiplying Decimals

You probably know an algorithm for multiplying decimal numbers by hand. But if you think carefully about the algorithm, it should **make sense** based on what the decimal numbers represent and what it means to multiply. Let's start by using number sense to think about multiplying whole numbers by decimals.

Think / Pair / Share

Consider the expression

$$16 \times \square.$$

Fill in the box with a whole number or decimal so that the product is:

- Greater than 100.
- Greater than 64 but less than 100.
- At least 17, but less than 32.
- Equal to 16.
- Greater than 8 but less than 16.
- Less than 8, but greater than 0.

Be sure to justify your answers. You should use your number sense rather than computing by hand or with a calculator!

One way to multiply decimal numbers by converting them to fractions and then using what you know about multiplying fractions. There are other ways to think about multiplying that focus on number sense and place value rather than on the mechanics of computation.

Example: 321 × 0.4

Suppose a student wanted to compute 321×0.4, but he didn't already know the standard algorithm. What might she do? Here is one idea:

> I know that $321 \times 4 = 1284$. Since I want to multiply by 0.4 and not by 4, my answer should be $\frac{1}{10}$ of this one. So
>
> $$321 \times 0.4 = 128.4.$$

You should notice that the student is using the **associative property** of multiplication:

$$321 \times 0.4 \;=\; 321 \times \frac{4}{10} \;=\; 321 \times \left(4 \times \frac{1}{10}\right) \;=\; (321 \times 4) \times \frac{1}{10}.$$

Problem 15

For each computation below, the result of the computation is shown correctly, but the decimal point is missing. Use number sense and reasoning to correctly place the decimal point, and briefly justify how you know you're right.

(Don't use a calculator, don't work out the multiplication by hand, and don't use the trick of "counting the number of decimal places." Use your number sense!)

$(a)\ 855 \times 1.7 = 14535$ $(b)\ 549 \times 0.33 = 18117$

$(c)\ 2.03 \times 1028 = 208684$ $(d)\ 999 \times 0.53 = 52947$

$(e)\ 30.02 \times 472 = 1416944$ $(f)\ 173 \times 0.09 = 1557.$

On Your Own

Work on the following exercises on your own or with a partner.

1. Write each number given as a fraction. (Write them as "improper fractions," not "mixed numbers.")

$$(a)\ 15.2 \qquad (b)\ 3.43$$
$$(c)\ 0.0021 \qquad (d)\ 13.02026.$$

2. In exercise (1) above, how does the number of digits to the right of the decimal point compare to the number of zeros in the denominator? Use what you know about place value to explain why your answer is always true (not just for the examples above).
3. Find each product.

$$(a)\ 10 \times 10000 \qquad (b)\ 100 \times 1000$$
$$(c)\ 100000 \times 1000 \qquad (d)\ 10^m \times 10^n.$$

4. In exercise (3) above, how is the number of zeros in the product

related to the number of zeros in the two factors? Use what you know about place value to explain why your answer is always true (not just for the examples above).

1. If you write 0.037 as a fraction, how many zeros would be in the denominator?
2. What if you write 0.59 as a fraction,how many zeros would be in the denominator?
3. How many zeros would be in the denominator of the product of 0.037 and 0.59? (Don't compute the product to answer this question!)

5. Use the fact that $37 \times 59 = 2183$ and your answers to the exercises above to find 0.037×0.59. Explain how you got your answer.

Standard multiplication algorithm

The standard algorithm for multiplying decimal numbers can be described this way:

Step 1

Compute the product as if the two factors were whole numbers. (Ignore the decimal points.)

Step 2

Count the number of digits to the right of the decimal point in each factor, and add those numbers together. Call the result n.

Step 3

The sum n that you found in Step 2 will be the number of digits to the right of the decimal point in the product. So place the decimal point according by counting the appropriate number of places from the right.

Think / Pair / Share

- Write down two examples of multiplying decimal numbers using the standard algorithm above.
- Use what you know about place value, fractions, and multiplication to **carefully explain why** the standard algorithm described above makes sense.

Dividing Decimals

As you might expect, dividing decimals is more complicated to explain than any of the other operations. It's hard to adapt our "Dots & Boxes" model for division. Suppose we want to compute

$15.37 \div 0.013$. We can certainly draw the picture for 15.37, but how could we make groups of 0.013 dots?

Think / Pair / Share

Let's start by sharing what you already know. Perform this computation (by hand, not with a calculator), showing all of your work. Explain your method to a partner, and see if your partner computed the same way.

$$0.0351 \div 0.074.$$

On Your Own

Work on the following exercises on your own or with a partner.

1. Explain why these two fractions are equivalent.

$$\frac{12.33}{44.1} \quad \text{and} \quad \frac{123.3}{441}.$$

2. Explain why these two division computations give the same result.

$$12.33 \div 44.1 \quad \text{and} \quad 123.3 \div 441.$$

3. Explain why these three fractions are equivalent.

$$\frac{325.5}{75.133}, \quad \frac{3255}{751.33}, \quad \text{and} \quad \frac{32550}{7513.3}.$$

4. Explain why these three division computations give the same result.

$$325.5 \div 75.133, \qquad 3255 \div 751.33, \qquad \text{and} \qquad 32550 \div 7513.3.$$

5. Fill in the box to make the equation true. Be sure to justify your answer.

$$\frac{325.5}{75.133} = \frac{\square}{75133}.$$

Standard division algorithm

The standard algorithm for dividing numbers represented by finite decimal expansions is something like this:

Step 1

Move the decimal point of the divisor to the end of the number.

Step 2

Move the decimal point of the dividend the same number of positions (the same distance and direction).

Step 3

Divide the new decimal dividend (from Step 2) by the new whole number divisor (from Step 1). Since we're dividing by a whole number, our standard methods make sense.

This is a pretty mechanical description, and doesn't give a lot of insight into **why** this algorithm works.

Think / Pair / Share

Write down at least two examples of computing with the algorithm described above. (Make up your own numbers to test. Be sure to show every step clearly.) You can do the division by drawing a "Dots & Boxes" picture or by another method (but don't use a calculator). Then answer these more general questions.

- Suppose you want to compute $a \div b$ where a and b are decimal numbers. Carefully explain why $(10 \cdot a) \div (10 \cdot b)$ will give the same result.
- Suppose you want to compute $a \div b$ where a and b are decimal numbers. Carefully explain why $(100 \cdot a) \div (100 \cdot b)$ will give the same result.
- Suppose you want to compute $a \div b$ where a and b are decimal numbers. Carefully explain why $(10^k \cdot a) \div (10^k \cdot b)$ will give the same result.
- Suppose b has a finite decimal expansion. Carefully explain why you can find a power of 10 so that $10^k \cdot b$ is a whole number.

Problem 16

Carefully explain **why** the algorithm described above in three steps works for computing division of decimal numbers. You need to explain what is going on when you "move the decimal point" in Steps 1 and 2, and why the result you compute in Step 3 is the same as the original problem.

56. Orders of Magnitude

Problem 17

How old were you when you were one million seconds old? (That's $1,000,000$.)

- Before you figure it out, write down a guess. What's your gut instinct? About a day? A week? A month? A year? Have you already reached that age? Or maybe you won't live that long?
- Now figure it out! When was / will be your million-second birthday?

Problem 18

How old were you when you were one *billion* seconds old? (That's $1,000,000,000$.)

- Again, before you figure it out, write down a guess.
- Now figure it out! When was / will be your billion-

second birthday?

Were you surprised by the answers? People (most people, anyway) tend to have a very good sense for small, everyday numbers, but have very bad instincts about big numbers. One problem is that we tend to think *additively*, as if one billion is about a million plus a million more (give or take). But we need to think *mulitplicatively* in situations like this. One billion is $1,000 \times \text{ a million}$.

So you could have just taken your answer to Problem 17 and multiplied it by $1,000$ to get your answer to Problem 18. Of course, you would probably still need to do some calculations to make sense of the answer.

Think / Pair / Share

When is your one trillion second birthday? What will you do to celebrate?

Think / Pair / Share

The US debt is total amount the government has borrowed. (This borrowing covers the *deficit* – the difference between what the government spends and what it collects in taxes.) In summer of 2013, the US debt was *on the order of* 10 trillion dollars. (That means more than 10 trillion but less than 100 trillion. If you were to write out the dots-and-boxes picture, the dots would be as far left as the $10,000,000,000$ place.)

- If the US pays back one penny every second, will the national debt be paid off in your lifetime? Explain your answer.
- A headline from April 2013 said, "US to Pay Down $35 billion in Quarter 2." Suppose the US pays down $35 billion dollars *every* quarter (so four times per year). About how many years would it take to pay of the total national debt?

Here are some big-number problems to think about. Can you solve them?

Problem 19

1. Suppose you have a million jelly beans, and you tile the floor with them. How big of an area will they cover? The classroom? A football field? Something bigger? What if it was a billion jelly beans?
2. Suppose you have a million jelly beans and you stack them up. How tall would it be? As tall as you? As a tree? As a skyscraper? What if it was a billion jelly beans? About how many jelly beans (what *order of magnitude*) would you need to stack up to reach the moon? Explain your answers.

Fermi Problems

James Boswell wrote,

> Knowledge is of two kinds. We know a subject ourselves, or we know where we can find information upon it.

But math proves this wrong. There is actually a third kind of knowledge: Knowledge that you *figure out for yourself*. In fact, this is what scientists and mathematicians do for a living: they create new knowledge! Starting with what is already known, they ask "what if…" questions. And eventually, they figure out something new, something no one ever knew before!

Even for knowledge that you *could* look up (or ask someone), you can often figure out the answer (or a close approximation to the answer) on your own. You need to use a little knowledge, and a little ingenuity.

Fermi problems, named for the physicist Enrico Fermi, involve using your knowledge, making educated guesses, and doing reasonable calculations to come up with an answer that might at first seem unanswerable.

Example

Here's a classic Fermi problem: How many elementary school teachers are there in the state of Hawaii?

You might think: How could I possibly answer that? Why not just google it? (But some Fermi problems we meet will have – gasp! – non-googleable answers.)

First let's define our terms. We'll say that we care about classroom teachers (not administrators, supervisors, or other school personnel) who have a permanent position (not a sub, an aide, a resource room teacher, or a student teacher) in a grade K–5 classroom.

But let's stop and think. Do you know the population of Hawaii? It's about $1,000,000$ people. (That's not exact, of course. But this is an exercise is estimation. We're trying to get at the *order of magnitude* of the answer.)

How many of those people are elementary school students? Well, what do you know about the population of Hawaii? Or what do you *suspect* is true? A reasonable guess would be that the population is evenly distributed across all age groups. That would give a population that looks something like this:

age range	# of people
0 – 9	125,000
10 – 19	125,000
20 – 29	125,000
30 – 39	125,000
40 – 49	125,000
50 – 59	125,000
60 – 69	125,000
70 – 79	125,000

We'll assume people don't live past 80. Of course some people do! But we're all about making simplifying assumptions right now. That gives us eight age categories, with about 125,000 people in each category.

An even better guess (since we have a large university that draws lots of students) is that there's a "bump" around college age. And some people live past 80, but there are probably fewer people in the older age brackets. Maybe the breakdown is something like this? (If you have better guesses, use them!)

age range	# of people
0 – 9	125,000
10 – 19	130,000
20 – 29	140,000
30 – 39	125,000
40 – 49	125,000
50 – 59	125,000
60 – 69	120,000
> 70	105,000

So, how many K–5 students are in Hawaii? That covers about six years of the 0–9 range. If we are still going with about the same number of people at each age, there should be about 12,500 in each grade for a total of $12,500 \times 6 = 75,000$ K–5 students.

OK, but we really wanted to know about K–5 *teachers*. One nice thing about elementary school: there tends to be just one teacher per class. So we need an estimate of how many classes, and that will tell us how many teachers.

So, how many students in each class? It probably varies a bit, with smaller kindergarten classes (since they are more rambunctious and need more attention), and larger fifth grade classes. There are also smaller classes in private schools and charter schools, but larger classes in public schools. A reasonable average might be 25 students per class across all grades K–5 and all schools.

So that makes $75,000 \div 25 = 3,000$ K–5 classrooms in Hawaii. And that should be the same as the number of K–5 teachers.

How good is this estimate? Can you think of a way to check and find out for sure?

So now you see the process for tackling a Fermi problem:

- Define your terms.
- Write down what you know.
- Make some reasonable guesses / estimates.
- Do some simple calculations.

Try your hand at some of these:

Problem 20

How much money does your university earn in parking revenue each year?

Problem 21

How many tourists visit Waikiki in a year?

Problem 22

How much gas would be saved in Hawaii if one out of every ten people switched to a carpool?

Problem 23

How high can a climber go up a mountain on the energy in one chocolate bar?

Problem 24

How much pizza is consumed each month by students at your university?

Problem 25

How much would it cost to provide free day care to every four-year-old in the US?

Problem 26

How many books are in your university's main library?

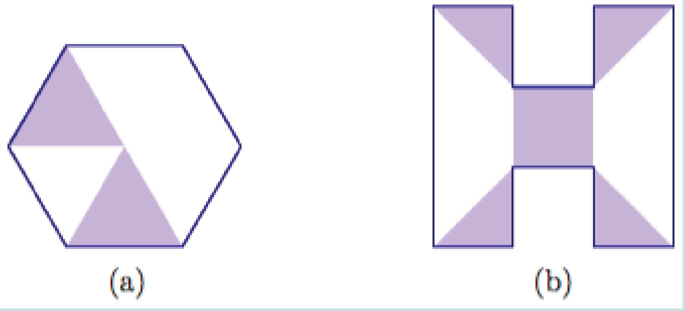
Problem 27

Make up your own Fermi problem… what would you be interested in calculating? Then try to solve it!

57. Problem Bank

Problem 28

Express the shaded portion of each figure as both a fraction and as a decimal. Justify your answers.



Problem 29

Which number is bigger: 0.135 or 0.14? Justify your answer.

Problem 30

Arrange the digits 1, 2, 3, and 4 in the boxes to create the smallest possible sum. Use each digit exactly once. Justify that your answer is as small as possible.

$$\frac{\square}{\square} + \frac{\square}{\square}$$

Problem 31

Arrange the digits 1, 2, 3, and 4 in the boxes to create the smallest possible (positive) difference. Use each digit exactly once. Justify that your answer is as small as possible.

$$\frac{\square}{\square} - \frac{\square}{\square}$$

Problem 32

Use the "Dots & Boxes" model to show that $\frac{1}{9} = 0.\overline{1}$. Then use this fact to answer these questions and justify your answers.

1. What fraction is given by $0.\overline{2}$?
2. What fraction is given by $0.\overline{5}$?
3. What fraction is given by $0.\overline{6}$?
4. What fraction is given by $0.\overline{8}$?
5. What fraction is given by $0.\overline{9}$?

Problem 33

In this problem, you will focus on the calculation

$$170 \times \square.$$

Your goal is to get a product that is close to 200.

1. Will you multiply 170 by a number greater or less than 1? Greater or less than 2? Justify your answers.
2. Suppose you can use only one decimal place. Fill in the box with a number that gets as close to 200 as possible.

3. Suppose you can use only two decimal places. Fill in the box with a number that gets as close to 200 as possible.
4. Suppose you can use only three decimal places. Fill in the box with a number that gets as close to 200 as possible.

Problem 34

Do each computation below without using a calculator. Explain your thinking.

1. $(23 \times 0.1) + (0.001 \times 55)$.
2. $18.45 \div (0.63 \div 0.7)$.
3. $22.65 - (0.03 \cdot 10)$.

Problem 35

Without actually calculating anything (just use your number sense!), order x, y, and z from smallest to largest. Explain your ordering.

$$x = 0.07 + 0.000001$$
$$y = 0.07 \times 0.000001$$
$$z = 0.07 \div 0.000001$$

Problem 36

For each question below, choose the correct calculation and explain your choice. Then estimate the answer (don't calculate it exactly) and explain why your estimate is a good one.

1. A large pizza has eight slices and costs $15.95. How much does each slice of pizza cost? Should you calculate 15.95×8 or $15.95 \div 8$?
2. There are 2.54 centimeters in an inch. A standard sheet of notebook paper is $8\frac{1}{2}$ inches wide and 11 inches long. How many centimeters wide is the page? Should you

calculate 8.5×2.54 or 11×2.54 or $8.5 \div 2.54$ or $11 \div 2.54$?

3. In a model train set, 1.38 inches represents one foot in real life. The height of One World Trade Center in New York City is 1776 feet. How tall would a scale model of the building be? Should you calculate 1776×1.38 or $1776 \div 1.38$?
4. Eight-tenths of a jumprope is 1.75 meters long. How long is the whole rope? Should you calculate 0.8×1.75 or $0.8 \div 1.75$ or $1.75 \div 0.8$?

Problem 37

Kaimi had no money at all when he cashed his paycheck. As he left the bank, he bought a piece of candy for a nickel from a machine. Later, he realized that the money in his pocket was equal to twice his paycheck. After a quick calculation, he figured out what happened: the teller accidentally switched the dollars and cents. How much was Kaimi supposed to be paid, and what did the teller give him? Justify your answer.

Problem 38

Here are the rules to a card game. Read the rules carefully and then answer the questions below.

- Each player starts with 10 points. The goal is to score as close to 100 points as possible without going over.
- On your turn: draw two cards, which will each have a decimal number on them. Using estimation (no computation), you can choose to multiply or divide your current score by one of the decimal numbers.
- After you decide, compute your new score exactly using a calculator. If your new score is over 100, you lose. If not, the other player takes a turn.
- At the end of your turn, you can decide to end the game. If you do, the other player gets one more turn. Then, the player with the score that is closest to 100 without going over wins the game.

Here are the questions:

1. On your turn, your score is 50. You draw the cards 0.2 and 1.75. Remember that your choices are:

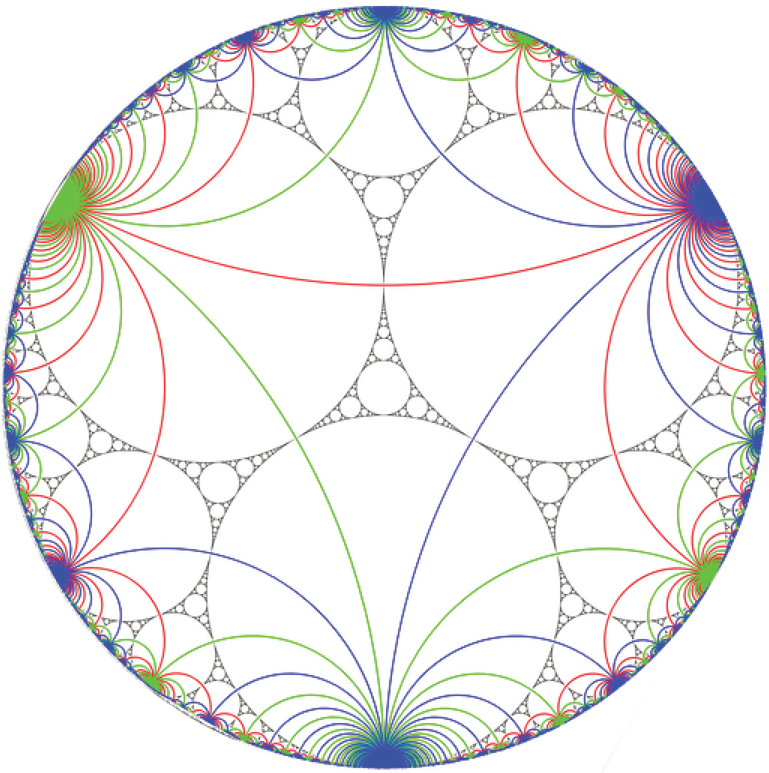
$$\text{divide by } 0.2 \qquad \text{multiply by } 0.2$$

$$\text{divide by } 1.75 \qquad \text{multiply by } 1.75.$$

What is your best move and why?

2. On your turn, your score is 88. You draw 1.3 and 0.6. What is your best move and why?
3. Your partner has a score of 57, and your score is 89. On her turn, your partner draws 0.8 and 1.8. She says she wants to end the game. On your final turn, you draw 0.7 and 1.2. If you both make the best possible move, who will win the game? Justify your answer.

PART VII
GEOMETRY



1

1. Image by Tomruen (Own work) [CC BY-SA 3.0 (https://creativecommons.org/licenses/by-sa/3.0)], via Wikimedia Commons.

Geometry is the art of good reasoning from bad drawings.

– Henri Poincaré

58. Introduction

The word “geometry” comes from the ancient Greek words “geo” meaning Earth and “metron” meaning measurement. It is probably the oldest field of mathematics, because of its usefulness in calculating lengths, areas, and volumes of everyday objects.

The study of geometry has evolved a great deal during the last 3,000 years or so. Like all of mathematics, what’s really important in geometry is reasoning, making sense of problems, and justifying your solutions.

The mathematician Henri Poincaré said that

> *Geometry is the art of good reasoning from bad drawings.*

This insight should guide your study in this chapter. You should never trust a drawing. You might find that one line segment appears to be longer than another, or an angle looks like it might be 90 degrees. But “appears to be” and “looks like” are simply not good enough. You have to reason through the situation and figure out what you know for sure and why you know it.

Think / Pair / Share

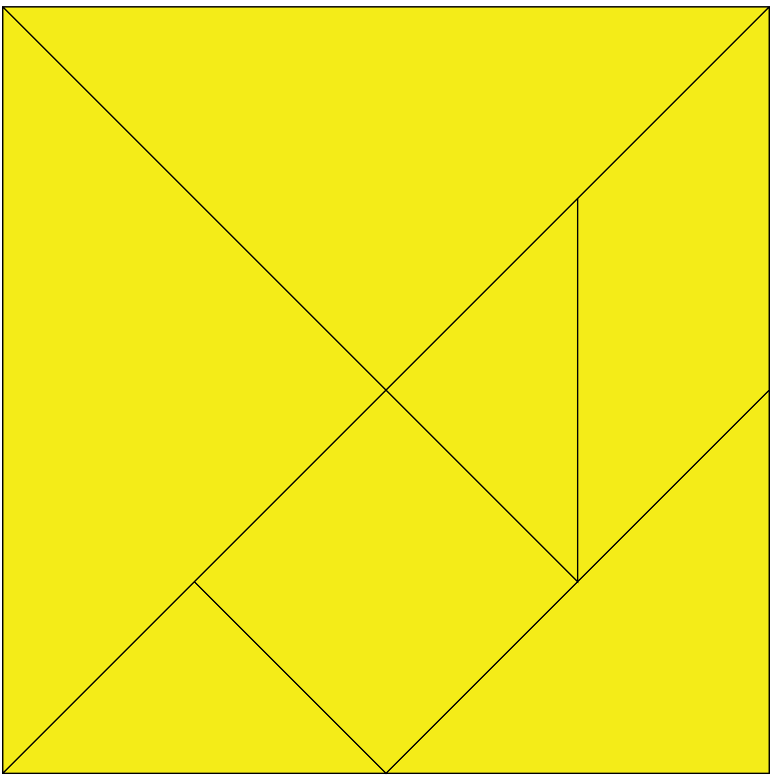
Reflect on your learning of geometry in the past. What is geometry really about? Also think about these questions:

- What is a point?
- What is a line? A segment? A ray?

- What is a plane?
- What is a circle?
- What is an angle?
- Which of these basic objects can be measured? How are they measured? What kinds of tools are useful?

59. Tangrams

Tangrams are a seven-piece geometric puzzle that dates back at least to the Song Dynasty in China (about 1100 AD). Below[1] you will find the seven puzzle pieces. Make a careful copy (a photocopy or printout is best), cut out the puzzle pieces, and then use them to solve the problems in this section.



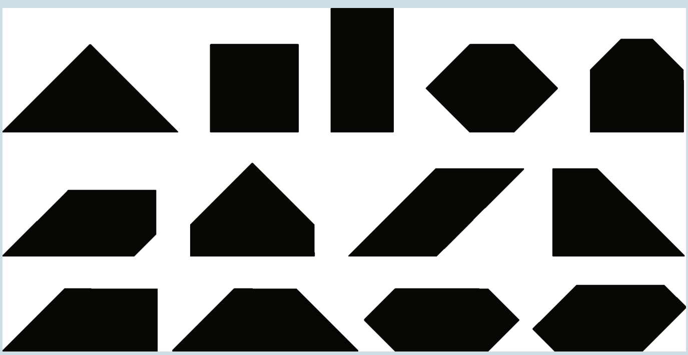
1. Image of tangram puzzle from Wikimedia Commons, public domain.

Whenever you solve a tangram puzzle, your job is to **use all seven pieces to form the shape**. They should fit together like puzzle pieces, sitting flat on the table; **no overlapping** of the pieces is allowed.

You can trace around your solutions to remember what you have done and to have a record of your work.

Problem 1

Use your tangram pieces to build the following designs[2]. How many can you make?

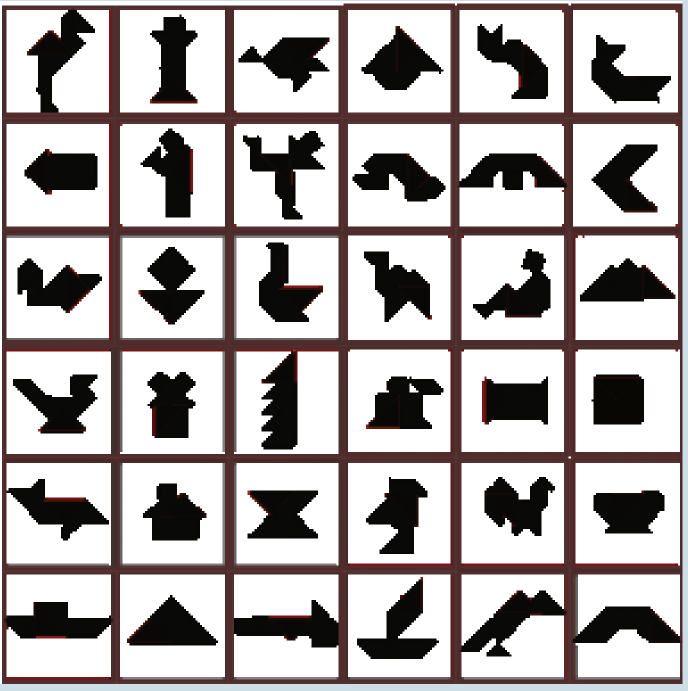


(These are all separate challenges. Each one requires all seven pieces. Once you solve one, trace your solution. Then try to solve another one.)

2. Tangram puzzles from Wikimedia Commons, public domain.

Problem 2

Use your tangram pieces to build the following designs[3]. How many can you make?



(These are all separate challenges. Each one requires all seven pieces. Once you solve one, trace your solution. Then try to solve another one.)

3. Tangram puzzles from pixababy.com,CC0 Creative Commons.

Think / Pair / Share

- Which tangram problems were easier and which were harder: making "real life" objects like cats and people, or purely mathematical objects like the rectangle?
- What do you think made one kind of problem easier or harder?

60. Triangles and Quadrilaterals

Think / Pair / Share

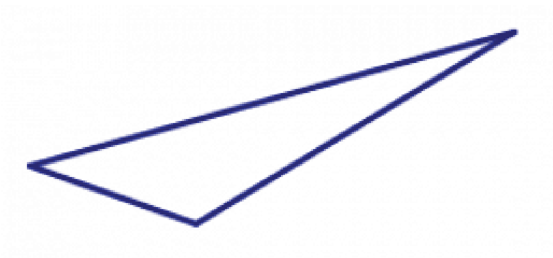
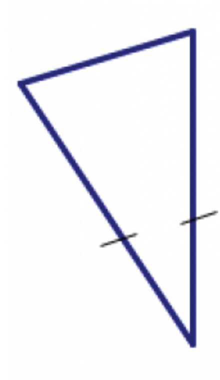
Follow these directions on your own:

- Draw any triangle on your paper.
- Draw a second triangle that is different in some way from your first one. Write down a sentence or two to say how it is different.
- Draw a third triangle that is different from both of your other two. Describe how it is different.
- Draw two more triangles, different from all the ones that came before.

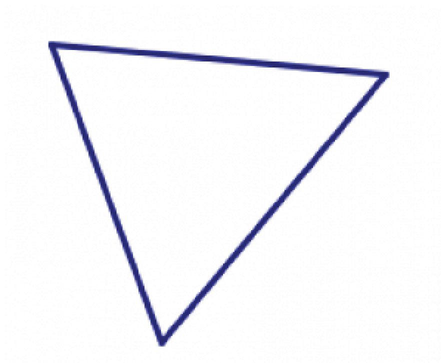
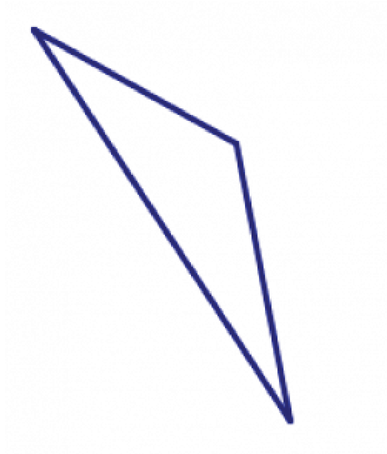
Compare your triangles and descriptions with a partner. To make "different" triangles, you have to change some feature of the triangle. Make a list of the features that you or your partner changed.

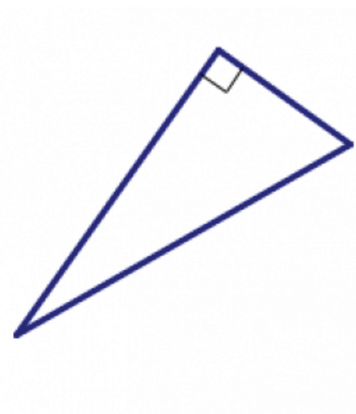
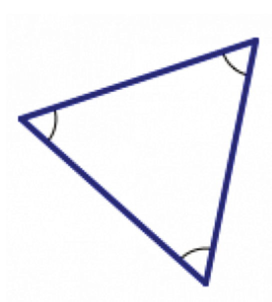
Triangles are classified according to different properties. The point of learning geometry is not to learn a lot of vocabulary, but it's useful to use the correct terms for objects, so that we can communicate clearly. Here's a quick dictionary of some types of triangles.

Classification by sides

scalene	isosceles	
		
all sides have different lengths	*two sides have the same length*	*all three*

Classification by angles

acute	obtuse
	
all interior angles measure less than 90°	*one interior angle measures more than 90°*

right	equiangular
	
one interior angle measures exactly 90°	*all interior angles have the same measure*

Remember that "geometry is the art of good reasoning from bad drawings." That means you can't always trust your eyes. If you look at a picture of a triangle and one side looks like it's longer than another, that may just mean the drawing was done a bit sloppily.

Notation: Tick marks

Mathematicians either write down measurements or use tick marks to indicate when sides and angles are supposed to be equal.

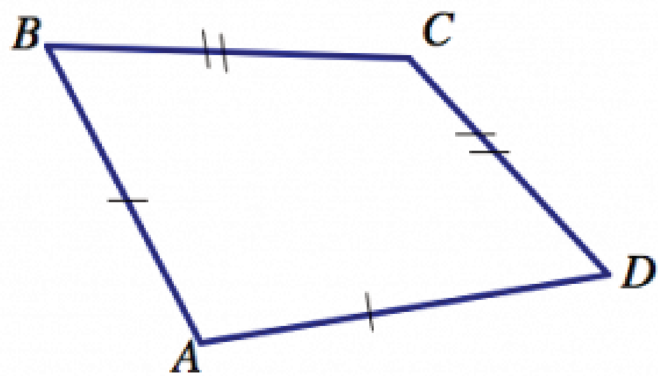
If two sides have the same measurement or the same number of tick marks, you must believe they are equal and work out the problem accordingly, *even if it doesn't look that way to your eyes.*

You can see examples of these in some of the pictures above. Another example is the little square used to indicate a right angle in the picture of the right triangle.

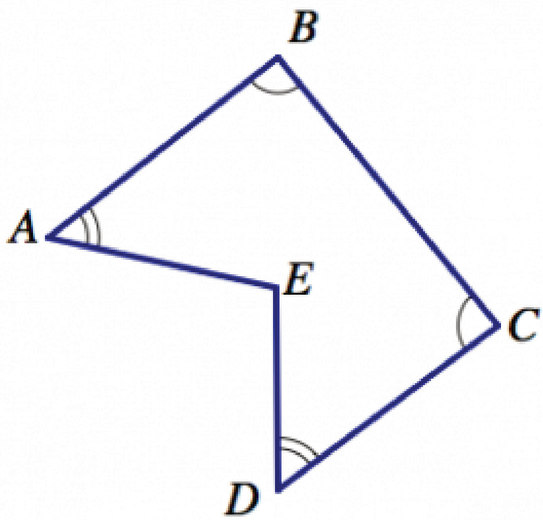
On Your Own

Work on the following exercises on your own or with a partner.

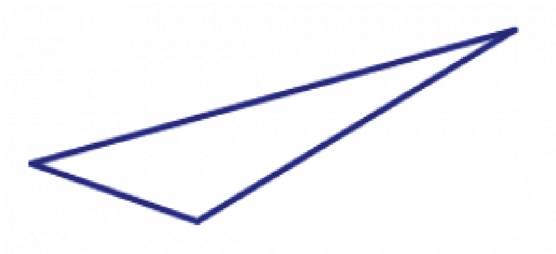
1. In the picture below, which sides are understood to have the same length (even if it doesn't look that way in the drawing)?



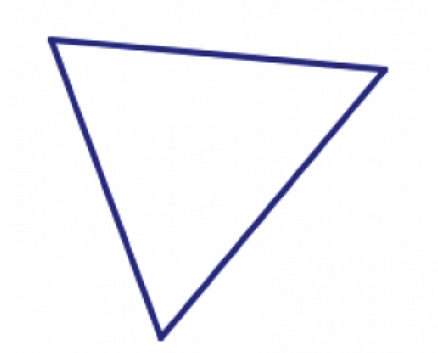
2. In the picture below, which angles are understood to have the same measure (even if if doesn't look that way in the drawing)?



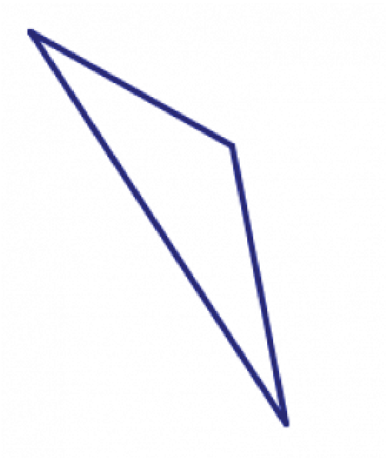
3. Here is a scalene triangle. Sketch two more scalene triangles, each of which is different from the one shown here in some way.



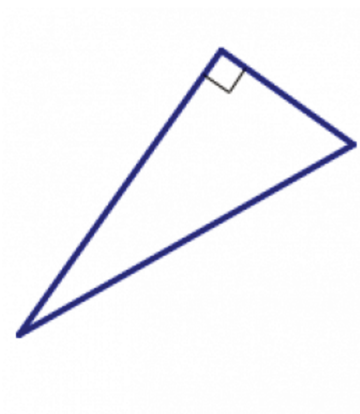
4. Here is an acute triangle. Sketch two more acute triangles, each of which is different from the one shown here in some way.



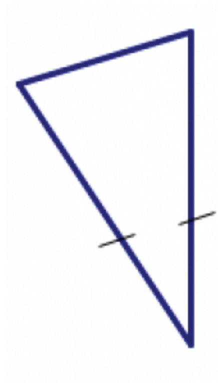
5. Here is an obtuse triangle. Sketch two more obtuse triangles, each of which is different from the one shown here in some way.



6. Here is a right triangle. Sketch two more right triangles, each of which is different from the one shown here in some way. Be sure to indicate which angle is 90°.



7. Here is an isosceles triangle. Sketch two more isosceles triangles, each of which is different from the one shown here in some way. Use tick marks to indicate which sides are equal.

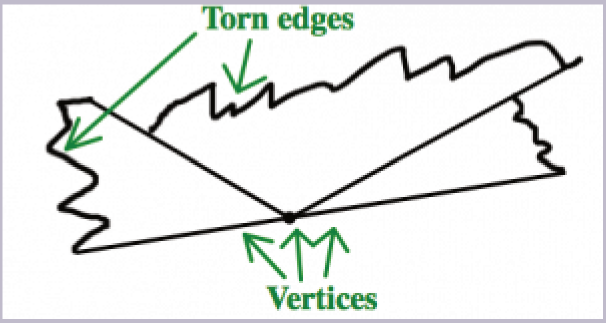


Angle Sum

Think / Pair / Share

By now, you have drawn several different triangles on your paper. Choose one of your triangles, and follow these directions:

- Using scissors, cut the triangle out.
- Tear (do not cut) off the corners, and place the three vertices together. Your should have something that looks a bit like this picture:



What do you notice? What does this suggest about the angles in a triangle?

You may remember learning that the sum of the angles in any triangle is 180°. In your class, you now have lots of examples of triangles where the sum of the angles seems to be 180°. But remember, our drawings are not exact. How can we be sure that our eyes are not deceiving us? How can we be sure that the sum of the angles in a triangle isn't 181° or 178°, but is really 180° on the nose in every case?

Think / Pair / Share

What would convince you beyond all doubt that the sum of the angles in any triangle is 180°? Would testing lots of cases be enough? How many is enough? Could you ever test every possible triangle?

History: Euclid's axioms

Often high school geometry teachers prove the sum of the angles in a triangle is 180°, usually using some facts about parallel lines. But (maybe surprisingly?) it's just as good to take this as an *axiom*, as a given fact about how geometry works, and go from there. Perhaps this is less satisfying than proving it from some other statement, and if you're curious you can certainly find a proof or your instructor can share one with you.

In about 300BC, Euclid[1]Creative Commons Attribution 4.0 International license. was the first mathematician (as far as we know) who tried to write down careful axioms and then build from those axioms rigorous proofs of mathematical truths.

1. Portrait of Euclid from Wikimedia Commons, licensed under the

Euclid

Euclid had five axioms for geometry, the first four of which seemed pretty obvious to mathematicians. People felt they were reasonable assumptions from which to build up geometric truths:

1. Given two points, you can connect them with a straight line segment.

2. Given a line segment, you can extend it as far as you like in either direction, making a line.

3. Given a line segment, you can draw a circle having that segment as a radius.

4. All right angles are congruent.

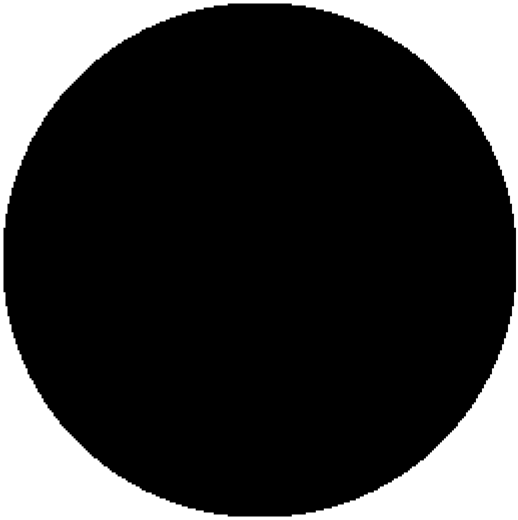
The fifth postulate bothered people a bit more. It was originally stated in more flowery language, but it was equivalent to this statement:

5. The sum of the angles in a triangle is 180°.

It's easy to see why this fifth axiom caused such a ruckus in mathematics. It seemed much less obvious than the other four, and mathematicians felt like they were somehow cheating if they just assumed it rather than proving it had to be true. Many mathematicians spent many, many years trying to prove this fifth axiom from the other axioms, but they couldn't do it. And with good reason: There are other kinds of geometries where the first four axioms are true, but the fifth one is not!

For example, if you do geometry on a sphere — like a basketball or more importantly on the surface of the Earth — rather than on a flat plane, the first four axioms are true. But triangles are a little strange on the surface of the earth. Every triangle you can draw on the surface of the earth has an angle sum strictly *greater than* 180°. In fact, you can draw a triangle on the Earth that has three right angles[2] Creative Commons Attribution-Share Alike 3.0 Unported., making an angle sum of 270°.

2. Image by Coyau / Wikimedia Commons, via Wikimedia Commons, licensed under



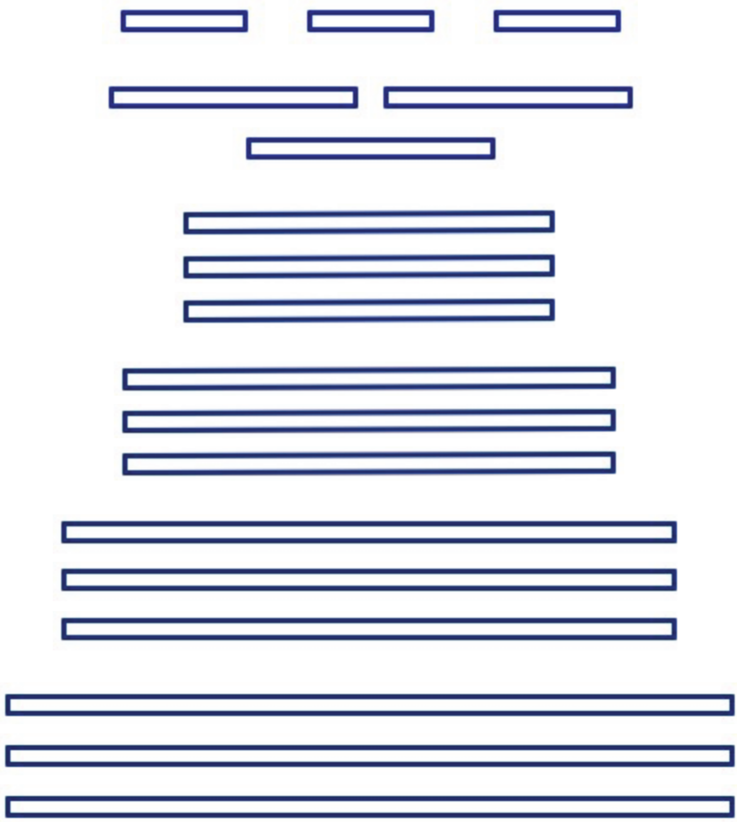
Triangle with three right angles on a sphere.

On a sphere like the Earth, the angle sum is not constant among all triangles. Bigger triangles have bigger angle sums, and smaller triangles have smaller angle sums, but even tiny triangles have angle sums that are greater than 180°.

The geometry you study in school is called *Euclidean geometry*; it is the geometry of a flat plane, of a flat world. It's a pretty good approximation for the little piece of the Earth that we see at any given time, but it's not the only geometry out there!

Triangle Inequality

Make a copy of these strips of paper and cut them out. They have lengths from 1 unit to 6 units. You may want to color the strips, write numbers on them, or do something that makes it easy to keep track of the different lengths.



Problem 3

Repeat the following process several times (at least 10) and keep track of the results (a table has been started for you).

- Pick three strips of paper. (The lengths do not have to be all different from each other; that's why you have multiple copies of each length.)
- Try to make a triangle with those three strips, and decide if you think it is possible or not. (Don't overlap the strips, cut them, or fold them. The length of the strips should be the length of the sides of the triangle.)

Length 1	Length 2	Length 3	Triangle?
4	3	2	yes
4	2	1	no
4	2	2	??

Your goal is to come up with a **rule that describes when three lengths will make a triangle** and when they will not. Write down the rule in your own words.

Think / Pair / Share

Compare your rule with other students. Then use your rule to answer the following questions. Keep in mind the goal is not to try to build the triangle, but to **predict the outcome** based on your rule.

- Suppose you were asked to make a triangle with sides 40 units, 40 units, and 100 units long. Do you think you could do it? Explain your answer.
- Suppose you were asked to make a triangle with sides 2.5 units, 2.6 units, and 5 units long. Do you think you could do it? Explain your answer.

You probably came up with some version of this statement:

Theorem: Triangle Inequality

The sum of the lengths of two sides in a triangle is greater than the length of the third side.

Of course, we know that in geometry we should not believe our eyes. You need to look for an *explanation*. Why does your statement make sense?

Remember that "geometry is the art of good reasoning from bad drawings." Our materials weren't very precise, so how can we be sure this rule we've come up with is is correct?

Well in this case, the rule is really just the same as the saying "the shortest distance between two points is a straight line." In fact, this is exactly what we mean by the words *straight line* in geometry.

SSS Congruence

We say that two triangles (or any two geometric objects) are *congruent* if they are exactly the same shape and the same size. That means that if you could pick one of them up and move it to put down on the other, they would exactly overlap.

Problem 4

Repeat the following process several times and keep track of the results.

- Pick three strips of paper that will definitely form a triangle.
- Try to make two *different* (non-congruent) triangles with the same three strips of paper. Record if you were able to do so.

Problem 5

Repeat the following process several times and keep track of the results.

- Pick four strips of paper and form a quadrilateral with them. (If your four strips do not form a quadrilateral, pick another four strips.)
- Try to make two *different* (non-congruent) quadrilaterals with the same four strips of paper. Record if you were able to do so.

Think / Pair / Share

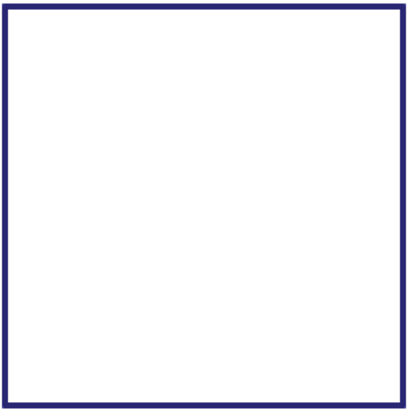
What do you notice from Problems 4 and 5? Can you make a general statement to describe what's going on? Can you explain why your statement makes sense?

You probably came up with some version of this statement:

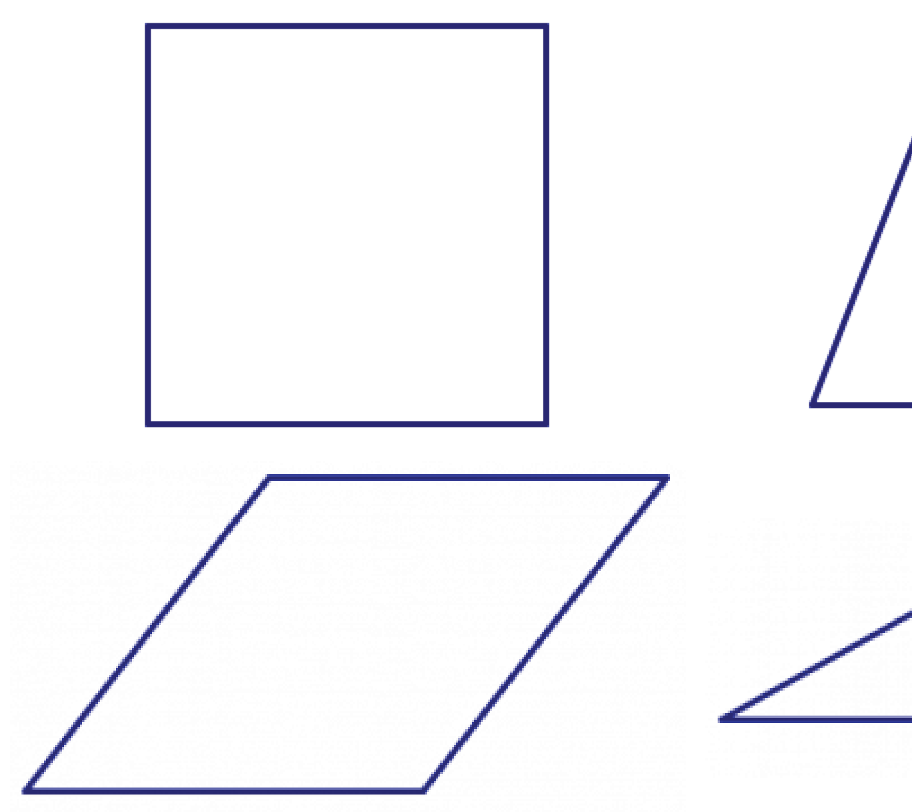
Theorem: SSS (side-side-side) Congruence

If two triangles have the same side lengths, then the triangles are congruent.

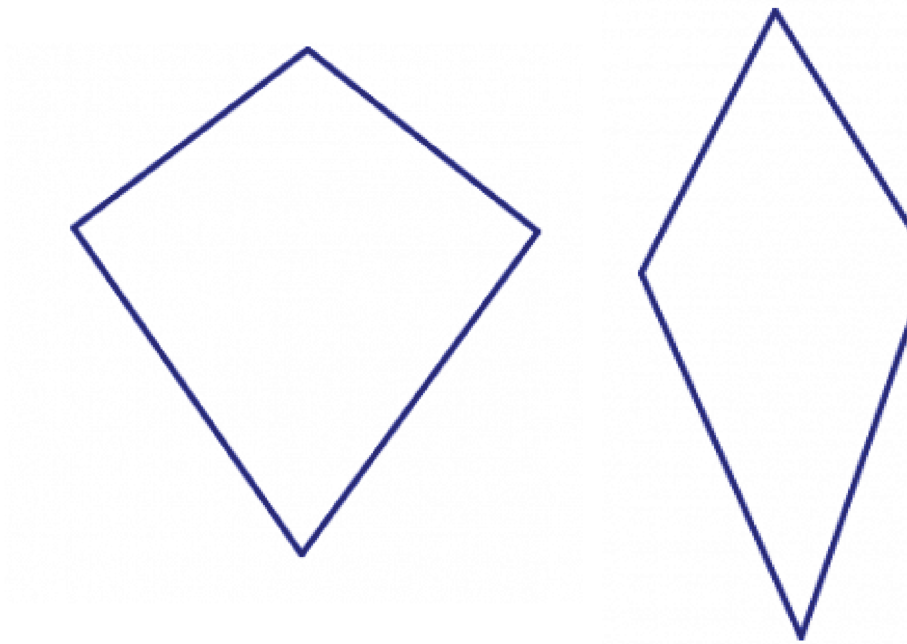
This most certainly is not true for quadrilaterals. For example, if you choose four strips that are all the same length, you can make a square:



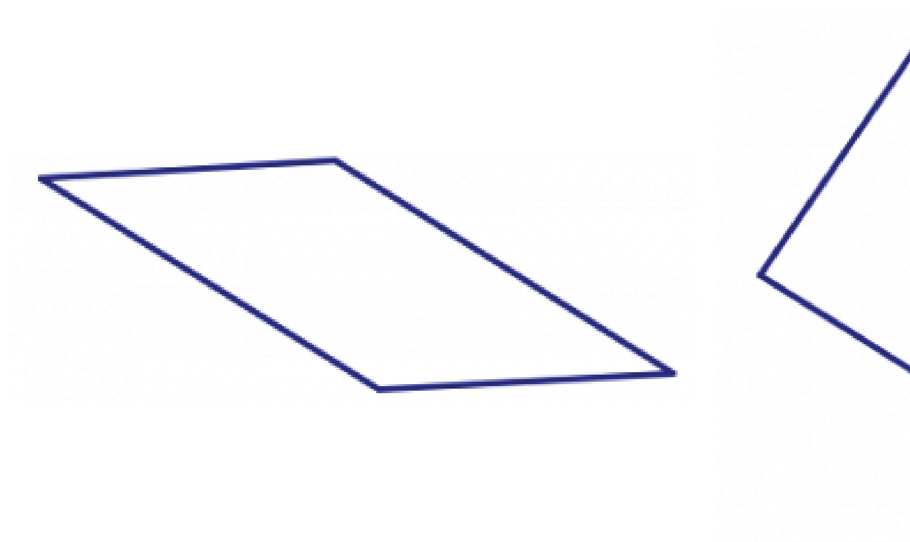
But you can also squish that square into a non-square rhombus. (Try it!)



If you don't choose four lengths that are all the same, in addition to "squishing" the shape, you can rearrange the sides to make different (non-congruent) shapes. (Try it!)



These two quadrilaterals have the same four side lengths in the same order.



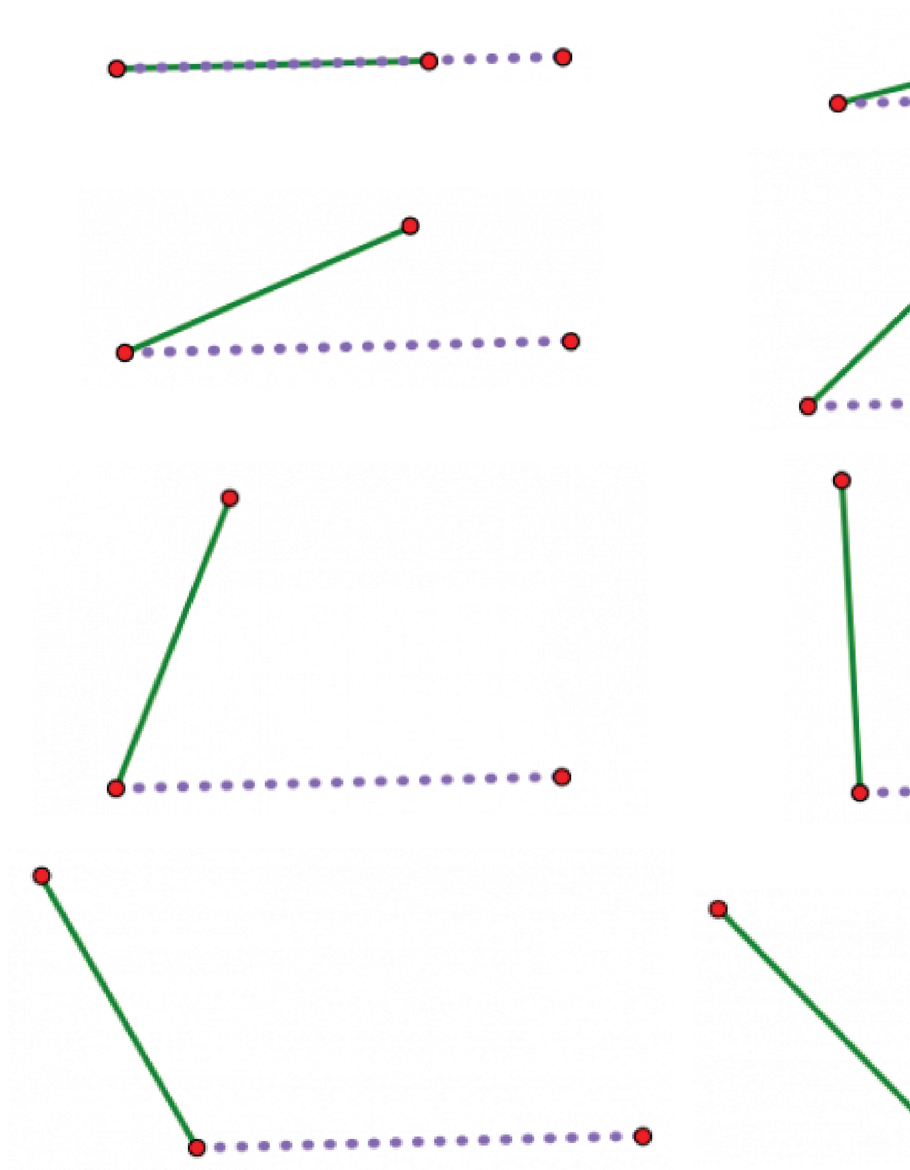
These two quadrilaterals have the same four side lengths as t
but the sides are in a different order.

But this can't happen with triangles. Why not? Well, certainly you can't rearrange the three sides. That would be just the same as rotating the triangle or flipping it over, but not making a new shape.

Why can't the triangles "squish" the way a quadrilateral (and other shapes) can? Here's one way to understand it. Imagine you pick two of your three lengths and lay them on top of each other, hinged at one corner.

This shows a longer purple dashed segment and a shorter green segment.
The two segments are hinged at the red dot on the left.

Now imagine opening up the hinge a little at a time.



As the hinge opens up, the two non-hinged endpoints get farther and farther apart. Whatever your third length is (assuming you are actually able to make a triangle with your three lengths), there is **exactly one position** of the hinge where it will just exactly fit to close off the triangle. No other position will work.

61. Polygons

It can seem like the study of geometry in elementary school is nothing more than learning a bunch of definitions and then classifying objects. In this part, you'll explore some problem solving and reasoning activities that are based in geometry. But definitions are still important! So let's start with this one.

Definition

A **polygon** is:

1. a plane figure
2. that is bounded by a finite number of straight line segments
3. in which each segment meets exactly two others, one at each of its endpoints.

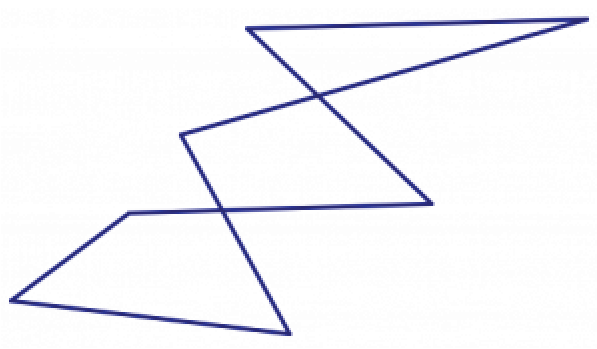
Think / Pair / Share

Just as the first step in problem solving is to understand the problem, the first step in reading a mathematical definition is to understand the definition.

- Use the definition above to draw several examples of figures that are definitely polygons. (You should be able to say why your example fits the definition.)
- Draw several non-examples as well: shapes that are definitely not polygons. (You should be able to say which part of the definition fails for your non-examples.)

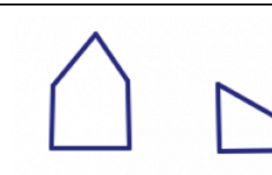
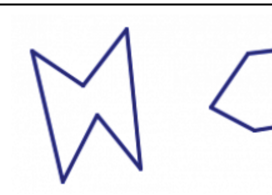
A few comments about polygons:

- The line segments that make up a polygon are called its **edges** and the points where they meet are called its **vertices** (singular: **vertex**).
- Because of properties (2) and (3) in the definition, the boundaries of polygons are not self-intersecting.



Not a polygon.

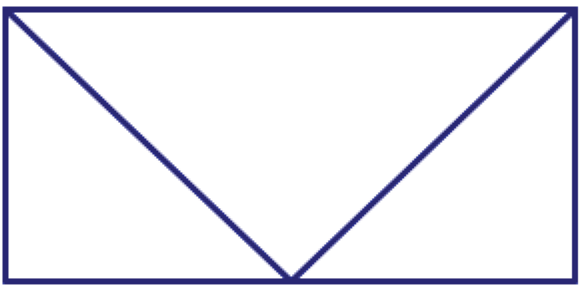
- Polygons are named based on the number of sides they have.

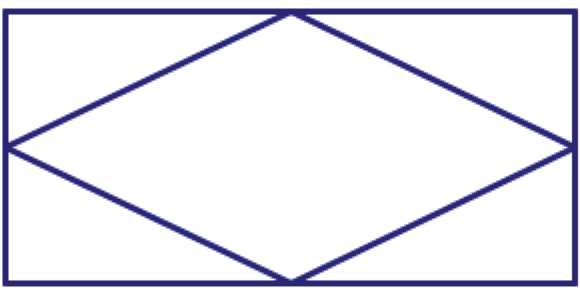
name	# of sides	examples
triangle	3	
quadrilateral	4	
pentagon	5	
hexagon	6	
heptagon	7	
octagon	8	
nonagon	9	
decagon	10	

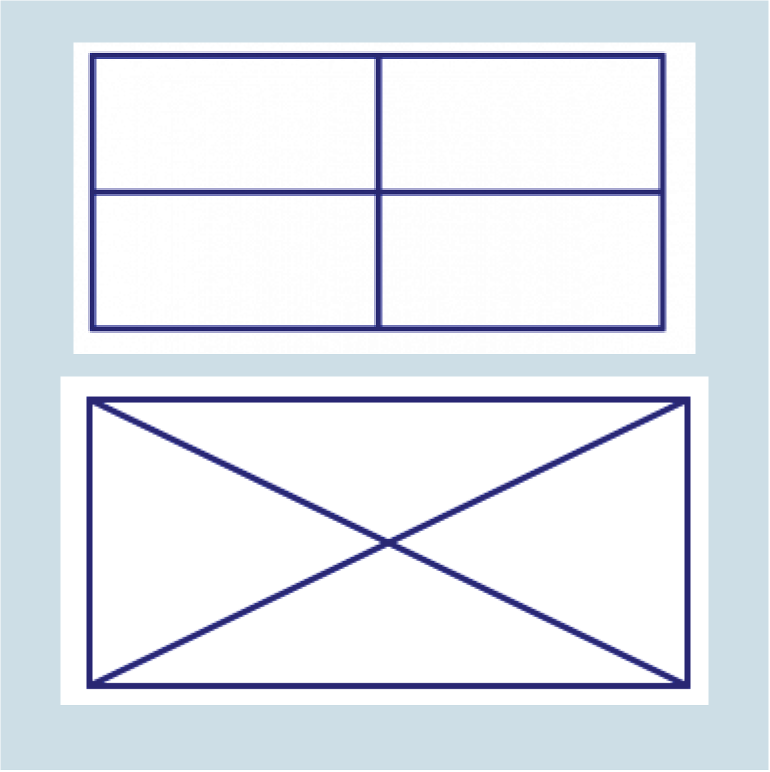
- In general, we call a polygon with n sides an n-*gon*.

Problem 6

In the pictures below, there are polygons hidden in the design. In each design, find all of the triangles, quadrilaterals, pentagons, and hexagons. How can you be sure you've found them all and haven't counted any twice?







Angle Sum

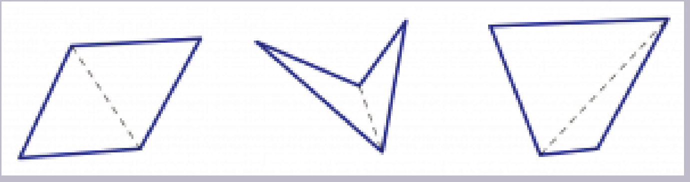
You know that the sum of the interior angles in any triangle is 180°. Can you say anything about the angles in other polygons?

You probably know that rectangles have four 90° angles. So *if if all quadrilaterals have the same interior angle sum*, it must be 360° (since 4 × 90° = 360°).

But notice: We don't necessarily have any reason to believe this constant sum would be true. Remember that SSS congruence is true for triangles, but not for any other polygons. Triangles are special, and we shouldn't assume that true statements about triangles will hold true for other shapes.

Think / Pair / Share

Any quadrilateral can be split into two triangles, where the vertices of the triangles all coincide with the vertices of the quadrilateral:



Use the pictures above to carefully explain why all quadrilaterals do, indeed, have an angle sum of 360°.

On Your Own

Work on the following exercises on your own or with a partner.

1. Draw several different pentagons on your paper. Show that each of them can be split into exactly three triangles in such a way that the vertices of the triangles all coincide with the vertices of the pentagon.
2. Use the fact that every pentagon can be split into three triangles in this way to find the sum of the angles in any pentagon.
3. Draw several different hexagons on your paper. Show that each of them can be split into exactly four triangles so that the

vertices of the triangles all coincide with the vertices of the hexagon.

4. Use the fact that every hexagon can be split into four triangles in this way to find the sum of the angles in any hexagon.

Problem 7

Use your work on the exercises above to complete this general statement:

Angle Sum in Polygons

The sum of the interior angles in an n-gon (a polygon with n sides) is

____________________________.

Explain how you know your statement is true.

Definition

A **regular polygon** has all sides the same length and all angles the same measure.

For example, squares are regular quadrilaterals – all four sides are the same length, and all four angles measure 90°. But a non-square rectangle is *not regular*. Even though all of the angles are 90°, the sides are not all the same length. Similarly, a non-square rhombus is *not regular*. Even though the sides of a rhombus are all the same length, the angles can be different.

Problem 8

Since a square is a regular quadrilateral, you know that every angle in a regular quadrilateral measure 90°. What about angles in other regular polygons?

1. What is the measure of each angle in a regular triangle? Explain how you know you are right.
2. What is the measure of each angle in a regular pentagon? Explain how you know you are right.
3. What is the measure of each angle in a regular hexagon? Explain how you know you are right.
4. What is the measure of each angle in a regular *n-gon*? Explain how you know you are right.

62. Platonic Solids

Of course, we live in a three-dimensional world (at least!), so only studying flat geometry doesn't make a lot of sense. Why not think about some three-dimensional objects as well?

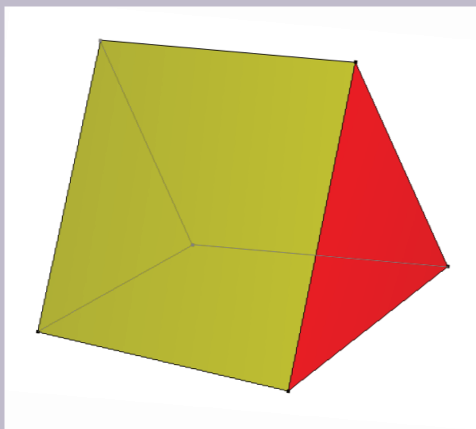
Definition

A **polyhedron** is a solid (3-dimensional) figure bounded by polygons. A polyhedron has **faces** that are flat polygons, straight **edges** where the faces meet in pairs, and **vertices** where three or more edges meet.

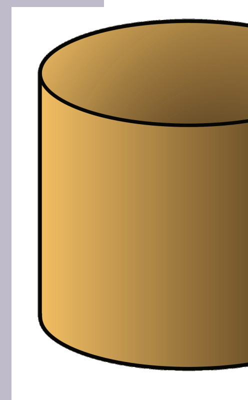
The plural of polyhedron is **polyhedra.**

Think / Pair / Share

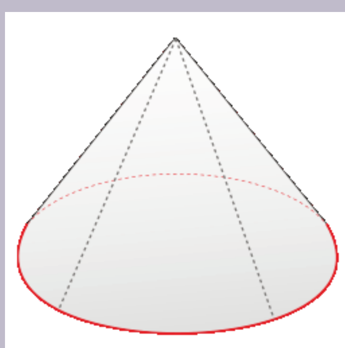
Look at the pictures of solids below, and decide which are polyhedra and which are not. You should be able to say why each figure does or does not fit the definition.



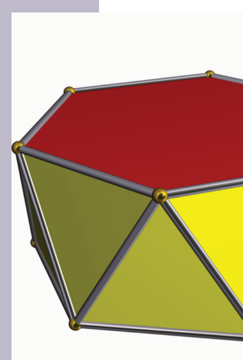
(a)[1]



(b)[2]

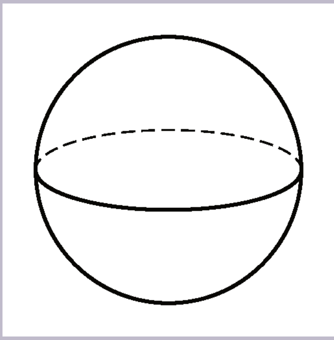


(d)[4]



(e)[5]

1. Image by Tom Ruen [Public domain], via Wikimedia Commons
2. Image via pixababy.com, CC0 Creative Commons license.



(g)[7]

(h)[8]

Remember that a *regular polygon* has all sides the same length and all angles the same measure. There is a similar (if slightly more complicated) notion of *regular* for solid figures.

3. Image by Aldoaldoz (Own work) [CC BY-SA 3.0, via Wikimedia Commons.
4. Image by By Thinkingarena (Own work) [CC BY-SA 4.0], via Wikimedia Commons
5. Image by Robert Webb's Stella software: http://www.software3d.com/Stella.php, via Wikimedia Commons.
6. Image DTR CC-BY-SA-3.0], via Wikimedia Commons
7. Imgae by Stephen.G.McAteer (Own work) [CC BY-SA 3.0], via Wikimedia Commons.
8. Imgae via Wikimedia Commons [Public domain].
9. Image by self [CC BY-SA 3.0], via Wikimedia Commons.

Definition

A **regular polyhedron** has faces that are all *identical (congruent) regular polygons*. All vertices are also identical (the same number of faces meet at each vertex).

Regular polyhedra are also called **Platonic solids** (named for Plato).

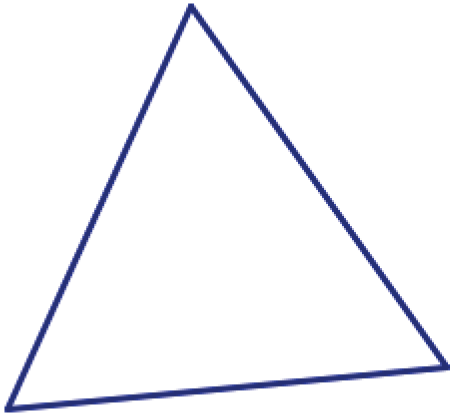
If you fix the number of sides and their length, there is one and only one regular polygon with that number of sides. That is, every regular quadrilateral is a square, but there can be different sized squares. Every regular octagon looks like a stop sign, but it may be scaled up or down. Your job in this section is to figure out what we can say about regular polyhedra.

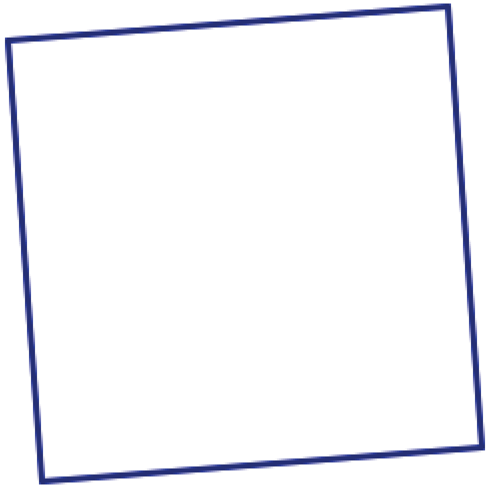
On Your Own

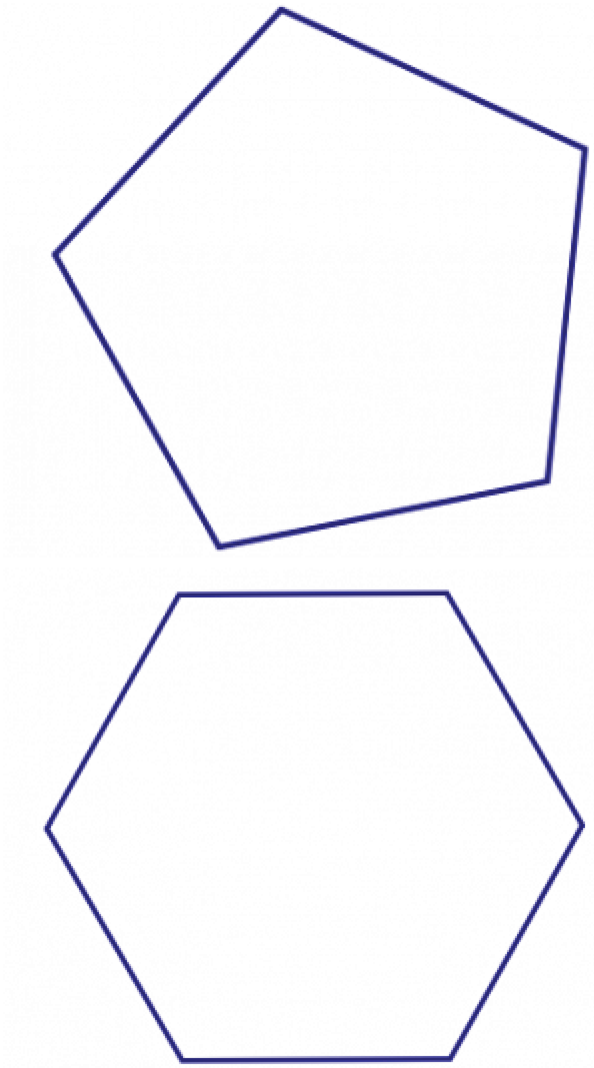
Work on the following exercises on your own or with a partner. You will need to make lots of copies of the regular polygons below. Copy and cut out at least:

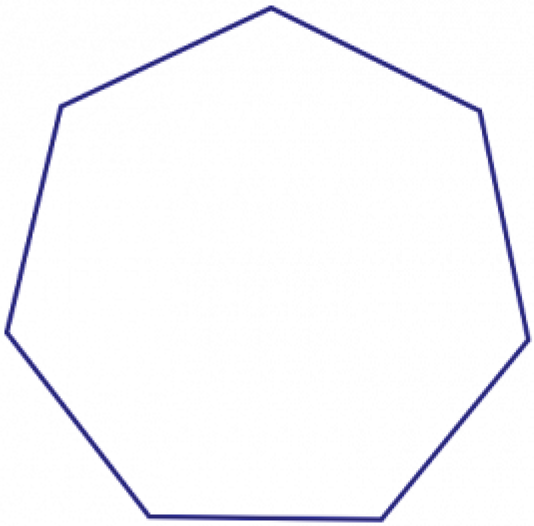
- 40 copies of the equilateral triangle,
- 15 copies of the square,
- 20 copies of the regular pentagon, and
- 10 copies each of the hexagon, heptagon, and octagon.

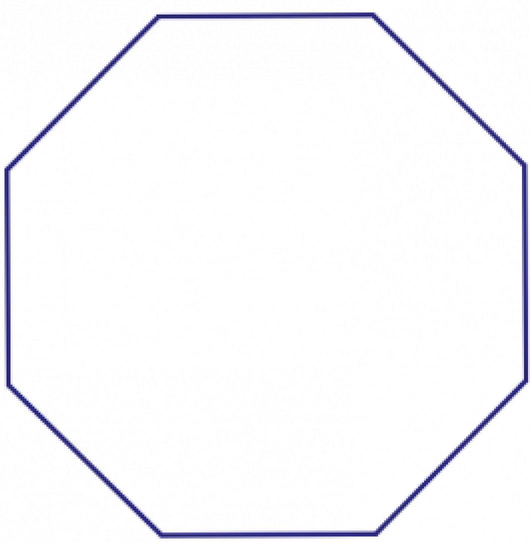
You will also need some tape.











1. In any polyhedron, at least three polygons meet at each vertex.

Start with the equilateral triangles: Put **three** of them together meeting at a vertex and tape them together. Then close them up so they form a solid shape. Can you complete this shape into a platonic solid? Be sure to check that at every vertex you have **exactly three** triangles meeting.

2. Now repeat this process, but start with **four** equilateral triangles around a single vertex. Then close them up so they form a solid shape. Can you complete this into a platonic solid? Be sure to check that at every vertex you have **exactly four** triangles meeting.
3. Repeat this process with **five** equilateral triangles, then six, then seven, and so on. Keep going until you are convinced you understand what's happening with Platonic solids that have triangular faces.
4. When you are done with triangular faces, move on to square faces. Work systematically: Try to build a Platonic solid with three squares at each vertex, then four, then five, etc. Keep going until you can make a definitive statement about Platonic solids with square faces.
5. Repeat this process with the other regular polygons you cut out: pentagons, hexagons, heptagons, and octagons.

You must have noticed that the situation for Platonic solids is quite different from the situation for regular polygons. There are *infinitely many* regular polygons (even if you don't account for size). There is a regular polygon with n sides for every value of n bigger than 2. But for solids, we have the following (perhaps surprising) result.

Theorem

There are exactly five Platonic solids.

The key fact is that for a three-dimensional solid to close up and form a polyhedron, there must be less than 360° around each vertex. Otherwise, it either lies flat (if there is exactly 360°) or folds over on itself (if there is more than 360°).

Problem 9

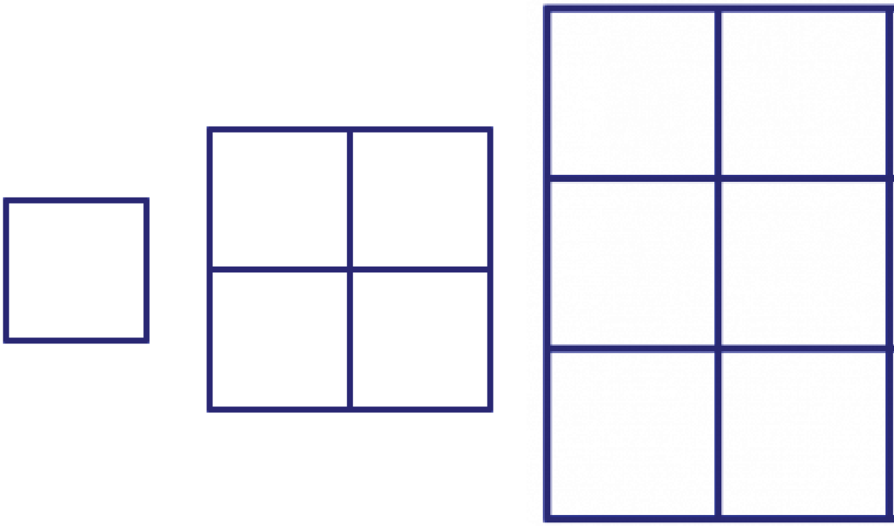
Based on your work in the exercises, you should be able to write a convincing justification of the Theorem above. Here's a sketch, and you should fill in the explanations.

1. If a Platonic solid has faces that are equilateral triangles, then fewer than 6 faces must meet at each vertex. Why?
2. If a Platonic solid has square faces, then three faces can meet at each vertex, but not more than that. Why?
3. If a Platonic solid has faces that are regular pentagons, then three faces can meet at each vertex, but not more than that. Why?
4. Regular hexagons cannot be used as the faces for a Platonic solid. Why?
5. Similarly, regular n-gons for n bigger than 6 cannot be

used as the faces for a Platonic solid. Why?

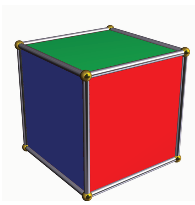
63. Painted Cubes

You can build up squares from smaller squares:

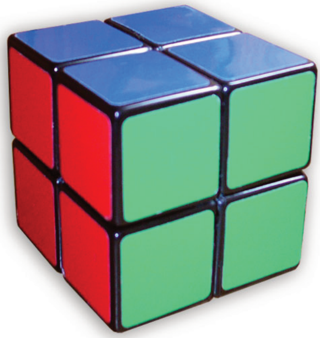


1 × 1 square 2 × 2 square 3 × 3 square

In a similar way, you can build up cubes from smaller cubes:



1 × 1 × 1 cube[1]



2 × 2 × 2 cube[2]

3 ×

Think / Pair / Share

We call a 1 × 1 × 1 cube a **unit cube**.

1. Image by Robert Webb's Stella software: http://www.software3d.com/Stella.php, via Wikimedia Commons.
2. Image by Mike Gonzalez (TheCoffee) (Work by Mike Gonzalez (TheCoffee)) [CC BY-SA 3.0], via Wikimedia Commons.
3. Image by Mike Gonzalez (TheCoffee) (Work by Mike Gonzalez (TheCoffee)) [CC BY-SA 3.0], via Wikimedia Commons.

- How many unit cubes are in a $2 \times 2 \times 2$ cube?
- How many unit cubes are in a $3 \times 3 \times 3$ cube?
- How many unit cubes are in a $n \times n \times n$ cube?

Explain your answers.

Problem 10

Imagine you build a $3 \times 3 \times 3$ cube from 27 small white unit cubes. Then you take your cube and dip it into a bucket of bright blue paint. After the cube dries, you take it apart, separating the small unit cubes.

1. After you take the cube apart, some of the unit cubes are still all white (no blue paint). How many? How do you know you are right?
2. After you take the cube apart, some of the unit cubes have blue paint on just one face. How many? How do you know you are right?
3. After you take the cube apart, some of the unit cubes have blue paint on two faces. How many? How do you know you are right?
4. After you take the cube apart, some of the unit cubes have blue paint on three faces. How many? How do you

know you are right?

5. After you take the cube apart, do any of the unit cubes have blue paint on more than three faces? How many? How do you know you are right?

Problem 11

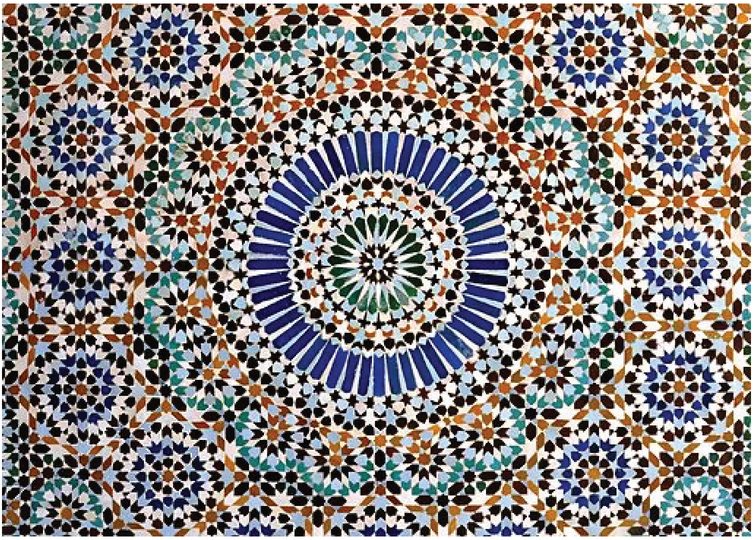
Generalize your work on Problem 10. What if you started with a 2 × 2 × 2 cube? Answer the same questions. What about a 4 × 4 × 4 cube? How about an $n \times n \times n$ cube? Be sure to justify what you say.

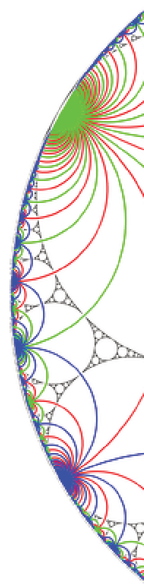
64. Symmetry

Mathematicians use symmetry in all kinds of situations. There can be symmetry in calculations, for example. But the most recognizable kinds of symmetry are those in geometric designs.

Geometric and real-world objects can have different kinds of symmetries[1].

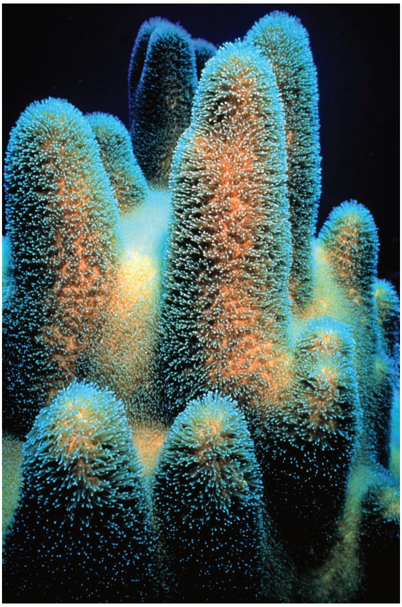
1. Mosaic image by MarcCooperUK (Flickr: Paris central mosque) [CC BY 2.0], via Wikimedia Commons. Apollonian Circle Packing by Tomruen (Own work) [CC BY-SA 3.0], via Wikimedia Commons. Butterfly by Bernard DUPONT from FRANCE (Swallowtail Butterfly (Papilio oribazus)) [CC BY-SA 2.0], via Wikimedia Commons. Starfish by Paul Shaffner [CC BY 2.0], via Wikimedia Commons. Normal distribution from Wikimedia Commons [Public domain]. Water drop from pixababy.com [CC0 Creative Commons].

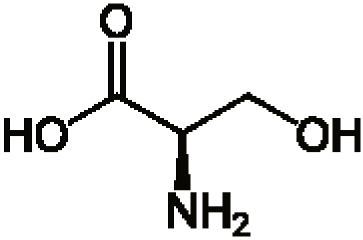




Or they might have no symmetry[2] at all.

2. Pillar coral, wave, and molecule from Wikimedia commons [Public domain]. Head of a woman by Pablo Picasso, image from Gandalf's Gallery on flickr [CC-BY-NC-SA 2.0]





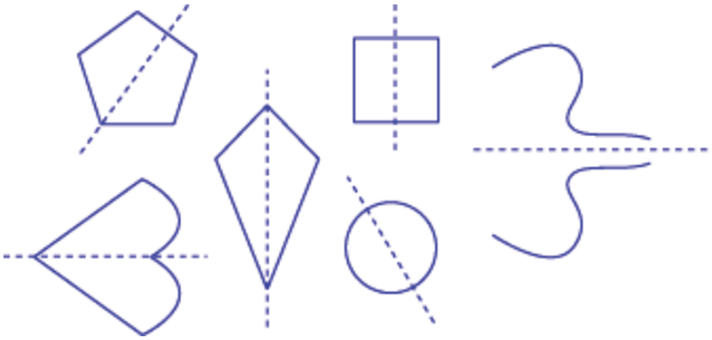
O
HO
OH
NH2

Think / Pair / Share

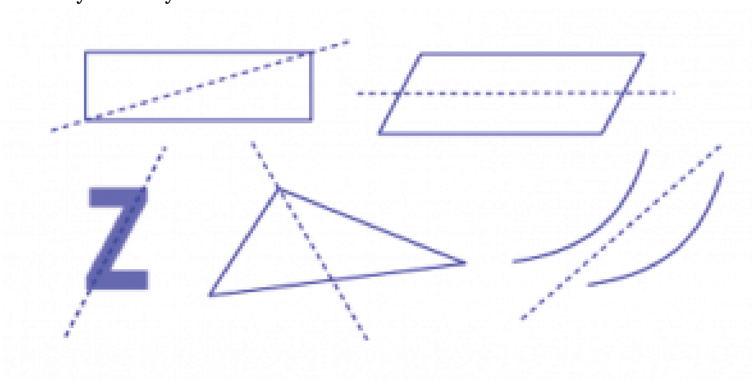
- What do you already know about the idea of *symmetry*? What does it mean to say a design is *symmetric*?
- Do you know about different types of symmetry? What types?
- Can you give examples of real-world objects that are symmetric? What about objects that are not symmetric?

Line Symmetry

If you can flip a figure over a line – this is called *reflecting* the figure – and then it appears unchanged, then the figure has **reflection symmetry** or **line symmetry**. A **line of symmetry** divides an object into two mirror-image halves. The dashed lines below are lines of symmetry:



Compare with the dashed lines below. Though they do cut the figures in half, they don't create mirror-image halves. These are **not** lines of symmetry:



Think / Pair / Share

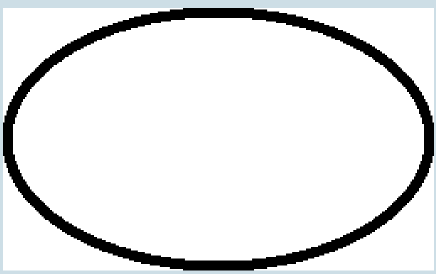
Look at the first set of pictures at the start of this chapter. Do any of them have lines of symmetry? How can you tell?

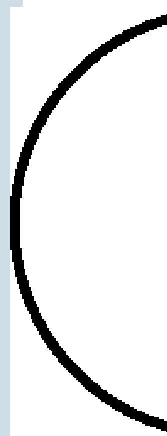
Problem 12

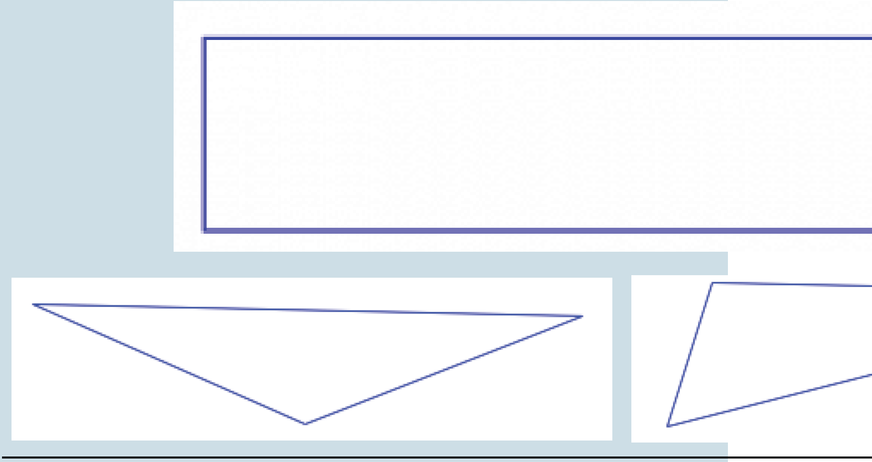
For each of the figures[3] below:

1. Decide if it has any lines of symmetry. If not, how do you know?
2. If it does have one or more lines of symmetry, find / describe all of them. Explain how you did it.

3. Circle and ellipse by Paris 16 (Own work) [CC BY-SA 4.0], via Wikimedia Commons

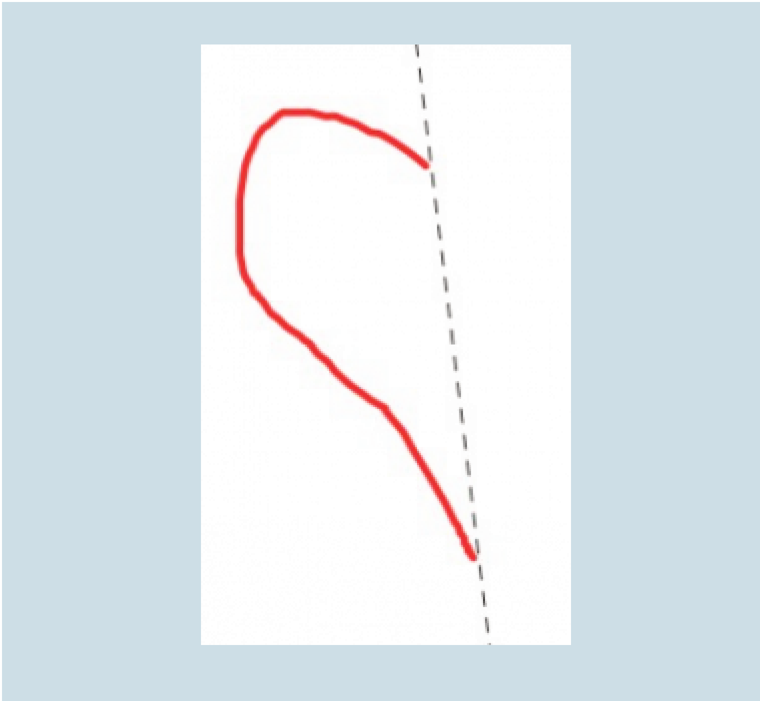


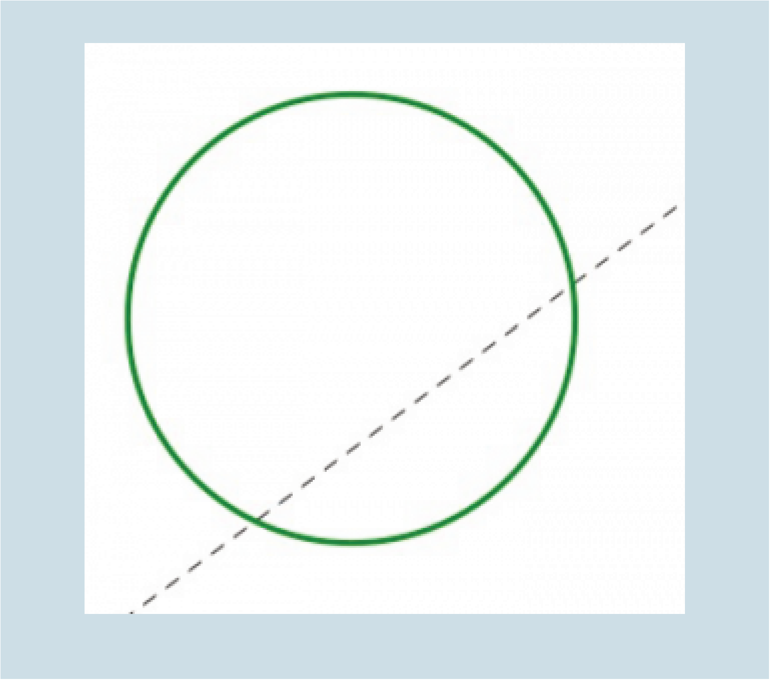




Problem 13

Each picture below shows **half** of a design with line symmetry. The line of symmetry (dashed) is shown. Can you complete the design? Explain how you did it.

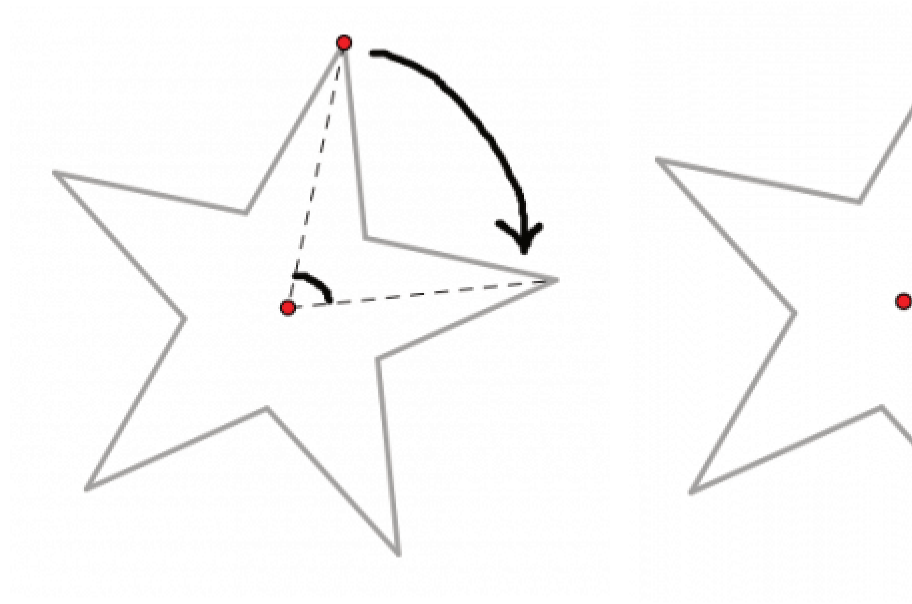




Rotational Symmetry

If you can turn a figure around a center point less than a full circle – this is called a *rotation* – and the figure appears unchanged, then the figure has **rotational symmetry**. The point around which you rotate is called the center of rotation, and the smallest angle you need to turn is called the angle of rotation.

This star has rotational symmetry of 72°, and the center of rotation is the center of the star. One point is marked to help you visualize the rotation.



Think / Pair / Share

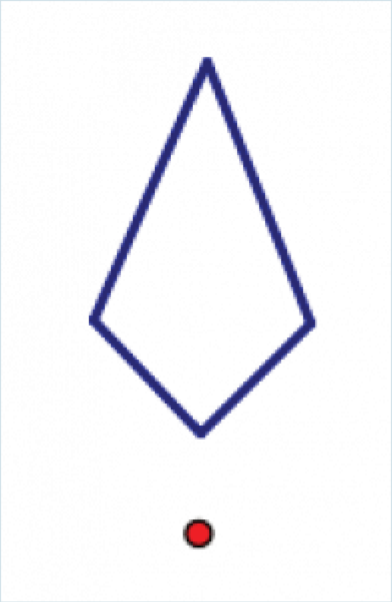
- How can you be certain that the angle of rotation for the star is exactly 72°?
- Look at the first set of pictures at the start of this chapter. Do any of them have rotational symmetry? How can you tell?

Problem 14

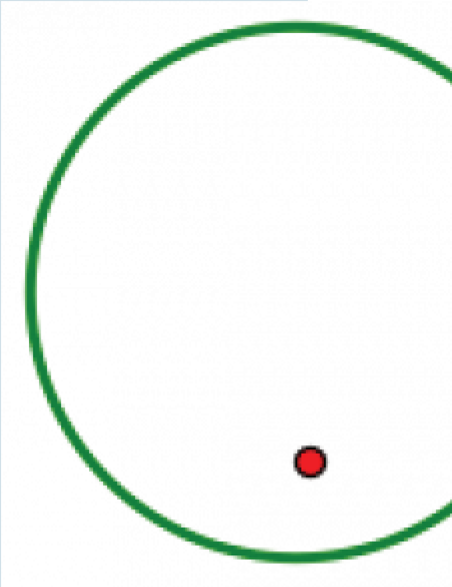
Each of the figures below has rotational symmetry. Find the center of rotation and the angle of rotation. Explain your thinking.

Problem 15

Each picture below shows part of a design with a marked center of rotation and an angle of rotation given. Can you complete the design so that it has the correct rotational symmetry? Explain how you did it.



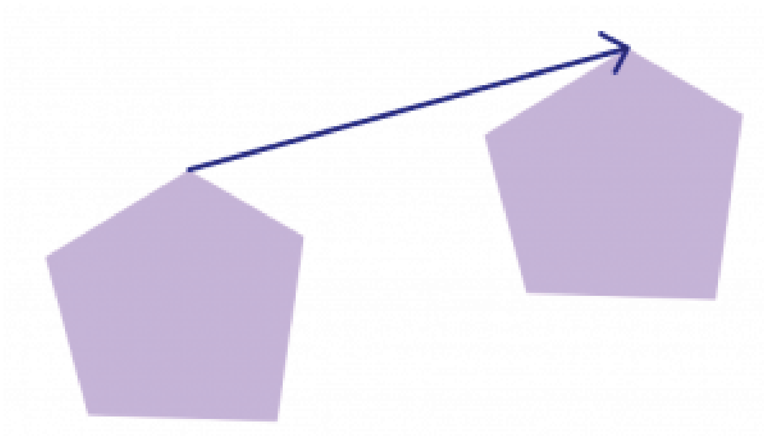
90°



45°

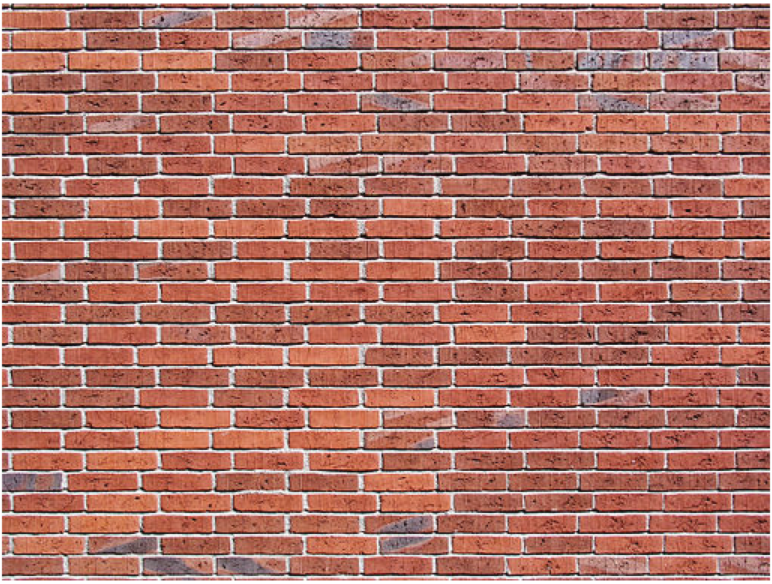
Translational Symmetry

A **translation** (also called a slide) involves moving a figure in a specific direction for a specific distance. A **vector** (a line segment with an arrow on one end) can be used to describe a translation, because the vector communicates both a distance (the length of the segment) and a direction (the direction the arrow points).

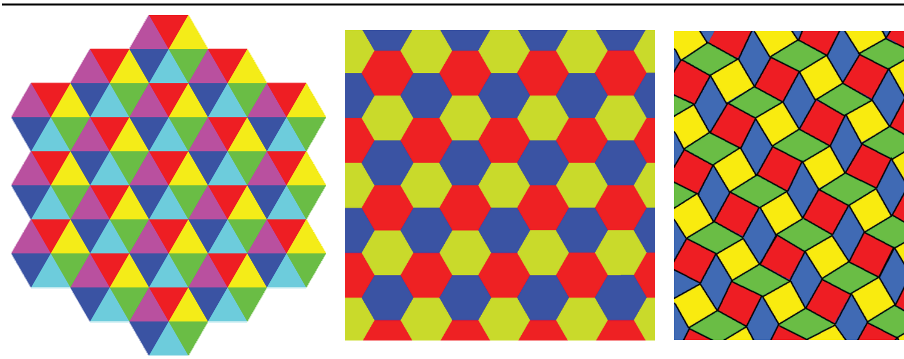


A design has **translational symmetry** if you can perform a translation on it and the figure appears unchanged. A brick wall[4] has translational symmetry in lots of directions!

4. Image by I, Xauxa [CC-BY-SA-3.0], via Wikimedia Commons



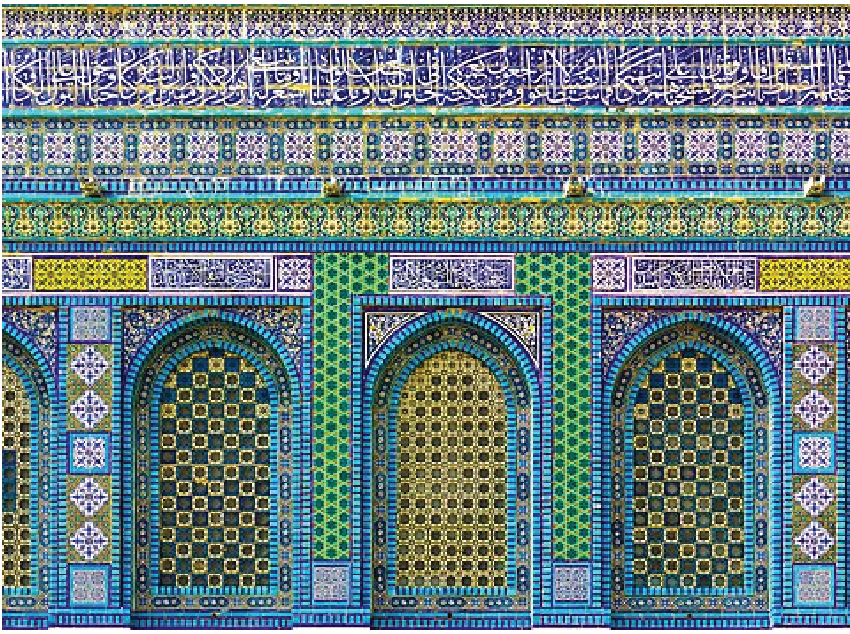
The brick wall is one example of a *tessellation*[5], which you'll learn more about in the next chapter.

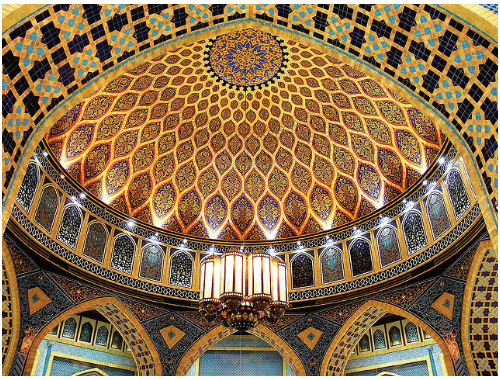


5. Triangular tessellation from pixababy [CC0]. Hexagonal and rhombic tessellations from Wikimedia Commons [Public domain].

You can see translation symmetry in lots of places. It's in architecture and design[6].

6. Tile at Jerusalem temple by Andrew Shiva / Wikipedia, via Wikimedia Commons [CC BY-SA 4.0]. Mosque by Hisham Binsuwaif via flickr [CC BY-SA 2.0]. British Museum great court by Andrew Dunn, http://www.andrewdunnphoto.com/ (Own work) [CC BY-SA 2.0], via Wikimedia Commons





It's in art, most famously that by M.C. Escher. (You might want to visit http://www.mcescher.com/gallery/symmetry/ and browse the "Symmetry" gallery.)

And it appears in traditional Hawaiian and other Polynesian tattoo[7] designs.

7. Royal Hawaiian officer via Wikimedia Commons [Public domain]. Shoulder and arm tattoos by Micael Faccio on flicker [CC BY-2.0].

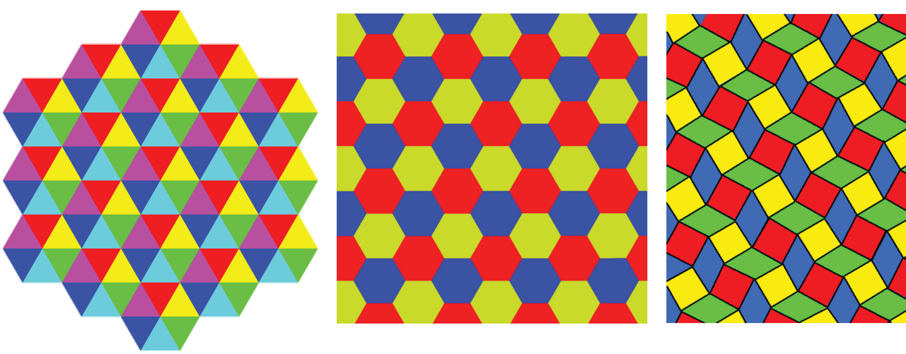
Think / Pair / Share

- On each of the pictures with translational symmetry above, sketch a vector to indicate the direction and distance of the translational symmetry.
- Create your own design with translational symmetry. Explain how you did it.

65. Geometry in Art and Science

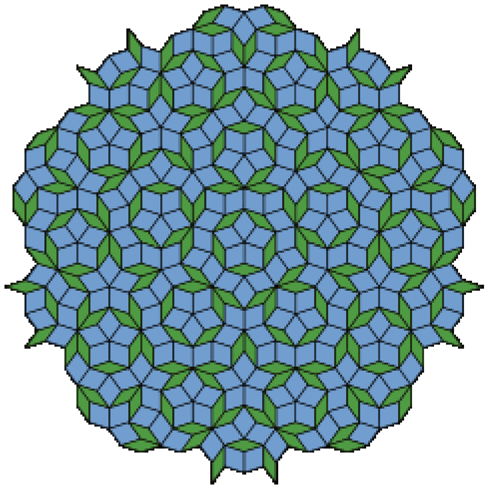
Tessellations

A *tessellation*[1] is a design using one ore more geometric shapes with no overlaps and no gaps. The idea is that the design could be continued infinitely far to cover the whole plane (though of course we can only draw a small portion of it).



Many tessellations have translational symmetry, but it's not strictly necessary. The *Penrose tiling* shown below[2] does not have any translational symmetry.

1. *Triangular tessellation from pixababy [CC0]. Hexagonal and rhombic tessellations from Wikimedia Commons [Public domain].*
2. Image via Wikimedia Commons [Public donmain].

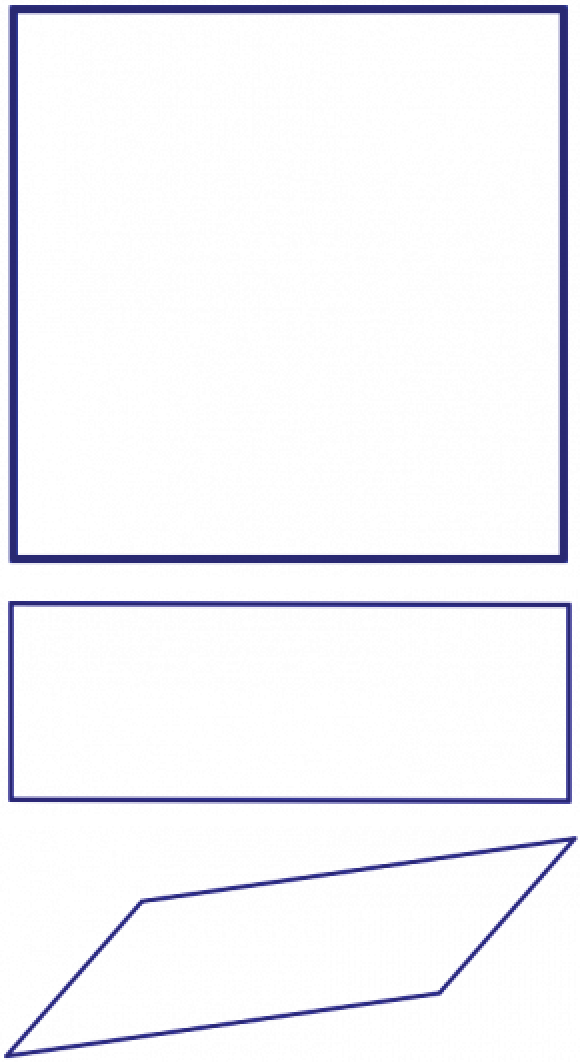


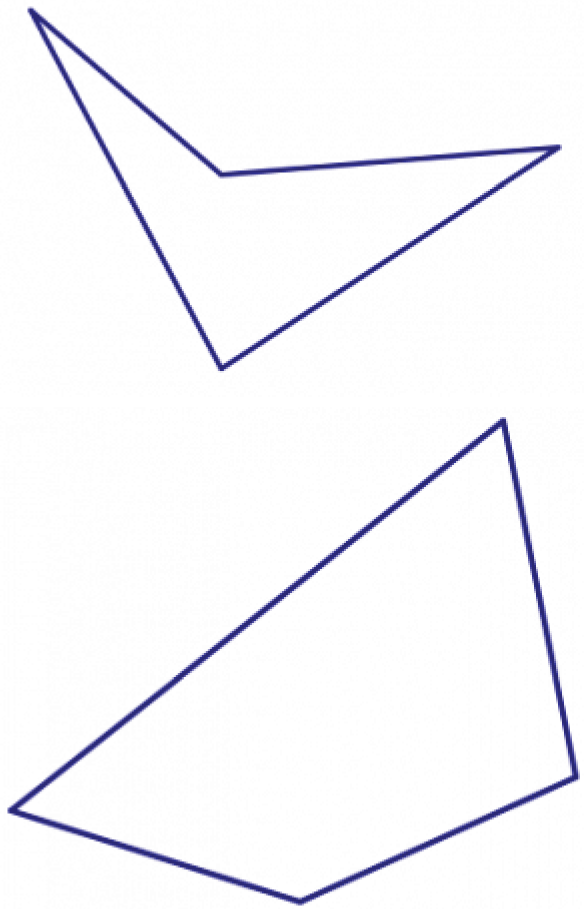
It's actually much harder to come up with these "aperiodic" tessellations than to come up with ones that have translational symmetry. So we'll focus on how to make symmetric tessellations.

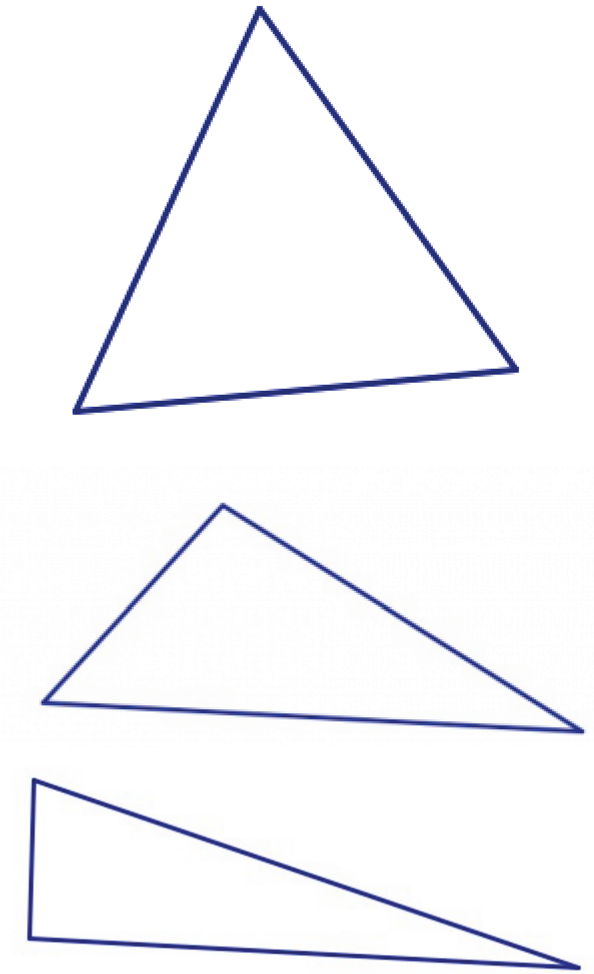
The first two tessellations above were made with a single geometric shape (called a *tile*) designed so that they can fit together without gaps or overlaps. The third design uses two basic tiles. Tessellations are often called *tilings*, and that's what you should think about: If I had tiles made in this shape, could I use them to tile my kitchen floor? Or would it be impossible?

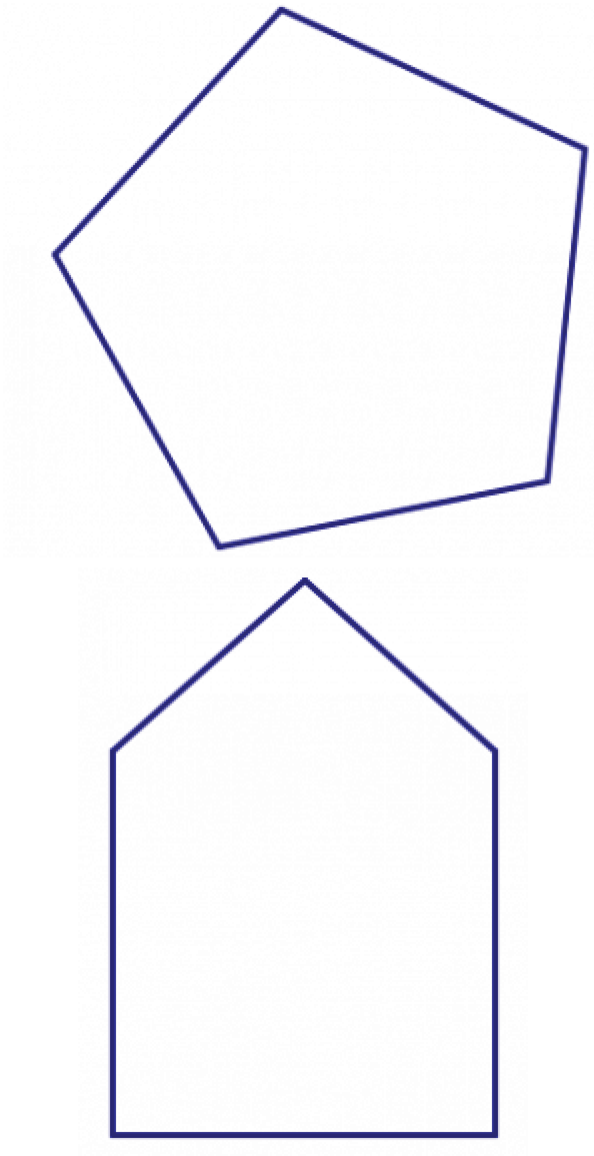
On Your Own

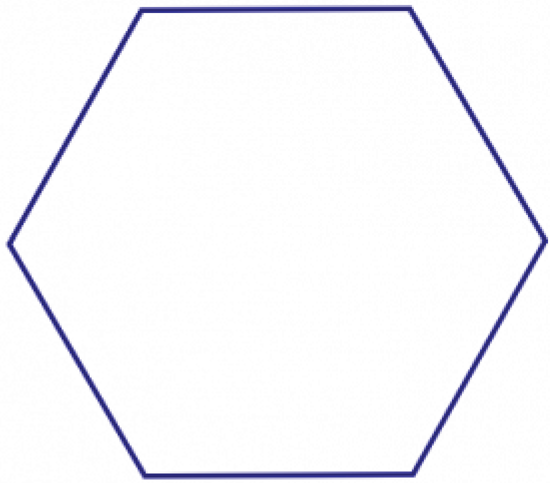
Work on these exercises on your own or with a partner. You will need lots of copies (maybe 10–15 each) of each shape below. In each problem, focus on just a single tile for making your tessellation.

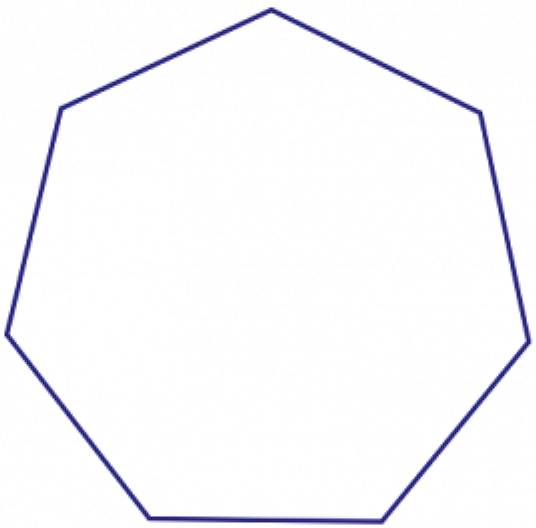


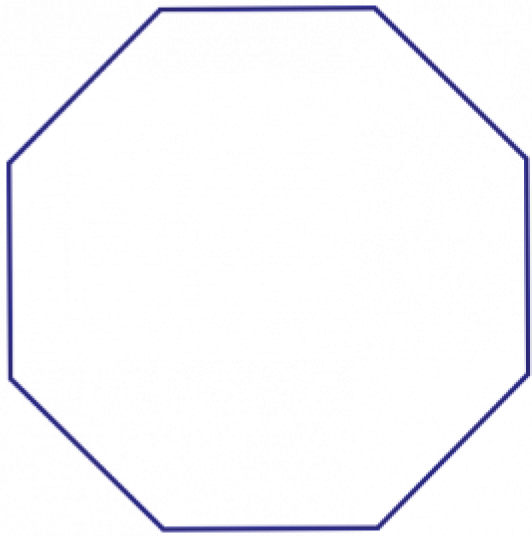












1. Start with the square tile. Can you fit the squares together in a pattern that could be continued forever, with no gaps and no overlaps? Can you do it in more than one way?
2. Now try one of the triangular tiles. Can you use many copies of a single triangle to tessellate the plane?
3. Repeat this process with each of the other tiles. Keep track of your findings.

Think / Pair / Share

- Which of the tiles given above tessellate, and which do not?
- Do you have any conjectures based on this experience, about which shapes will tessellate and which will not?

Escher Drawings

The artist M.C. Escher created many works of art inspired by mathematics, including some very beautiful tessellations. Below you will see some images[3] inspired by his work. You can view the real thing at http://www.mcescher.com/ in the “Symmetry” gallery.

3. Images from flickr [CC BY-NC-SA 2.0]. Birds by Sharon Drummond. Lizard tiles by Ben Lawson.

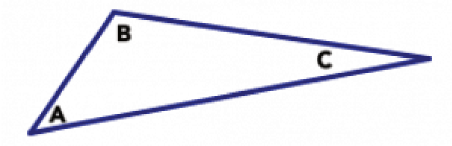
 

You can make your own Escher-like drawings using some facts that you learned while studying tessellations.

Theorem: Tessellations

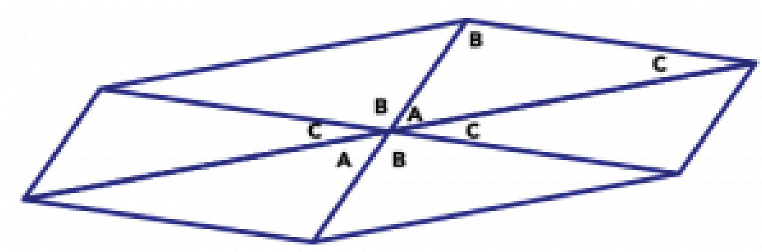
Any triangle will tessellate. So will any quadrilateral.

The explanation for this comes down to what you know about the sums of angles. The sum of the angles in a triangle is 180°.

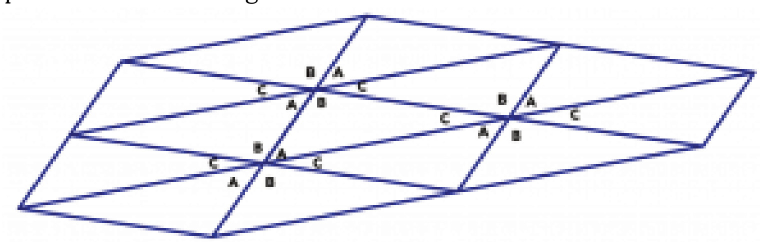


So if you make six copies of a single triangle and put them together at a point so that each angle appears twice, there will be a total of

360° around the point, meaning the triangles fit together perfectly with no gaps and no overlaps.



You can then repeat this at every vertex, using more and more copies of the same triangles.



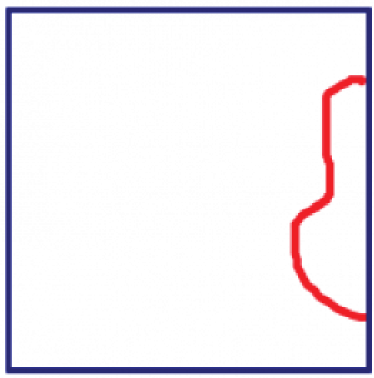
Think / Pair / Share

- Use the fact that the sum of the angles in any quadrilateral is 360° to explain why every quadrilateral will tessellate.
- Use angles to explain why regular hexagons will tessellate.
- Explain why regular pentagons will not tessellate.

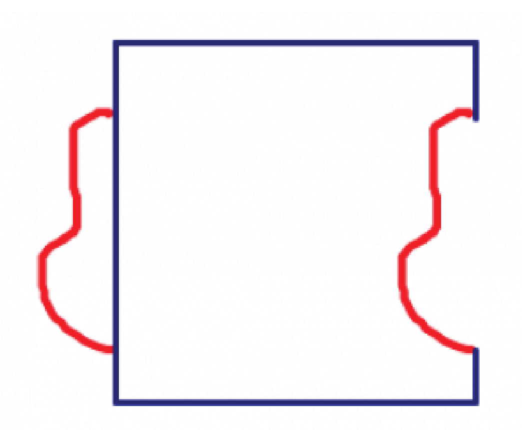
On Your Own

Work on the following exercises on your own or with a partner. Here's how you can create your own Escher-like drawings.

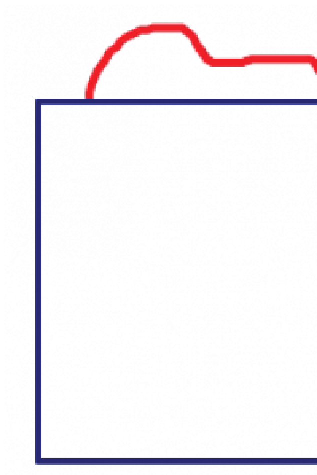
1. Select your basic tile. The first time you do this, it's easiest to start with a simple shape that you know will tessellate, like an equilateral triangle, a square, or a regular hexagon.
2. Draw a "squiggle" on one side of your basic tile.



3. Cut out the squiggle, and move it to another side of your shape. You can either translate it straight across or rotate it.

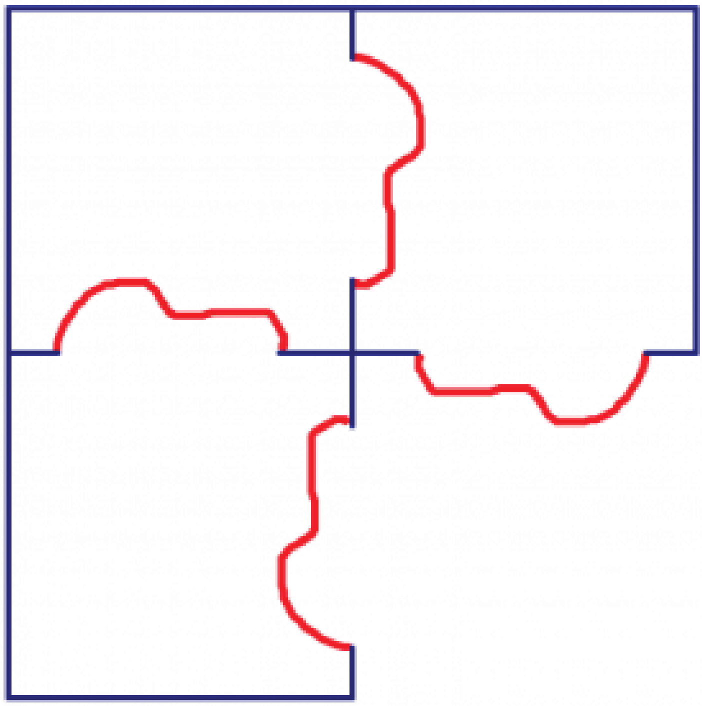


or



4. It's important that the cut-out lines up along the new edge in the same place that it appeared on its original edge.

5. Tape the squiggle into its new location. This is your basic tile. On a large piece of paper, trace around your tile. Then move it the same way you moved the squiggle (translate or rotate) so that the squiggle fits in exactly where you cut it out.



6. The shape will still tessellate, so go ahead and fill up your paper.

7. Now get creative. Color in your basic shape to look like something – an animal? a flower? a colorful blob? Add color and design throughout the tessellation to transform it into your own Escher-like drawing.

8. If you want to try a more complicated version, cut two different squiggles out of two different sides, and move them both.

Building Towers

For this activity, you will need some construction materials:

- You'll need lots of toothpicks.
- You'll also need something to connect the toothpicks together. The best material for this is mini marshmallows; you can stick the ends of the toothpicks into the marshmallows to connect them. You can also use pieces of clay, bits of gummy candies, or other similar (sticky) material.

Problem 16

Try this as a warm-up activity. Grab exactly six toothpicks. Your job is to make four triangles using all six toothpicks. You cannot break any of the toothpicks or add any other materials besides the marshmallow connectors.

Problem 17

Now comes the main challenge. You have ten minutes to build the *tallest free-standing structure* that you can make. "Free-standing" means that it will stand up on its own. You can't hold it up or lean it against something. When the ten minutes are up, back away from your tower and measure its height.

Think / Pair / Share

Look at your own tower and at other students' towers. Talk about these questions:

- What design choices led to taller free-stranding structures? Why do you think that is?
- If you had another ten minutes to try this activity again, what would you do differently and why?

66. Problem Bank

Problem 18

In the Tangrams chapter, you first saw all 7 tangram pieces arranged into a square.

1. If the large square you made with all seven pieces is one whole, assign a (fractional) value to each of the seven tangram pieces. Justify your answers.
2. The tangram puzzle contains a small square. If the small square (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.
3. The tangram set contains two large triangles. If a large triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.
4. The tangram set contains one medium triangle. If the medium triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.
5. The tangram set contains two small triangles. If a small triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.

Problem 19

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. A right triangle that is scalene.
2. A right triangle that is isosceles.
3. A right triangle that is equilateral.

Problem 20

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. An acute triangle that is scalene.
2. An acute triangle that is isosceles.
3. An acute triangle that is equilateral.

Problem 21

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. An obtuse triangle that is scalene.
2. An obtuse triangle that is isosceles.
3. An obtuse triangle that is equilateral.

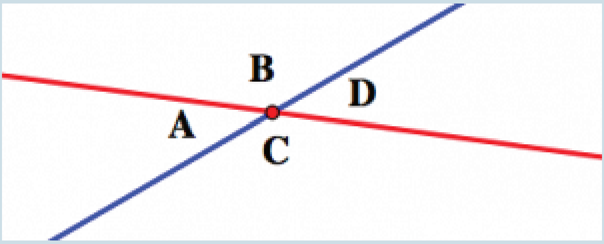
Problem 22

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. An equiangular triangle that is scalene.
2. An equiangular triangle that is isosceles.
3. An equiangular triangle that is equilateral.

Problem 23

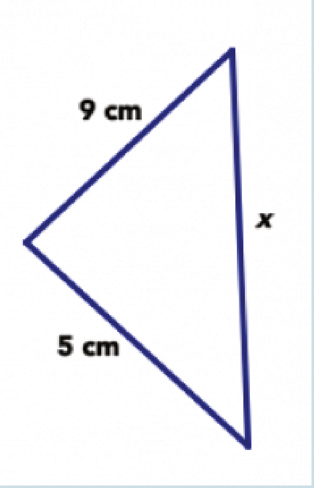
Look at the picture below, which shows two lines intersecting. Angles A and D are called "vertical angles," and so are angles B and C.



Use this drawing to explain why vertical angles must have the same measure. (Hint: what is the sum of the measures of angle A angle B? How do you know?)

Problem 24

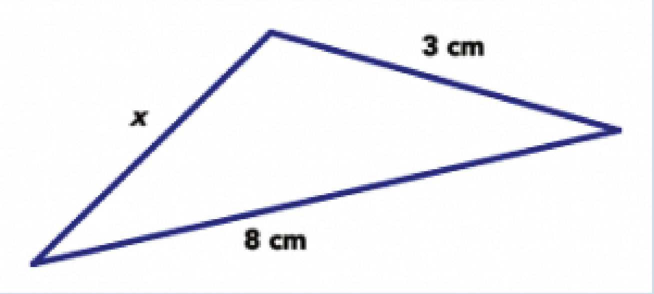
Answer the following questions about the triangle below. Be sure to focus on what you *know for sure* and not what the picture *looks like.*



1. Could it be true that x = 4 cm? Explain your answer.
2. Could it be true that x = 20 cm? Explain your answer.
3. Give three possible values of x, based on the information in the picture.

Problem 25

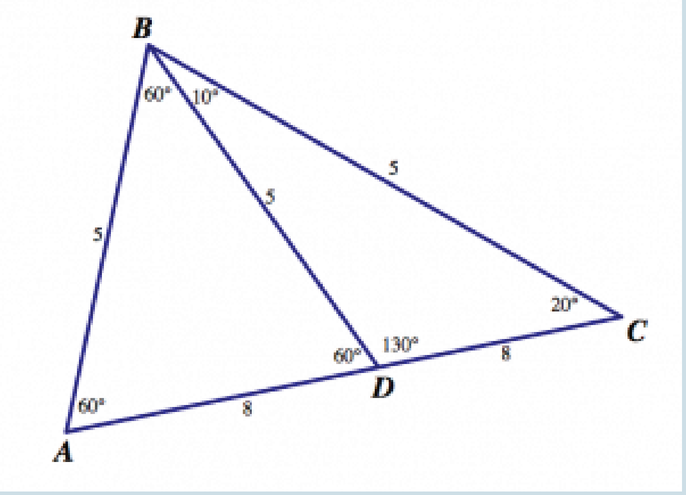
Answer the following questions about the triangle below. Be sure to focus on what you know for sure and not what the picture looks like.



1. If x = 3 cm, the triangle is isosceles. Is this possible? Explain your answer.
2. If x = 8 cm, the triangle is isosceles. Is this possible? Explain your answer.
3. Give three *impossible* values of x, based on the information in the picture.

Problem 26

Prof. Faber drew this picture on the board, saying it showed three triangles: △ABC, △ABD, and △CBD. Side lengths and angle measurements are shown for each of the triangles.



There are **lots of mistakes** in this picture. Use what you know about side lengths and angles in triangles to find all the mistakes you can. For each mistake, say what is wrong with the picture, and why it's a mistake. Explain your thinking as clearly as you can.

Problem 27

Because of SSS congruence, triangles are exceptionally sturdy. This means they are used frequently in architecture and design to provide supports for buildings, bridges, and other man-made objects. Take your camera with you, and find several places in your neighborhood or near your campus that use triangular supports. Snap a picture, and describe what the structure is and where you see the triangles.

Problem 28

It is possible to create designs that have multiple symmetries. See if you can find images (or create your own!) that have both:

1. reflection symmetry and rotational symmetry,
2. reflection symmetry and translational symmetry, and
3. rotational symmetry and translational symmetry.

PART VIII

VOYAGING ON HŌKŪLE`A

> We sail for peace, for the love of our planet and with the desire to leave the children of the world a hopeful and healthy future.
> -Hōkūle`a crew

Unless otherwise noted, photos and drawings in this Part come from the "Press Room at Outreach Tools" at http://hokulea.com and are used here non-commercially in accordance with their agreement.

67. Introduction

In the 1950's and 1960's, historians couldn't agree on how the Polynesian islands – including the Hawaiian islands – were settled. Some historians insisted that Pacific Islanders sailed deliberately around the Pacific Ocean, relocating as necessary, and settling the islands with purpose and planning. Others insisted that such a navigational and voyaging feat was impossible thousands of years ago, before European sailors would leave the sight of land and sail into the open ocean. These historians believed that the Polynesian canoes were caught up in storms, tossed and turned, and eventually washed up on the shores of faraway isles.

Think / Pair / Share

- How could such a debate ever be settled one way or the other, given that we can't go back in time to find out what happened?
- What kinds of evidence would support the idea of "intentional voyages"? What kinds of evidence would support the idea of "accidental drift"?
- What do you already know about how this debate was eventually settled?

68. Hōkūle`a

The Polynesian Voyaging Society (PVS) was founded in 1973 for scientific inquiry into the history and heritage of Hawai`i: How did the Polynesians discover and settle these islands? How did they navigate without instruments, guiding themselves across ocean distances of 2500 miles or more?

In 1973–1975, PVS built a replica of an ancient double-hulled voyaging canoe to conduct an experimental voyage from Hawai`i to Tahiti. The canoe was designed by founder Herb Kawainui Kāne and named Hōkūle`a ("Star of Gladness").

On March 8th, 1975, Hōkūle`a was launched. Mau Piailug, a master navigator from the island of Satawal in Micronesia, navigated her to Tahiti using traditional navigation techniques (no modern instruments at all).

Think / Pair / Share

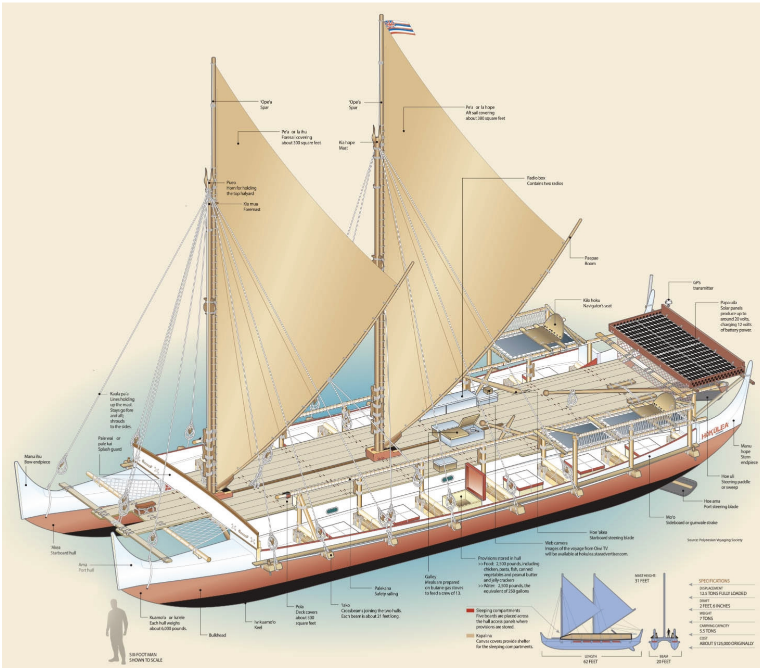
- What are some mathematical questions you can ask about voyaging on Hōkūle`a?
- What kinds of problems (especially mathematics problems) did the crew have to solve before setting off on the voyage to Tahiti?
- What are you curious about, with respect to voyaging on Hōkūle`a?

When you teach elementary school, you will mostly likely be teaching all subjects to your students. One thing you should think about as a teacher: How can you connect the different subjects together? Specifically, how can you see mathematics in other fields of study, and how can you draw out that mathematical content?

In this chapter, you'll explore just a tiny bit of the mathematics involved in voyaging on a traditional canoe. You will apply your knowledge of geometry to create scale drawings and make a star compass. And you'll use your knowledge of operations and algebraic thinking to plan the supplies for the voyage. The focus here is on applying your mathematical knowledge to a new situation.

One of the first things to know about Hōkūle`a[1] is what she looks like. You can find more pictures at http://hokulea.com.

1. Hokulea homecoming picture by Michelle Manes.



Problem 1

Here's some information about the dimensions of Hōkūle`a. Your job is to draw a good scale model of the canoe, like a floor plan.

- Hōkūle`a is 62 feet 4 inches long. (This is "LOA" or "length overall" in navigation terms. It means the maximum length measured parallel to the waterline.)

- Hōkūle`a is 17 feet 6 inches wide. (This is “at beam” meaning at the widest point.)
- You can see from the picture that Hōkūle`a has two hulls, connected by a rectangular deck. The deck is about 40 feet long and 10 feet wide.

Imagine you are above the canoe looking down at it. Draw a scale model of the hulls and the deck. Do not include the sails or any details; you are aiming to convey the overall *shape* in a scale drawing.

You will use this scale drawing several times in the rest of this unit, so be sure to do a good job and keep it somewhere that you can find it later.

Note: You don’t have all the information you need! So you either need to find out the missing information or make some reasonable estimates based on what you do know.

Problem 2

Crew for a voyage is usually 12–16 people. During meal times, the whole crew is on the deck together. About how much space does each person get when they’re all together on the deck?

69. Worldwide Voyage

To Prepare for next activity:

1. Read this description of the daily life on Hōkūle`a: http://pvs.kcc.hawaii.edu/ike/canoe_living/daily_life.html.
2. Watch the video about the Worldwide Voyage:

A Vimeo element has been excluded from this version of the text. You can view it online here: http://pressbooks.oer.hawaii.edu/mathforelementaryteachers/?p=654

From the webpage above, you learned:

> *The quartermaster is responsible for provisioning the canoe – loading food, water and all needed supplies, and for maintaining Hōkūle`a's inventory. While this is not an on board job, it is critical to the safe and efficient sailing of the canoe.*

Problem 3

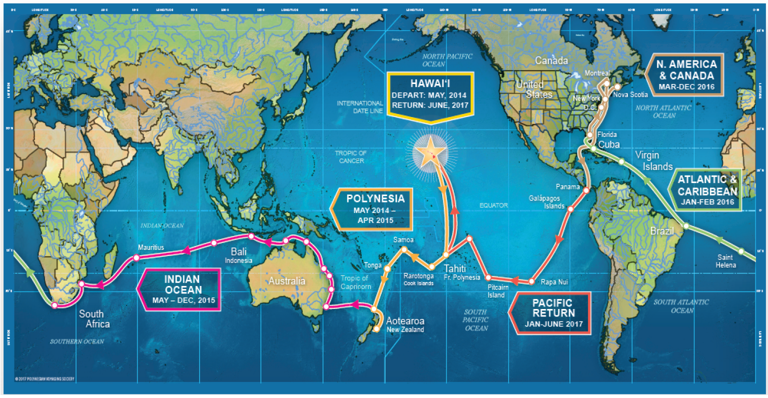
Imagine that you are part of the crew for the Worldwide Voyage, and you are going to help the quartermaster and the captain with provisioning the canoe for one leg of the voyage. You need to write a preliminary report for the quartermaster, documenting:

1. Which leg of the trip are you focused on? (See the map below.)
2. How long will that leg of the trip take? Explain how you figured that out.
3. How much food and water will you need for the voyage? Explain how you figured that out.

The rest of this section contains pointers to information that may or may not be helpful to you as you make your plans and create your report. Your job is to do the relevant research and then write your report. You should include enough detail about how you came to your conclusions that the quartermaster can understand your reasoning.

Pick a leg of the route:

Here's a picture of the route planned for the Worldwide Voyage, which you can find at the Worldwide Voyage website: http://www.hokulea.com/worldwide-voyage/ and a full-sized map here: https://tinyurl.com/WWVmap. On the map, the different colors correspond to different years of the voyage. A "leg" means a dot-to-dot route on the map.



After you pick a leg of the voyage, you'll need to figure out the total distance of that leg. This tool might help (or you can find another way): http://www.acscdg.com/.

Here is some relevant information to help you figure out how long it will take Hōkūle`a to complete your chosen leg:

- The first trip from Hawai`i to Tahiti in 1976 took a total of 34 days. (You probably want to use the tool above to compute the number of nautical miles.)

Plan the provisions:

Here is some information about provisions.

- Hōkūle`a can carry about 11,000 pounds, including the weight of the crew, provisions, supplies, and personal gear.
- The supplies (sails, cooking equipment, safety equipment, communications equipment, etc.) account for about 3,500 pounds.
- The crew eats three meals per day and each crew member gets 0.8 gallons of water per day.
- For a trip that is expected to take 30 days, the quartermaster

plans for 40 days' worth of supplies, in case of bad weather and other delays.

70. Navigation

The following is from http://pvs.kcc.hawaii.edu/ike/hookele/modern_wayfinding.html.

> *A voyage undertaken using modern wayfinding has three components:*
>
> *Design a course strategy, which includes a reference course for reaching the vicinity of one's destination, hopefully upwind, so that the canoe can sail downwind to the destination rather than having to tack into the wind to get there. (Tacking involves sailing back and forth as closely as possible into the wind to make progress against the wind; its very arduous and time-consuming, something to be avoided if at all possible, particularly at the end of a long, difficult voyage.)*
>
> *During the voyage, holding as closely as possible to the reference course while keeping track of (1) distance and direction traveled; (2) one's position north and south and east and west of the reference course and (3) the distance and direction to the destination.*
>
> *Finding land after entering the vicinity of the destination, called a target screen or 'the box'.*

So how is the navigation done – especially component (2) – through thousands of miles of open ocean? You can't see land. How can you hold closely to the reference course? How can you keep track of distance and direction traveled? How can you even know if you're going in the right direction if all you can see is blue ocean and blue sky?

By day, the navigators use their deep knowledge of the oceans. Which way do the winds blow? Which way do the prevailing currents move? Clouds in the sky, flotsam in the water, and animal behaviors give them great insight into where land might be, and where they are in relation to it.

By night, they use the stars. In this section, you'll learn just a tiny fraction of what these master navigators know about the stars.

Think / Pair / Share

Here is a time-lapse picture[1] of the stars in the night sky:

- Describe what you see happening in this picture.
- What can you conclude about how the stars move through the night sky?
- How might that help a navigator find his way?

1. Image from pixababy [CC0 Creative Commons].

Star Compass

A fundamental tool for navigators on Hōkūle`a and other voyaging canoes is a star compass. Here's a picture of Mau Piailug[2] and a star compass[3] he used in his teaching.

The object in the center of the circle represents the canoe. The shells along the outside represent directional points. The idea is to imagine the stars rising up from the horizon in the east, traveling through

2. Picture by Maiden Voyage Productions [CC BY-SA 3.0], via Wikimedia Commons
3. Image by Newportm (Own work) [CC BY-SA 3.0], via Wikimedia Commons.

the night sky, and setting past the horizon in the west. They move like they're on a sphere surrounding the Earth (it's called the celestial sphere).

Problem 4

Nainoa Thompson developed a star compass with 32 equidistant points around a circle. (Note this is different than the points in Mau's star compass pictured above.) You will first try to make a rough sketch of Nainoa's star compass based on this information.

- Place 32 points around the circle so they are equally spaced.
- The arcs between these equidistant points are called "houses." You will label each house with its Hawaiian name. Start with the four cardinal directions:

`Ākau: North.

Hema: South.

Hikina: East.

Komohana: West.

- The four quadrants also get names. (These cover all of the houses in the quadrant, so label them in the appropriate place inside the compass.)

Ko`olau: northeast.

Malani: southeast.

Kona: southwest.

Ho`olua: northwest.

- Moving from `Ākau to Hikina (clockwise), there are seven houses. They are labeled in order as you move away from `Ākau:

Haka: “empty,” describing the skies in this house.

Nā Leo: “the voices” of the stars speaking to the navigator.

Nālani: “the heavens.”

Manu: “bird,” the Polynesian metaphor for a canoe.

Noio: the Hawaiian tern (a bird).

`Āina: “land.”

Lā: “sun,” which stays in this house most of the year.

- The compass has a vertical line of symmetry, so there are the same seven houses in the same order as you move from `Ākau to Komohana (counterclockwise).
- The compass also has a horizontal line of symmetry. Use that fact to label the houses from Hema to Hikina (counterclockwise) and from Hema to Komohana (clockwise).

How is the star compass used in navigation? There are lots of ways. Here’s a (very!) quick overview:

- The canoe is pictured in the middle of the star compass, with all of the houses around.

- Winds and ocean swells move directly across the star compass from north to south or vice versa.

  - If the swells are coming from `Āina Ko`olau, they will be heading in the direction `Āina Kona. (Look at your star compass and trace out this path.)
  - If the wind is coming from Nālani Malani, it will be heading towards Nālani Ho`olua. (Look at your star compass and trace out this path.)

- Stars stay in their houses, but also in their hemisphere. They do not move across the center of the circle.
- Just like the sun, they rise in the east and set in the west.

  - `A`ā (Sirius) rises in Lā Malanai and sets in Lā Kona. (Look at your star compass and trace out this path.)
  - Hōkūle`a rises in `Āina Ko`olau and sets in `Āina Ho`olua. (Look at your star compass and trace out this path.)

A navigator memorizes the houses of over 200 stars. At sunrise and sunset (when the sun or the stars are rising), the navigator can use the star compass to memorize which way the wind is moving and which way the currents are moving. The navigator can then use that information throughout the day or night to ensure the canoe stays on course.

Think / Pair / Share

Look again at the time-lapse picture of the stars:

- Describe how this shows that stars “stay in their houses” and in their hemisphere as they move through the night sky.
- The star Ke ali`i o kona i ka lewa (Canopus), rises in Nālani Malanai. Where does it set?

When teaching navigation while sitting on land, it’s perfectly fine to have a rough sketch or model of the star compass. But if you really have to do the navigation, you need to make a very, very precise star compass.

Imagine Nainoa Thompson, who navigated Hōkūle`a on the final leg of her journey from Hawai`i to Rapa Nui, an island even smaller and lower than Ni`ihau. You have to be within 30 miles of Rapa Nui to see it. But a mistake of even one degree would have led to Hōkūle`a being 60 miles off course. And if you end up drifting in the open ocean and supplies run out? Well…

Nainoa Thompson

Problem 5

Now that you have a rough sketch of the star compass and know what it should look like, your job is to draw one that's as perfect as possible. That means you want to draw:

1. A perfect circle (well, as perfect as possible). What tools can you use to do that? What tools would ancient Polynesian navigators have had to use?
2. Thirty-two points around the circle that are exactly evenly spaced apart. (What tools would help you? What tools would ancient Polynesian navigators have had to use?)
3. When you have finished, label your perfectly drawn star compass with the houses.

Of course, a star compass on a piece of paper isn't so useful when you're out on a canoe. How do you position it properly? And how do you keep it from getting lost, damaged, or soaking wet? You paint it on the rails of the canoe, permanently!

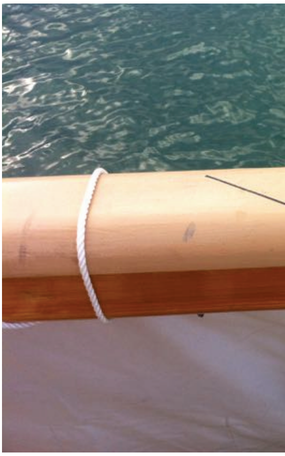
Look back at the drawing of Hōkūle`a. Find the "kilo" (navigator's seat) in the rear (aft) of the canoe. There is actually one navigator's seat on either side of the deck.

Problem 6

Go back to the scale drawing of Hōkūle`a that you made in Problem 1. Add the navigator's seats to your drawing. You will then add the star compass to the rails as follows:

1. Start with the kilo (seat) on the left (port) side of the canoe. That will be the center of your star compass. Imagine looking to the right. You want to see the star compass markings on the rails when you look to the right. Of course, Hōkūle`a is not a circular canoe, and the navigator doesn't sit at the center. So how can you make the markings in the right places?
2. Now repeat that process, using the seat on the right (starboard) side of the canoe.

The kilo onHōkūle`a[4].



Compass markings on the r
before v

Nainoa Thompson has said:

> *Initially, I depended on geometry and analytic mathematics to help me in my quest to navigate the ancient way. However as my ocean time and my time with Mau have grown, I have internalized this knowledge. I rely less on mathematics and come closer and closer to navigating the way the ancients did.*

Really he is still doing a lot of mathematics; it's just mathematics that he has internalized and that is now second nature to him. The ancient navigators may not have spoken of their navigation techniques in the same modern language we've been using – compass points and perfect circles and degrees. But their mathematical understanding was truly astonishing.

4. Photo by Michelle Manes.
5. Photo by Michelle Manes.